ERRATUM TO "ON THE p-PART OF THE BIRCH–SWINNERTON-DYER FORMULA FOR MULTIPLICATIVE PRIMES"

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Abstract. We fix a mistake in [Cas18] and prove a version of the main Theorem A in *op. cit.* that is both weaker and stronger than the original result.

1. Introduction

When $\mathbf{f} \in \mathbb{I}[[q]]$ is a Hida family passing through a p-new p-stabilized newform in weight 2, the existence of a point $\phi \in \mathcal{X}_{\mathbb{I}}^{a}$ as used in the proof of [Cas18, Thm. 4.2] is not guaranteed in general. This affects the proof of [*op. cit.*, Thm. 4.4].

In the case $p \| N$ (which is the case where such $\mathbf{f}$ arises, so we only consider this case below), Theorem 4.4 in [Cas18] should be replaced by Theorem 1.1 below, which we shall prove here without using [Cas18, Thm. 4.2] and allowing E to have primes of additive reduction.

Theorem 1.1. *Let $E/\mathbf{Q}$ be an elliptic curve of conductor N with multiplicative reduction at $p > 3$, and let K be an imaginary quadratic field such that there exists an ideal $\mathfrak{N} \subset \mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{N} \simeq \mathbf{Z}/N\mathbf{Z}$ and in which $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits. Assume that:*

(i) *$E[p]$ is irreducible as a $G_{\mathbf{Q}}$-module.*
(ii) *If 2 is nonsplit in K, then $2 \| N$.*
(iii) *E has nonsplit multiplicative reduction at each prime $q \| N$ which is nonsplit in K, and that there is at least one such prime q at which $E[p]$ is ramified.*
(iv) *$E(\mathbf{Q}_p)[p] = 0$.*

Then $Ch_{\Lambda}(X_{\rm ac}(E[p^\infty]))$ is Λ-torsion and

$$Ch_{\Lambda}(X_{\rm ac}(E[p^\infty]))\Lambda_{R_0} = (L_p(f)).$$

Using Theorem 1.1 in place of [Cas18, Thm. 4.4], the same argument as in [Cas18, §5] yields the following result, which should replace the main Theorem A in *op. cit.* when $p \| N$.

Theorem A'. *Let $E/\mathbf{Q}$ be an elliptic curve of conductor N with multiplicative reduction at $p > 3$. Assume that $E[p]$ is irreducible as a $G_{\mathbf{Q}}$-module, E has nonsplit multiplicative reduction at some prime $q \neq p$ where $E[p]$ is ramified, and $E(\mathbf{Q}_p)[p] = 0$. If ${\rm ord}_{s=1}L(E,s) = 1$, then*

$${\rm ord}_p\left(\frac{L'(E,1)}{{\rm Reg}(E/\mathbf{Q})\cdot\Omega_E}\right) = {\rm ord}_p\biggl(\#\text{Ш}(E/\mathbf{Q})\prod_{\ell \mid N} c_\ell(E/\mathbf{Q})\biggr),$$

where

- ${\rm Reg}(E/\mathbf{Q})$ *is the discriminant of the Néron–Tate height pairing on* $E(\mathbf{Q}) \otimes \mathbf{R}$*;*
- Ω_E *is the Néron period of* E*;*
- $\text{Ш}(E/\mathbf{Q})$ *is the Tate–Shafarevich group of* E*; and*
- $c_\ell(E/\mathbf{Q})$ *is the Tamagawa number of* E *at the prime* ℓ*,*

and hence the p-part of the Birch and Swinnerton-Dyer formula holds for E.

Remark. Compared to Theorem A of [Cas18] in the case $p \| N$, Theorem A' assumes in addition that

(1) the prime $q\|N$ different from p where $E[p]$ is required to ramify should be a prime of *nonsplit* multiplicative reduction for E;
(2) $E(\mathbf{Q}_p)[p]=0$,

but does not require the E to be semistable. These additional hypotheses arise from (ii), (iii), and (iv) in Theorem 1.1: (ii) has its origin in [Wan20, CLW22], where it is used to overcome some technicality at 2, while (iii) mostly arises from [Hsi14]; both hypotheses should have been included in [Cas18, Thm. 4.4]—they are not intrinsically needed for the argument presented here, but rather inherited from (the final versions of) the results quoted in *loc. cit.* (and here). On the other hand, the new hypothesis (2) will be forced on us to show that the Selmer groups $\mathrm{Sel}_{\mathfrak{p}}^{\Sigma}(K_\infty, A_f)$ behave well under congruences.

Remark. Using Tate's p-adic uniformization, it is easy to see that (2) is equivalent to condition (b) in [SZ14, Thm. 1.1] when E has split multiplicative reduction at p (i.e. the condition that $p\nmid \mathrm{ord}_p(q_E)$ and $\log_p(q_E)\in p\mathbf{Z}_p^\times$, where $q_E\in\mathbf{Q}_p^\times$ is the Tate period of $E/\mathbf{Q}_p$).

2. Proof of Theorem 1.1

We refer to [Cas18] for any unexplained notation. Fix an embedding of $\imath_p:\overline{\mathbf{Q}}\hookrightarrow\overline{\mathbf{Q}}_p$, and let $g\in S_k(\Gamma_0(M))$ be a p-ordinary newform of even weight $k\geq 2$ and level $M\geq 3$ with $p\nmid M$ defined over $\mathscr{O}$, the ring of integers of a finite extension of $\mathbf{Q}_p$. Let

$$\bar\rho_g: G_{\mathbf{Q}}\to \mathrm{GL}_2(\mathscr{O}/(\varpi))$$

be the associated (semi-simple) residual Galois representation, where $\varpi\in\mathscr{O}$ is a uniformizing parameter. Let V_g be the self-dual Tate twist of the Galois representation associated to g by Deligne, let $T_g\subset V_g$ be the $G_{\mathbf{Q}}$-stable $\mathscr{O}$-lattice in [Nek92, §3], and put $A_g:=V_g/T_g$.

Let K be an imaginary quadratic field in which $p=\mathfrak{p}\overline{\mathfrak{p}}$ splits, with $\mathfrak{p}$ the prime of K above p induced by $\imath_p$. Similarly as in [Cas18, §2.1], put $\Lambda_{\mathscr{O}}=\mathscr{O}[[\Gamma]]$ and

$$M_g:=T_g\otimes_{\mathscr{O}}\Lambda_{\mathscr{O}}^*,$$

where G_K acts on $\Lambda_{\mathscr{O}}^*$ via Ψ^{-1}. For any finite set Σ of primes $v\nmid p$ of K, and for $m\geq 1$, set

$$\mathrm{Sel}_{\mathfrak{p}}^{\Sigma}(K,M_g[\varpi^m]):=\ker\left\{H^1(K,M_g[p^m])\to H^1(K_{\mathfrak{p}},M_g[\varpi^m])\times\prod_{\substack{w\nmid p\\ w\notin\Sigma}}H^1(K_w,M_g[\varpi^m])\right\},$$

dropping Σ from the notation when $\Sigma=\emptyset$. On the other hand, define $\mathrm{Sel}_{\mathfrak{p}}^{\Sigma}(K,M_g)$ following [Cas18, Def. 2.2].

Lemma 2.1. *Suppose Σ contains all primes $v\nmid p$ where T_g is ramified, $\bar\rho_g|_{G_K}$ is irreducible, and $H^0(K_{\mathfrak{p}},A_g[\varpi])=0$. Then the inclusion $M_g[\varpi^m]\subset M_g$ induces an isomorphism*

$$\mathrm{Sel}_{\mathfrak{p}}^{\Sigma}(K,M_g[\varpi^m])\simeq\mathrm{Sel}_{\mathfrak{p}}^{\Sigma}(K,M_g)[\varpi^m].$$

Proof. Let S_p be the set of primes of K above p, put $S=\Sigma\cup S_p$, and denote by $G_{K,S}$ the Galois group of the maximal algebraic extension of K unramified outside S. By our assumption on Σ, the Selmer groups $\mathrm{Sel}_{\mathfrak{p}}^{\Sigma}(K,M_g)$ and $\mathrm{Sel}_{\mathfrak{p}}^{\Sigma}(K,M_f[\varpi^m])$ are submodules of $H^1(G_{K,S},M_g)$ and $H^1(G_{K,S},M_g[\varpi^m])$, respectively. Since $H^0(K,M_g)=H^0(K_\infty,A_g)=0$ by Shapiro's lemma and the irreducibility of $\bar\rho_g|_{G_K}$, the inclusion $M_g[\varpi^m]\to M_g$ induces an isomorphism

$$H^1(G_{K,S},M_g[\varpi^m])\simeq H^1(G_{K,S},M_g)[\varpi^m].$$

By definition, under the above identification $\mathrm{Sel}_{\mathfrak{p}}^{\Sigma}(K,M_g)[\varpi^m]$ is the kernel of the composite map

$$H^1(G_{K,S},M_g[p^m])\to H^1(K_{\mathfrak{p}},M_g[\varpi^m])\to H^1(K_{\mathfrak{p}},M_g)[\varpi^m].$$

Since the kernel of the second arrow is given by $H^0(K_{\mathfrak{p}},M_g)/\varpi^m H^0(K_{\mathfrak{p}},M_g)$ and this vanishes when so does $H^0(K_{\mathfrak{p}},A_g[\varpi])$, the proof concludes. □

Write $X^{\Sigma}_{\rm ac}(A_g)=\mathrm{Sel}^{\Sigma}_{\mathfrak{p}}(K,M_g)^*$ for the Pontryagin dual of $\mathrm{Sel}^{\Sigma}_{\mathfrak{p}}(K,M_g)$.

Lemma 2.2. *Suppose $X_{\rm ac}(A_g)$ is $\Lambda_{\mathscr{O}}$-torsion. Then for any finite set Σ of primes $v\nmid p$ of K, $\mathrm{Sel}^{\Sigma}_{\mathfrak{p}}(K,M_g)$ has no proper finite index $\Lambda_{\mathscr{O}}$-submodules.*

Proof. This is a special case of Greenberg's general results [Gre16]. For $\Sigma=\emptyset$, the details are given in [HL19, Lem. 3.12], and the case of arbitrary Σ follows as in [Ski16, Prop. 2.3.3(ii)]. □

Suppose now that in addition K satisfies the following *Heegner hypothesis*:

$$\text{there exists an ideal } \mathfrak{M}\subset\mathcal{O}_K \text{ with } \mathcal{O}_K/\mathfrak{M}\simeq\mathbf{Z}/M\mathbf{Z}.$$

Let $\pi(g)=\otimes_v\pi(g)_v$ be the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by g, and put $\Lambda_{\mathscr{O}^{\rm ur}}=\Lambda_{R_0}\otimes_{\mathbf{Z}_p}\mathscr{O}$.

The key result we need is the following higher weight extension of [Cas18, Thm. 4.1].

Theorem 2.3. *Let $g\in S_k(\Gamma_0(M))$ be a p-ordinary newform of weight $k\geq 2$ and level $M\geq 3$ with $p\nmid M$. Assume that:*

(i) *$\bar\rho_g|_{G_K}$ is irreducible.*
(ii) *If 2 is nonsplit in K, then $2\|M$.*
(iii) *There is a prime $q\|M$ which is nonsplit in K.*
(iv) *If $\ell\|M$ is nonsplit in K, then the local component $\pi(f)_\ell$ is the special representation twisted by the unramified character sending $\ell\mapsto-\ell^{k/2-1}$.*

If Σ is any finite set of primes $v\nmid p$ of K, then $X^{\Sigma}_{\rm ac}(A_g)$ is $\Lambda_{\mathscr{O}}$-torsion, and

$$Ch_{\Lambda_{\mathscr{O}}}(X^{\Sigma}_{\rm ac}(A_g))\Lambda_{\mathscr{O}^{\rm ur}}=\big(L^{\Sigma}_p(g)\big),$$

where $L^{\Sigma}_p(g)$ is as in [Cas18, (5.1)].

Proof. The argument goes along the same lines as the proof of [Cas18, Thm. 4.1] (contained in [Cas17] and [Wan21]) in the weight 2 case. Let $z_{g,c}\in\mathrm{Sel}(H_c,T_g)$ be the system of generalized Heegner classes defined in [CH18, (4.7)] (taking $\chi=1$ in *loc. cit.*), where c runs over the positive integers prime to M, and H_c is the ring class field of K of conductor c. Put

$$\mathbf{T}_g:=T_g\otimes_{\mathscr{O}}\Lambda_{\mathscr{O}},$$

where G_K acts on $\Lambda_{\mathscr{O}}$ through Ψ, and let $\kappa_{g,\infty}\in H^1(K,\mathbf{T}_g)$ be the Heegner class constructed in [CH18, §5.2] from the classes z_{g,p^m} for varying $m\geq 0$. By [LV19, Thm. 4.7], there is a Kolyvagin system

$$\kappa^{\rm Hg}_g=\{\kappa^{\rm Hg}_{g,n}\}_{n\in\mathcal{N}}\in\mathbf{KS}(\mathbf{T}_g,\mathcal{F}_\Lambda,\mathcal{L}) \tag{2.1}$$

for the Selmer structure $\mathcal{F}_\Lambda$ in [*op. cit.*, §3.3], where $\mathcal{L}$ is a certain set of primes inert in K (see [LV19, §4.1]), and $\mathcal{N}$ is the set of squarefree products of primes $\ell\in\mathcal{L}$. In the same way as in [CGLS22, Rem. 4.1.3], we see that $\kappa^{\rm Hg}_{g,1}$ agrees with $\kappa_{g,\infty}$ up to a p-adic unit. Since $\kappa_{g,\infty}$ is not $\Lambda_{\mathscr{O}}$-torsion by [CH18, Thm. 6.1], the Kolyvagin system (2.1) is non-trivial, and so by [CGS23, Thm. 5.5.1] the modules $X_{\rm ord}(A_g):=H^1_{\mathcal{F}_\Lambda}(K,M_g)^*$ and $H^1_{\mathcal{F}_\Lambda}(K,\mathbf{T}_g)$ both have $\Lambda_{\mathscr{O}}$-rank one, and we have the divisibility

$$Ch_{\Lambda_{\mathscr{O}}}(X_{\rm ord}(A_g)_{\rm tors})\supset Ch_{\Lambda_{\mathscr{O}}}\big(H^1_{\mathcal{F}_\Lambda}(K,\mathbf{T}_g)/\Lambda_{\mathscr{O}}\kappa_{g,\infty}\big)^2, \tag{2.2}$$

where the subscript tors denotes the maximal $\Lambda_{\mathscr{O}}$-torsion submodule. The divisibility directly obtained in *loc. cit.* is up to powers of p, an ambiguity that can be removed if the constants C_1 and C_2 defined as in [CGLS22, §3.3.1] can both be taken to be zero. As noted in [*loc. cit.*, Rem. 3.3.5], the irreducibility of $\bar\rho_g$ implies that $C_2=0$. On the other hand, C_1 is a p-power exponent sufficient to annihilate the kernel of the restriction map in the proof of [CGLS22, Prop. 3,3,6], and it follows from [Cha05, Thm. 2] (see also [MN19, (0.9)]) that under hypothesis (i) one may take $C_2=0$. Thus the divisibility (2.2) holds in $\Lambda_{\mathscr{O}}$. Using the explicit reciprocity law for $\kappa_{g,\infty}$ in [CH18, Thm. 5.7], the same global duality argument as in [BCK21, Thm. 5.2] shows that the module $X_{\rm ac}(A_g)$ is $\Lambda_{\mathscr{O}}$-torsion, and that (2.2) implies the divisibility

$$Ch_{\Lambda_{\mathscr{O}}}(X_{\rm ac}(A_g))\Lambda_{\mathscr{O}^{\rm ur}} \supset \big(L_p(g)\big) \tag{2.3}$$

in $\Lambda_{\mathscr{O}^{\rm ur}}$.

Conversely, let $\widetilde{\Lambda}_{\mathscr{O}} = \mathscr{O}[[{\rm Gal}(\widetilde{K}_\infty/K)]]$ be the Iwasawa algebra for the $\mathbf{Z}_p^2$-extension $\widetilde{K}_\infty/K$, and put $\widetilde{\Lambda}_{\mathscr{O}^{\rm ur}} := \widetilde{\Lambda}_{\mathscr{O}}\hat{\otimes}_{\mathbf{Z}_p}R_0$. For any finite set Σ of primes of K away from p, let $X_K^\Sigma(A_g)$ denote the Σ-imprimitive Selmer group defined as in [Cas18, §2.1] with $\widetilde{K}_\infty$ in place of K_∞, omitting Σ from the notation if $\Sigma=\emptyset$. By [FW21, Thm. 4.41], we then have the divisibility

$$Ch_{\widetilde{\Lambda}_{\mathscr{O}}}(X_K(A_g))\widetilde{\Lambda}_{\mathscr{O}^{\rm ur}} \subset \big(L_p^{\rm Gr}(g)\big) \tag{2.4}$$

in $\widetilde{\Lambda}_{\mathscr{O}^{\rm ur}}$, where $L_p^{\rm Gr}(g)$ is a certain two-variable p-adic L-function deduced from [EW16]. (Note that the proof of this integral divisibility uses the $\mu=0$ result of [Hsi14, Thm. B].) By [FO12, Cor. 7.2.1], $L_p^{\rm Gr}(g)$ agrees (up to a unit) with the product of a two-variable Hida p-adic Rankin L-series, an anticyclotomic Katz p-adic L-function, and the class number of K. As a result, by the same calculation as in [CGS23, Prop. 1.4.5], letting $L_p^{{\rm Gr},\Sigma}(g)_{\rm ac}$ denote the image of the Σ-imprimitive $L_p^{{\rm Gr},\Sigma}(g)$ (defined in the same manner as in [Cas18, (3.1)]) under the natural projection $\widetilde{\Lambda}_{\mathscr{O}} \to \Lambda_{\mathscr{O}}$, we have

$$L_p^{{\rm Gr},\Sigma}(g)_{\rm ac} = L_p^\Sigma(g)$$

up to a p-adic unit. Taking a Σ that contains all primes dividing M, by [JSW17, Cor. 3.4.2] it follows that (2.4) yields the divisibility

$$Ch_{\Lambda_{\mathscr{O}}}(X_{\rm ac}^\Sigma(A_g))\Lambda_{\mathscr{O}^{\rm ur}} \subset \big(L_p^\Sigma(g)\big) \tag{2.5}$$

in $\Lambda_{\mathscr{O}^{\rm ur}}$. Since we have seen that $X_{\rm ac}^\Sigma(A_g)$ is $\Lambda_{\mathscr{O}}$-torsion, by the same argument as in [JSW17, Thn. 6.1.6] the divisibility (2.5) for Σ containing all the bad primes implies the same divisibility for any Σ. Together with (2.3), this concludes the proof. □

Proof of Theorem 1.1. The result can now be deduced following the approach in [Ski16, §3.1]. Put $M=N$ if $p\nmid N$, and $M=N/p$ if $p\| N$. As in the proof of Theorem 2.3, it suffices to prove the result for $X_{\rm ac}^\Sigma(E[p^\infty])$ and $L_p^\Sigma(f)$ for Σ a finite set of primes $v\nmid p$ of K containing the primes dividing M. We claim that, after possibly enlarging $\mathscr{O}$, for each $m\geq 1$ there exists

(a) a p-ordinary newform $g_m\in S_{k_m}(\Gamma_0(M))$ defined over $\mathscr{O}$ of weight $k_m>2$ with $k_m\equiv 2 \pmod{p-1}$;
(b) a $G_{\mathbf{Q}}$-stable lattice $T_{g_m}\subset V_{g_m}$ and an isomorphism $T_{g_m}/p^mT_{g_m}\simeq T/p^mT$ as $\mathscr{O}[G_{\mathbf{Q}}]$-modules;
(c) an equality $(L_p^\Sigma(g_m),p^m)=(L_p^\Sigma(f),p^m)\subset\Lambda_{\mathscr{O}^{\rm ur}}$.

Indeed, (a) and (b) follow from Hida theory (see the discussion in [Ski16, §2.6]), and (c) follows from [Cas20, Thm. 2.11]. By Theorem 2.3[1], the module $X_{\rm ac}^\Sigma(A_{g_m})$ is $\Lambda_{\mathscr{O}}$-torsion, with

$$Ch_{\Lambda_{\mathscr{O}}}(X_{\rm ac}^\Sigma(A_{g_m}))\Lambda_{\mathscr{O}^{\rm ur}} = (L_p^\Sigma(g_m)).$$

Moreover, by Lemma 2.2 we know that $Ch_{\Lambda_{\mathscr{O}}}(X_{\rm ac}^\Sigma(A_{g_m}))={\rm Fitt}_{\Lambda_{\mathscr{O}}}\big(X_{\rm ac}^\Sigma(A_{g_m})\big)$, and so from (b), Lemma 2.1, and basic properties of Fitting ideals we deduce the equality

$$({\rm Fitt}_{\Lambda_{\mathscr{O}}}(X_{\rm ac}^\Sigma(E[p^\infty])),p^m)\Lambda_{\mathscr{O}^{\rm ur}} = (L_p^\Sigma(f),p^m).$$

From this, the argument in [Ski16, p. 192] applies *verbatim*, using the nonvanishing of $L_p(f)$ that follows from the work of Cornut–Vatsal [CV07] and the explicit reciprocity law in [Cas20, Thm. 5.3] specialized to f (see also [BCK21, Cor. 4.5]). □

[1]Note that the hypothesis that $\bar{\rho}_{g_m}\simeq E[p]$ is irreducible as a $G_{\mathbf{Q}}$-module and ramified at some prime $q\|N$ nonsplit in K implies that $\bar{\rho}_{g_m}|_{G_K}$ is irreducible, see [Ski20, Lem. 2.8.1]. Moreover, by "rigidity of automorphic types" [FO12, Lem. 2.14], condition (iii) in Theorem 1.1 implies conditions (iii) and (iv) in Theorem 2.3.

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