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Abstract. In this paper the author examines Casteels' path model for H-prime ideals in quantum matrix algebras, introduces a graph theoretic analogue of polynormality and gives a new proofof the Goodearl-Lenagan polynormality conjecture for the algebras $\mathcal{O}_q(M_{2,N}(K))$.

1. Introduction

In recent decades it has been of much interest in the fields of mathematical physics and non-commutative geometry to study what are known as deformations of familiar algebraic objects. Perhaps the simplest example of this phenomenon is the idea of quasi-commuting variables. Two elements x, y of some algebra are said to q-commute if $xy = qyx$ where q is some scalar. For instance one can define an algebra over $\mathbb{Q}$ with generators x, y subject only to the relation that $xy = -yx$. This can be viewed as a deformation of the usual ring of rational two variable polynomials.

One would like to understand the structural changes that are made to algebras upon undergoing a deformation. The standard way to do this is to study a topological space associated with the new algebra called the spectrum given the so called Zariski topology. Within the field of classical commutative algebraic geometry, the spectrum of the coordinate ring of an algebraic variety in some sense models the same structure as the variety itself. This duality is the motivation behind non-commutative geometry where one in part studies the spectra of algebras as non-commutative versions of familiar objects like algebraic varieties.

Recently, there have arisen new methods of study in this front based around the links between combinatorics and algebra. Some of those methods like the one which will be elaborated upon below, make use of graph theoretic concepts to explain both commutative and non-commutative phenomena. One highly successful example of this idea is Casteels' model for Quantum Matrices. This model helped to prove many results about the structure of special prime ideals in quantum matrix algebras called H-primes but did not immediately yield a combinatorial proof of the Goodearl-Lenagan polynormality conjecture which was proven using Lie-algebraic methods by Milen Yakimov in 2011. Here we give a combinatorial proof of the polynormality conjecture in the subcase for $2 \times N$ quantum matrix algebras and introduce a dual notion of polynormality for digraphs.

2. Generalities on Quantum Matrix Algebras

Let K be a field. For the remainder of this paper fix a scalar q which is transcendental over K. The following is a list of relevant definitions for a problem which can only be stated after a sufficient coverage of certain details. For $n, m \in \mathbb{N}$ define the coordinate ring of n-by-m quantum matrices, notated as $\mathcal{O}_q(M_{n,m}(K))$, to be the K-algebra with matrix of generators

$$M = \begin{bmatrix} X_{11} & X_{12} & X_{13} & \dots & X_{1m} \\ X_{21} & X_{22} & X_{23} & \dots & X_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & X_{n3} & \dots & X_{nm} \end{bmatrix}$$

subject to the relation that if

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is any 2 by 2 submatrix of M then

(1) $ab = qba$

(2) $ac = qca$

(3) $bd = qdb$

(4) $cd = qdc$

(5) $bc = cb$

(6) $ad - da = \hat{q}bc$

where the notation $\hat{q} = q - q^{-1}$ is typically used for convenience.

One can see that variables in this algebra quasi commute along rows and columns up to q. This algebra is a deformation of the usually commutative coordinate ring of n by m matrices with coefficients in K. One can recover the commutative version through a sufficient localization and quotient but we will be interested in the more general non-commutative case. The quantum matrix algebra is meant to be modeled after the way normal matrices work but in a dual sense. The space of n-by-m matrices over a field has both left and right actions on column vectors and row vectors respectively and $\mathcal{O}_q(M_{n,m}(k))$ has left and right coactions on similar spaces known as quantum affine spaces. It is not immediately relevant here but for the interested reader, for $n = m$, $\mathcal{O}_q(M_{n,m}(k))$ admits a natural bialgebra structure with counit which annihilates all but the diagonal of M (which is mapped to 1) and coproduct given by

$$\Delta(X_{ij}) = \sum_{k=1}^{n} X_{ik} \otimes X_{kj}.$$

We will need the following class of special elements of $\mathcal{O}_q(M_{n,m}(K))$. Given two sets $I \subset \{1, ..., n\}$ and $J \subset \{1, ..., m\}$ with $|I| = |J| = r$ define a quantum minor $[I|J]$ by

$$[I|J] = \sum_{\sigma \in S_r} (-q)^{L(\sigma)} X_{i_1, j_{\sigma(1)}} \cdots X_{i_r, j_{\sigma(r)}}.$$

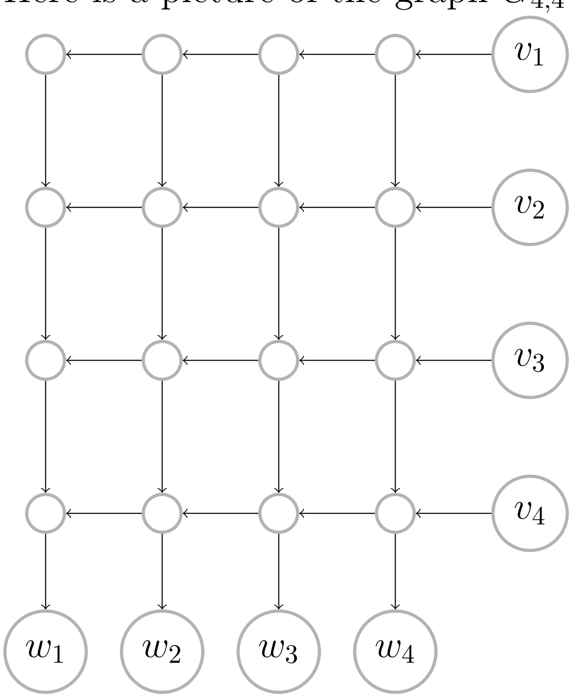
A prime ideal P in $\mathcal{O}_q(M_{n,m}(K))$ is an H-prime ideal iff for all $\gamma = (\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_m)$ the map defined by $h_\gamma(X_{ij}) = \alpha_i \beta_j X_{ij}$ fixes P i.e. $h_\gamma(P) = P$.

3. Path Model

What follows now is a brief outline of Casteels' model for H-prime ideals For our purposes an n by m grid graph, $G_{n,m}$ is the directed graph with vertices $\{(i,j) : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{v_1, ..., v_n\} \cup \{w_1, ..., w_m\}$ with directed edges

- $(i,j) \to (i, j-1)$ for $j \neq 1$
- $(i,j) \to (i-1, j)$ for $i \neq 1$
- $v_i \to (i, m)$
- $(n, j) \to w_j$

Here is a picture of the graph $G_{4,4}$



A Cauchon diagram on an n by m rectangular array is a subset of squares S such that for all $(i, j) \in S$ at least one of the following holds:

- $i = 1$
- $j = 1$
- $(i-1, j) \in S$
- $(i, j-1) \in S$.

As an example consider the simple 4×4 Cauchon diagram

$$D = \{(1,2),(2,1),(2,2),(3,2),(4,2),(2,3),(2,4)\}.$$

The importance of Cauchon diagrams in this context comes from a theorem of Casteels (4.20, [1]) that shows any H-prime ideal of $\mathcal{O}_q(M_{n,m}(K))$ will be generated from a Cauchon diagram in the following sense:

- Pick any n by m Cauchon diagram

- Alter the n by m grid graph $G_{n,m}$ by removing all vertices whose corresponding square is in the Cauchon diagram and extend edges across the removed vertices to reconnect the graph.

- For a set of sources I and a set of sinks J both with the same size, consider the set of vertex disjoint path systems from I to J in this new graph. Explicitly this means that if one enumerates both I and J in increasing order say $I = \{i_1, ..., i_r\}$, $J = \{j_1, ..., j_r\}$, consider only the collections of dipaths in this altered grid graph $p_1 : v_{i_1} \to w_{j_1}$, ... , $p_r : v_{i_r} \to w_{j_r}$ such that no two paths meet at any vertex.

- If there does not exist such a path system then add the minor $[I|J]$ to a list

- Continue this process for all possible I and J.
- The ideal generated by all quantum minors of the form $[I|J]$ found in the list made previously will be an H-prime

Now we move closer to the problem at hand. An element $x \in A$ is normal if $xA = Ax$. Given an ideal I of A, x is normal $modI$ if $x + I$ is normal in A/I. For instance, in $\mathcal{O}_q(M_{n,m}(k))$, the elements X_{n1} and X_{1m} are always normal.

A useful fact about normality which is essential to later arguments is the following:

Prop.

Suppose A is an algebra over a field with a finite generating set $X = \{x_1, ..., x_n\}$ and $y \in A$. If there exist nonzero scalars $r_1, ..., r_n$ such that $yx_i = r_i x_i y$ for all i then y is normal in A.

proof: First let $w = x_{i_1} \dots x_{i_m}$ be a string of generators. Then

$$yw = yx_{i_1} \dots x_{i_m} = r_{i_1} x_{i_1} y x_{i_2} \dots x_{i_m} = ... = r_{i_1} \dots r_{i_m} wy.$$

Denote by $r(w)$ the scalar $r_{i_1} \dots r_{i_m}$. Since X is a generating set for A, an arbitrary element of A is given by $b = \sum_i \alpha_i w_i$ where $\alpha' s$ are scalars, the $w' s$ are strings in the generators from X and the sum is over a finite list of terms. Then

$$yb = \sum_i \alpha_i y w_i = \sum_i \alpha_i r(w_i) w_i y = (\sum_i \alpha_i r(w_i) w_i) y.$$

Hence, $yA \subset Ay$. To see this in fact $yA = Ay$ take an arbitrary element of Ay say $cy = \sum_i \beta_i v_i y$ and consider $b := \sum_i \frac{\beta_i}{r(v_i)} v_i$. The above calculation shows that $yb = cy$. $\square$

For the heart of matter we need the next definition which extends the idea of normal elements. A sequence of elements $x_1, ..., x_n$ in A is polynormal if x_1 is normal in A and for all $i > 1$, x_{i+1} is normal $mod < x_1, ..., x_n >$.

For an example take the sequence $(ad - qbc, a)$ in $\mathcal{O}_q(M_{2,2}(K))$. It is a classical result that the quantum determinant $ad - qbc$ is in the center of $\mathcal{O}_q(M_{2,2}(k))$ and not hard to show by hand. Thus it is a normal element. Now by the proposition above it suffices to show that there are scalars r_b, r_c, r_d such that $ab = r_b ba + < ad - qbc >$, $ac = r_c ca + < ad - qbc >$ and $ad = r_d da + < ad - qbc >$. By the construction of this algebra we know that $ab = qba$ and $ac = qca$ so setting $r_a = r_c = q$ works. For d notice that since $ad = da + \hat{q}bc = da + (q - q^{-1})bc$, $ad - qbc = da - q^{-1}bc$ so that $qda = -bc + < ad - qbc >$. But this means that $ad - qbc = ad + q^2 da + < ad - qbc >$ so $r_d = -q^2$ works. Thus a is normal mod $< ad - qbc >$ and the sequence is polynormal.

We can now state the problem of interest.

<u>Main Problem</u>: Given any H-prime ideal P, prove that there is a polynormal sequence of quantum minors that generates P.

It is crucial to note here as in the introduction that the above problem was solved in 2011 by Milen Yakimov [2]. Here a different approach to that of Yakimov is taken to approach this problem.

4. Polynormality in Digraphs

To aid in the discussion of polynormality we introduce a helpful combinatorial version of this algebraic idea. Let S be a set and $A_1, ..., A_n$ nonempty subsets of S. Let G be a digraph on a vertex set which is a subset of $A_1 \times ... \times A_n$ with natural projection maps $\pi_1, ..., \pi_n$.

Pick any i.

- Given $B \subseteq A_i$ and $m \in A_i$ we say that *m is normal mod B in G* if either $m \in B$ or for all $v' \to v$ in G with $\pi_i(v) = m$ we have that

$$\{\pi_1(v'), \pi_2(v'), ..., \pi_n(v')\} \cap B \neq \emptyset.$$

- If $m \in A_i$ is normal mod $\emptyset$ (i.e. every preimage vertex of m in G is a source) then we call *m normal in G.*
- We say that a sequence $m_1, ..., m_r$ in A_i is *polynormal* if m_1 is normal mod $\emptyset$ and for all j we have m_{j+1} is normal mod $\{m_1, ..., m_j\}$.
- If there is an ordering of A_i that produces a polynormal sequence then say that G is polynormal with respect to A_i at the coordinate i (to avoid any confusion arising from the sets not necessarily being disjoint).
- If G is polynormal with respect to A_i for each i say that G itself is polynormal.

Note that in the case of just one set ($n = 1$) asking if G is polynormal with respect to A_1 we get an interesting problem asking whether or not G can be sequentially generated from a sequence of vertices where each consecutive vertex has all of its in-going neighbors in the previous set. For our purposes here we will only use two sets i.e. $n = 2$.

Now let A be an algebra over a field K and suppose A is finitely generated so that A has some finite generating set $X = \{x_1, ..., x_n\}$. Let $S_0 \subset S$ be finite nonempty subsets of nonzero elements of A. Suppose that for every $x_i \in X$ and $m \in S_0$ there exists some scalar $\alpha_{m,i} \neq 0$ and $w^{(i)}_{b,m,x_i} \in A$ for $b \in S \setminus \{m\}$ such that

$$x_i m - \alpha_{m,i} m x_i = \sum_{b \in S \setminus \{m\}} (w^{(1)}_{b,m,x_i} b + b w^{(2)}_{b,m,x_i})$$

i.e. $x_i m - \alpha_{m,i} m x_i$ is in the ideal generated by $S \setminus \{m\}$. These choices of scalars and elements of A may not be unique but choose some collection of $w's$ that make the above true and let W be the set of all such w corresponding to this choice. Define G to be a digraph with vertex set $(W \cup X) \times S$ with directed edges given by $(w^{(j)}_{b,m,x_i}, b) \rightarrow (x_i, m)$ whenever $w^{(j)}_{b,m,x_i}$ is nonzero. The following is an immediate consequence of this construction:

Prop.

If G is polynormal with respect to S_0, then S_0 has a polynormal ordering in A.

proof:

Assume that G is polynormal with respect to S_0. Suppose $m_1, ..., m_n$ is a polynormal ordering of S_0 in the graph theoretic sense. Then there are no arrows of the form $(w^{(j)}_{b,m_1,x_k}, b) \rightarrow (x_k, m_1)$ for any k and j. Hence, each $w^{(j)}_{b,m_1,x_k}$ is in fact 0 and thus for all k $x_k m_1 = \alpha_{m_1,k} m_1 x_k$. This implies that m_1 is normal in A. Now proceed inductively and assume that the subsequence

$m_1, ..., m_{r-1}$ is polynormal in A. We know that for every t

$$x_t m_r - \alpha_{m_r,t} m_r x_t = \sum_{b \in S \setminus \{m_r\}} (w^{(1)}_{b,m_r,x_t} b + b w^{(2)}_{b,m_r,x_t})$$

but by the definition of polynormality for digraphs it follows that for all nonzero $w^{(j)}_{b,m_r,x_t}$ appearing above $(w^{(j)}_{b,m_r,x_t}, b) \to (x_t, m_r)$ in G. By assumption, m_r is normal mod $\{m_1, ..., m_{r-1}\}$ in G for every nonzero $w^{(j)}_{b,m_r,x_t}$ mentioned before, either $b \in \{m_1, ..., m_{r-1}\}$ or $w^{(j)}_{b,m_r,x_t} \in \{m_1, ..., m_{r-1}\}$. But then the full sum

$$\sum_{b \in S \setminus \{m_r\}} (w^{(1)}_{b,m_r,x_t} b + b w^{(2)}_{b,m_r,x_t})$$

is in ideal generated by $\{m_1, ..., m_{r-1}\}$ so

$$x_t m_r = \alpha_{m_r,t} m_r x_t + < m_1, ..., m_{r-1} > .$$

Hence, m_r is normal modulo $< m_1, ..., m_{r-1} >$ in A. By induction, $m_1, ..., m_n$ is a polynormal ordering of S_0. $\square$

5. $2 \times N$ Quantum Matrices

We will now use this concept in action to give a combinatorial proof of the polynormality of H-prime ideals in $2 \times N$ quantum matrix algebras. To start here is an explicit form of the commutation relations for the relevant $2 \times N$ subcase.

Lemma

Suppose the following submatrix occurs in a quantum matrix algebra:

$$\begin{bmatrix} a & x & b \\ c & y & d \end{bmatrix}$$

Then the following hold:

- $x(ad - qbc) = q^{-1}(ad - qbc)x + \hat{q}b(ay - qxc)$
- $y(ad - qbc) = q(ad - qbc)y - \hat{q}c(xd - qby)$
- $a(xd - qby) = q(xd - qby)a$
- $c(xd - qby) = q(xd - qby)c$
- $b(ay - qxc) = q^{-1}(ay - qxc)b$
- $d(ay - qxc) = q^{-1}(ay - qxc)d$

proof: These are a direct consequence of the commutation relations for quantum minors due to Goodearl [3]. However, one sample calculation is written below to give the reader a feel for working with quantum matrices:

$$x(ad - qbc) = xad - qxbc = q^{-1}axd - q^2bxc$$

$$= q^{-1}a(dx + \hat{q}by) - q^2bcx$$

$$= q^{-1}adx + q^{-1}\hat{q}aby - q^2bcx$$

$$= q^{-1}adx + \hat{q}bay - q^2bcx$$

$$= q^{-1}adx - bcx + qbay - q^{-1}bay + bxc - q^2bxc$$

$$= q^{-1}adx - bcx + (q - q^{-1})bay - q(q - q^{-1})bxc$$

$$= q^{-1}(ad - qbc)x + \hat{q}b(ay - qxc).$$

For a 2 by N Cauchon Diagram define $\mathcal{M}_D$ to be the set of quantum minors contained in the H-prime generated from D. From these relations it is clear that $S_0 = \mathcal{M}_D$ and $S = \{$all quantum minors in $\mathcal{O}_q(M_{2,N}(K))$ $\}$ fit in the scheme of the last section.

Theorem

Fix a $2 \times N$ Cauchon diagram D. Then $\mathcal{M}_D$ will have a polynormal ordering in $G_{2,N}$.

Remark: As an immediate consequence this implies the polynormality conjecture for $\mathcal{O}_q(M_{2,N}(K))$.

Proof of the theorem:

We will be adding on a new column at each step of the induction process and verifying that if the original Cauchon diagram yields a polynormal sequence of quantum minors then the new sequence will as well no matter the choice of coloring of the new column.

Base Case: This is simply a 2x1 quantum affine plane so that this case is trivial as $G_{2,1}$ contains no edges.

Induction step: Fix a $2 \times N$ Cauchon diagram D and suppose that the corresponding set of minors $\mathcal{M}_D$ is normal in $G_{2,N}$. Now form a $2 \times (N+1)$ Cauchon diagram by adding a column on the right of the previous diagram and coloring the 2 new squares arbitrarily so that the new coloring on the $2 \times (N+1)$ grid is also a Cauchon diagram. Note that this will cover all possible cases of $2 \times (N+1)$ Cauchon diagrams as removing a column from a Cauchon diagram always produces a new one with smaller size. Call this new diagram D' and write $\mathcal{M}_{D'} = (\mathcal{M}_D \setminus \mathcal{M}_-) \cup \mathcal{M}_+$ where $\mathcal{M}_-$ is the set of minors lost by adding a column and $\mathcal{M}_+$ is the set of new minors gained by adding a column. We must now deal first with $\mathcal{M}_-$ to show that these minors can be removed without much of a problem.

By the path model, we know that the only case where the new column yields a nonempty $\mathcal{M}_-$ is where the new column has no colored in squares. Assume then that the new column has empty coloring. Then we know that $\mathcal{M}_-$ consists entirely of 1×1 minors of the form X_{1j} where $1 \leq j \leq N$, the square $(2, j)$ is not in the original Cauchon diagram, and either $j = N$ or $(1, i)$ is in the original Cauchon diagram D for all $i{>}j$. Now for all $X_{1j} \in \mathcal{M}_-$ we can safely remove X_{1j} from the polynormal sequence (given to us by induction) given that for every relevant outward neighbor of a preimage of X_{1j} in $G_{2,N}$ under either the π_X or $\pi_{\mathcal{M}}$ projection maps, (x, m) with $m \in \mathcal{M}_D$ say, for every minor m_0 with $(X_{1j}, m_0) \to (x, m)$ or $(m_0, X_{1j}) \to (x, m)$, m_0 is already in the list. This will end up being true! From the definition of the graph $G_{2,N}$ we know that the only edges of interest i.e. outward edges of preimages of some lost X_{1j} are of the form $(X_{1j}, X_{2r}) \to (X_{1r}, X_{2j})$ or $(X_{1j}, [12|ks]) \to (X_{1s}, [12|kj])$ with $j{>}s{>}k$ and $j{>}r$. We can immediately deal with the edges of the form $(X_j, X_{2r}) \to (X_{1r}, X_{2j})$ since if X_{1j} is a lost minor then there will be a path from 2 to j in the path model and hence X_{2j} will never have been in the sequence to begin with. We must deal with the vertices of the form $(X_{1j}, [12|kj]) \to (X_{1s}, [12|ks])$ with $j{>}s{>}k$ recursively.

First let j_0 be minimal such that X_{1j_0} is lost. Then either $(2, j_0)$ is in D or it is not in D. If it is in D then by the Cauchon property the whole bottom row before j is in D. We will trivially be done in this case. If $(2, j_0)$ is not in D then for any $[12|kj_0]$ appearing in the sequence we have that since $(2, j_0)$ is not in D, there must necessarily not exist a path from 1 to k. Thus $[12|ks]$

must be in the sequence somewhere. By the definition of the graph $G_{2,N}$ we know that there will not be a sequence of vertices that starts at a preimage of X_{1j} to a preimage of $[12|ks]$ under the $\pi_{\mathcal{M}}$ map. Thus we can remove X_{1j_0} from the list and simply rearrange to obtain a new polynormal sequence. Now order the j such that $X_{1j} \in \mathcal{M}_-$ by j. Suppose by induction that for all some $n \geq 0$ that the sequence of old minors in $\mathcal{M}_D$ can be rearranged in a way such that after removing $X_{1j_0}, ..., X_{1j_n}$ from the list, the result is still polynormal. First the vertices of the form $(X_{1j_{n+1}}, X_{2r}) \to (X_{1r}, X_{2j_{n+1}})$ are dealt with in the same way as before for j_0. As for the vertices of the form $(X_{1j_{n+1}}, [12|ks]) \to (X_{1s}, [12|kj_{n+1}])$ there are two possibilities. Either $k \geq j_0$ or $j_0 > k > j_{n+1}$. If $k \geq j_0$ then we can use the same argument as for j_0. Now suppose $j_0 > k > j_{n+1}$. Then $[12|ks]$ will already be in the sequence because the top row in D is from N to j_0. Thus we can reorder the original sequence and remove the lost minors so that the result is still polynormal. Now we deal with $\mathcal{M}_+$.

Suppose first that the new column has both squares not colored in. Then the only minors gained are of the form $[12|i, (N+1)]$ for i such that there is no path from 1 to i in the original diagram. These minors will be added to the previously constructed sequence in the order of decreasing i so that the first added is $[12|N, N+1]$ which is normal. At every step i the only new minor we have to worry about already appearing in the sequence is $[12|i, i+1]$ which is normal and must appear in the sequence as there is no path from 1 to i in the original diagram.

Now suppose one of the squares of the new column is colored in so that there are no lost minors. We just showed how to deal with all new minors of the form $[12|i, N+1]$ for i such that there is no path from 1 to i in the original diagram. The only new minors are either $X_{1,N+1}$ or $X_{2,N+1}$. We can always add $X_{1,N+1}$ whenever we want because it is normal. For $X_{2,N+1}$, we work recursively using the Cauchon property. In the case when the minor $X_{2,N+1}$ is gained, the square $(2, N+1)$ must be colored in so that either the full bottom row is colored in or $(1, N+1)$ is. In the first case we will trivially be able to deal with $X_{2,N+1}$ since minors of the form $X_{2,j}$ with $j<N+1$ will already occur in the sequence somewhere and by the definition of $G_{2,N+1}$ that is all we need. Otherwise, we can simply add the minor $X_{1,N+1}$ which will automatically cover all possible inward neighbors of preimages of $X_{2,N+1}$ in $G_{2,N+1}$ so we are finished. □

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7. References

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