ERRATUM TO “ON THE $p$-PART OF THE BIRCH–SWINNERTON-DYER FORMULA FOR MULTIPLICATIVE PRIMES”

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Abstract. We fix a mistake in [Cas18] and prove a version of the main Theorem A in op. cit. that is both weaker and stronger than the original result.

1. Introduction

When $f \in \mathbb{I}[ [q] ]$ is a Hida family passing through a $p$-new $p$-stabilized newform in weight 2, the existence of a point $\phi \in X_{\mathcal{T}}^a$ as used in the proof of [Cas18, Thm. 4.2] is not guaranteed in general. This affects the proof of [op. cit., Thm. 4.4].

In the case $p \nmid N$ (which is the case where such $f$ arises, so we only consider this case below), Theorem 4.4 in [Cas18] should be replaced by Theorem 1.1 below, which we shall prove here without using [Cas18, Thm. 4.2] and allowing $E$ to have primes of additive reduction.

Theorem 1.1. Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N$ with multiplicative reduction at $p > 3$, and let $K$ be an imaginary quadratic field such that there exists an ideal $\mathfrak{N} \subset \mathcal{O}_K$ with $\mathcal{O}_K / \mathfrak{N} \cong \mathbb{Z} / N \mathbb{Z}$ and in which $p = \mathfrak{p} \mathfrak{p}$ splits. Assume that:

(i) $E[p]$ is irreducible as a $G_{\mathbb{Q}}$-module.
(ii) If 2 is nonsplit in $K$, then $2 \nmid N$.
(iii) $E$ has nonsplit multiplicative reduction at each prime $q \nmid N$ which is nonsplit in $K$, and that there is at least one such prime $q$ at which $E[p]$ is ramified.
(iv) $E(\mathbb{Q}_p)[p] = 0$.

Then $\text{Ch}_\Lambda (X_{\text{ac}}(E[p^{\infty}]))$ is $\Lambda$-torsion and

$$\text{Ch}_\Lambda (X_{\text{ac}}(E[p^{\infty}])) \Lambda_{\mathcal{R}_0} = (L_p(f)).$$

Using Theorem 1.1 in place of [Cas18, Thm. 4.4], the same argument as in [Cas18, §5] yields the following result, which should replace the main Theorem A in op. cit. when $p \nmid N$.

Theorem A’. Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N$ with multiplicative reduction at $p > 3$. Assume that $E[p]$ is irreducible as a $G_{\mathbb{Q}}$-module, $E$ has nonsplit multiplicative reduction at some prime $q \neq p$ where $E[p]$ is ramified, and $E(\mathbb{Q}_p)[p] = 0$. If $\text{ord}_s L(E,s) = 1$, then

$$\text{ord}_p \left( \frac{L'(E,1)}{\text{Reg}(E/\mathbb{Q}) \cdot \Omega_E} \right) = \text{ord}_p \left( \# \text{III}(E/\mathbb{Q}) \prod_{\ell \mid N} c_\ell(E/\mathbb{Q}) \right),$$

where

- $\text{Reg}(E/\mathbb{Q})$ is the discriminant of the Néron–Tate height pairing on $E(\mathbb{Q}) \otimes \mathbb{R}$;
- $\Omega_E$ is the Néron period of $E$;
- $\text{III}(E/\mathbb{Q})$ is the Tate–Shafarevich group of $E$; and
- $c_\ell(E/\mathbb{Q})$ is the Tamagawa number of $E$ at the prime $\ell$,

and hence the $p$-part of the Birch and Swinnerton-Dyer formula holds for $E$.

Remark. Compared to Theorem A of [Cas18] in the case $p \nmid N$, Theorem A’ assumes in addition that
(1) the prime $q|N$ different from $p$ where $E[p]$ is required to ramify should be a prime of 
non-split multiplicative reduction for $E$;
(2) $E(Q_p)[p] = 0$,
but does not require the $E$ to be semistable. These additional hypotheses arise from (ii), (iii),
and (iv) in Theorem 1.1: (ii) has its origin in [Wan20, CLW22], where it is used to overcome
some technicality at $2$, while (iii) mostly arises from [Hsi14]; both hypotheses should have been
included in [Cas18, Thm. 4.4]—they are not intrinsically needed for the argument presented
here, but rather inherited from (the final versions of) the results quoted in loc. cit. (and here).
On the other hand, the new hypothesis (2) will be forced on us to show that the Selmer groups
$\text{Sel}_p^\Sigma(K_{\infty}, A_f)$ behave well under congruences.

Remark. Using Tate’s $p$-adic uniformization, it is easy to see that (2) is equivalent to condition
(b) in [SZ14, Thm. 1.1] when $E$ has split multiplicative reduction at $p$ (i.e. the condition that
$p \nmid \text{ord}_p(q_E)$ and $\log_p(q_E) \in p\mathbb{Z}_p^*$, where $q_E \in \mathbb{Q}_p^*$ is the Tate period of $E/Q_p$).

2. PROOF OF THEOREM 1.1

We refer to [Cas18] for any unexplained notation. Fix an embedding of $\iota_p : \overline{Q} \hookrightarrow \overline{Q}_p$, and
let $g \in S_k(\Gamma_0(M))$ be a $p$-ordinary newform of even weight $k \geq 2$ and level $M \geq 3$ with $p \nmid M$
defined over $\mathcal{O}$, the ring of integers of a finite extension of $\mathbb{Q}_p$. Let

$$\overline{\rho}_g : G_{\mathbb{Q}} \to \text{GL}_2(\mathcal{O}/(\varpi))$$

be the associated (semi-simple) residual Galois representation, where $\varpi \in \mathcal{O}$ is a uniformizing
parameter. Let $V_g$ be the self-dual Tate twist of the Galois representation associated to $g$ by
Deligne, let $T_g \subset V_g$ be the $G_{\mathbb{Q}}$-stable $\mathcal{O}$-lattice in [Nek92, §3], and put $A_g := V_g/T_g$.

Let $K$ be an imaginary quadratic field in which $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits, with $\mathfrak{p}$ the prime of $K$
above $p$ induced by $\iota_p$. Similarly as in [Cas18, §2.1], put $\Lambda_{\mathcal{O}} = \mathcal{O}[\Gamma]$ and

$$M_g := T_g \otimes_{\mathcal{O}} \Lambda_{\mathcal{O}}^*,$$

where $G_K$ acts on $\Lambda_{\mathcal{O}}^*$ via $\Psi^{-1}$. For any finite set $\Sigma$ of primes $v \nmid p$ of $K$, and for $m \geq 1$, set

$$\text{Sel}_p^\Sigma(K, M_g[\varpi^m]) := \ker \left\{ H^1(K, M_g[p^m]) \to H^1(K_g, M_g[\varpi^m]) \times \prod_{w \nmid p, w \notin \Sigma} H^1(K_w, M_g[\varpi^m]) \right\},$$

dropping $\Sigma$ from the notation when $\Sigma = \emptyset$. On the other hand, define $\text{Sel}_p^\Sigma(K, M_g)$ following
[Cas18, Def. 2.2].

Lemma 2.1. Suppose $\Sigma$ contains all primes $v \nmid p$ where $T_g$ is ramified, $\overline{\rho}_g|_{G_K}$ is irreducible,
and $H^0(K_g, A_g[\varpi]) = 0$. Then the inclusion $M_g[\varpi^m] \subset M_g$ induces an isomorphism

$$\text{Sel}_p^\Sigma(K, M_g[\varpi^m]) \cong \text{Sel}_p^\Sigma(K, M_g)[\varpi^m].$$

Proof. Let $S_p$ be the set of primes of $K$ above $p$, put $S = \Sigma \cup S_p$, and denote by $G_{K,S}$ the Galois
 group of the maximal algebraic extension of $K$ unramified outside $S$. By our assumption on $\Sigma$, the
Selmer groups $\text{Sel}_p^\Sigma(K, M_g)$ and $\text{Sel}_p^\Sigma(K, M_g[\varpi^m])$ are submodules of $H^1(G_{K,S}, M_g)$ and
$H^1(G_{K,S}, M_g[\varpi^m])$, respectively. Since $H^0(K_g, M_g) = H^0(K_{\infty}, A_g) = 0$ by Shapiro’s lemma
and the irreducibility of $\overline{\rho}_g|_{G_K}$, the inclusion $M_g[\varpi^m] \to M_g$ induces an isomorphism

$$H^1(G_{K,S}, M_g[\varpi^m]) \cong H^1(G_{K,S}, M_g)[\varpi^m].$$

By definition, under the above identification $\text{Sel}_p^\Sigma(K, M_g)[\varpi^m]$ is the kernel of the composite map

$$H^1(G_{K,S}, M_g[p^m]) \to H^1(K_p, M_g[\varpi^m]) \to H^1(K_g, M_g)[\varpi^m].$$

Since the kernel of the second arrow is given by $H^0(K_p, M_g[\varpi^m])H^0(K_p, M_g)$ and this vanishes
when so does $H^0(K_p, A_g[\varpi])$, the proof concludes. □
Write $X^{\Sigma}_{ac}(A_g) = \text{Sel}_{\text{g}}^{\Sigma}(K, M_g)^*$ for the Pontryagin dual of $\text{Sel}_{\text{g}}^{\Sigma}(K, M_g)$.

**Lemma 2.2.** Suppose $X_{ac}(A_g)$ is $\Lambda_\Theta$-torsion. Then for any finite set $\Sigma$ of primes $v \nmid p$ of $K$, $\text{Sel}_{\text{g}}^{\Sigma}(K, M_g)$ has no proper finite index $\Lambda_\Theta$-submodules.

**Proof.** This is a special case of Greenberg’s general results [Gre16]. For $\Sigma = \emptyset$, the details are given in [HL19, Lem. 3.12], and the case of arbitrary $\Sigma$ follows as in [Ski16, Prop. 2.3.3(ii)].

Suppose now that in addition $K$ satisfies the following Heegner hypothesis:

there exists an ideal $\mathfrak{M} \subset \mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{M} \simeq \mathbb{Z}/M\mathbb{Z}$.

Let $\pi(g) = \bigotimes_v \pi(g)_v$ be the cuspidal automorphic representation of $\text{GL}_2(\mathbb{A})$ generated by $g$, and put $\Lambda_{g,\text{ur}} = \Lambda_{R_0} \otimes_{\mathbb{Z}} \mathbb{Q}_p \theta$.

The key result we need is the following higher weight extension of [Cas18, Thm. 4.1].

**Theorem 2.3.** Let $g \in S_k(\Gamma_0(M))$ be a $p$-ordinary newform of weight $k \geq 2$ and level $M \geq 3$ with $p \nmid M$. Assume that:

(i) $\bar{\rho}_g|_{G_K}$ is irreducible.
(ii) If $2$ is nonsplit in $K$, then $2\|M$.
(iii) There is a prime $q\|M$ which is nonsplit in $K$.
(iv) If $\ell\|M$ is nonsplit in $K$, then the local component $\pi(f)_{\ell}$ is the special representation twisted by the unramified character sending $\ell \mapsto -\ell^{k-2}/4$.

If $\Sigma$ is any finite set of primes $v \nmid p$ of $K$, then $X^{\Sigma}_{ac}(A_g)$ is $\Lambda_\Theta$-torsion, and

$$C_{h_{\Lambda_\Theta}}(X^{\Sigma}_{ac}(A_g))\Lambda_{g,\text{ur}} = (L^\Sigma_p(g)), $$

where $L^\Sigma_p(g)$ is as in [Cas18, (5.1)].

**Proof.** The argument goes along the same lines as the proof of [Cas18, Thm. 4.1] (contained in [Cas17] and [Wan21]) in the weight 2 case. Let $z_{g,c} \in \text{Sel}(H_c, T_g)$ be the system of generalized Heegner classes defined in [CH18, (4.7)] (taking $\chi = 1$ in loc. cit.), where $c$ runs over the positive integers prime to $M$, and $H_c$ is the ring class field of $K$ of conductor $c$. Put

$$T_g := T_g \otimes_{\mathbb{Q}} \Lambda_\Theta,$$

where $G_K$ acts on $\Lambda_\Theta$ through $\Psi$, and let $\kappa_{g,\infty} \in H^1(K, T_g)$ be the Heegner class constructed in [CH18, §5.2] from the classes $z_{g,p}^m$ for varying $m \geq 0$. By [LV19, Thm. 4.7], there is a Kolyvagin system

$$\kappa_{g,\infty}^H = \{\kappa_{g,n}^H\}_{n \in \mathcal{N}} \in \text{KS}(T_g, \mathcal{F}_\Lambda, \mathcal{L})$$

for the Selmer structure $\mathcal{F}_\Lambda$ in op. cit., §3.3, where $\mathcal{L}$ is a certain set of primes inert in $K$ (see [LV19, §4.1]), and $\mathcal{N}$ is the set of squarefree products of primes $\ell \in \mathcal{L}$. In the same way as in [CGLS22, Rem. 4.1.3], we see that $\kappa_{g,1}^H$ agrees with $\kappa_{g,\infty}$ up to a $p$-adic unit. Since $\kappa_{g,\infty}$ is not $\Lambda_\Theta$-torsion by [CH18, Thm. 6.1], the Kolyvagin system (2.1) is non-trivial, and so by [CGS23, Thm. 5.5.1] the modules $X_{\text{ord}}(A_g) := H^1_{\mathcal{F}_\Lambda}(K, M_g)^*$ and $H^1_{\mathcal{F}_\Lambda}(K, T_g)$ both have $\Lambda_\Theta$-rank one, and we have the divisibility

$$C_{h_{\Lambda_\Theta}}(X_{\text{ord}}(A_g)) \supset C_{h_{\Lambda_\Theta}}(H^1_{\mathcal{F}_\Lambda}(K, M_g)^*) \simeq (L^\Sigma_p(g)), $$

where the subscript tors denotes the maximal $\Lambda_\Theta$-torsion submodule. The divisibility directly

obtained in loc. cit. is up to powers of $p$, an ambiguity that can be removed if the constants $C_1$ and $C_2$ defined as in [CGLS22, §3.3.1] can both be taken to be zero. As noted in [loc. cit., Rem. 3.3.5], the irreducibility of $\bar{\rho}_g$ implies that $C_2 = 0$. On the other hand, $C_1$ is a $p$-power exponent sufficient to annihilate the kernel of the restriction map in the proof of [CGLS22, Prop. 3.3.6], and it follows from [Cha05, Thm. 2] (see also [MN19, (0.9)]) that under hypothesis (i) one may take $C_2 = 0$. Thus the divisibility (2.2) holds in $\Lambda_\Theta$. Using the explicit reciprocity
law for $\kappa_{g,\infty}$ in [CH18, Thm. 5.7], the same global duality argument as in [BCK21, Thm. 5.2] shows that the module $X_{ac}(A_g)$ is $\Lambda_\rho$-torsion, and that (2.2) implies the divisibility

$$
Ch_{\Lambda_\rho}(X_{ac}(A_g))\Lambda_{\rho^{ur}} \supset (L_p(g))
$$

in $\Lambda_{\rho^{ur}}$.

Conversely, let $\tilde{\Lambda}_\rho := \Theta([Gal(\bar{K}_\infty/K)])$ be the Iwasawa algebra for the $\mathbb{Z}_p^2$-extension $\bar{K}_\infty/K$, and put $\tilde{\Lambda}_{\rho^{ur}} := \tilde{\Lambda}_\rho \otimes_{\mathbb{Z}_p} R_0$. For any finite set $\Sigma$ of primes of $K$ away from $p$, let $X_{Gr}^\Sigma(A_g)$ denote the $\Sigma$-imprimitive Selmer group defined as in [Cas18, §2.1] with $\bar{K}_\infty$ in place of $K_\infty$, omitting $\Sigma$ from the notation if $\Sigma = \emptyset$. By [FW21, Thm. 4.41], we then have the divisibility

$$
Ch_{\tilde{\Lambda}_\rho}(X_K(A_g))\tilde{\Lambda}_{\rho^{ur}} \subset (L_p^{Gr}(g))
$$

in $\tilde{\Lambda}_{\rho^{ur}}$, where $L_p^{Gr}(g)$ is a certain two-variable $p$-adic $L$-function deduced from [EW16]. (Note that the proof of this integral divisibility uses the $\mu = 0$ result of [Hsi14, Thm. B].) By [FO12, Cor. 7.2.1], $L_p^{Gr}(g)$ agrees (up to a unit) with the product of a two-variable Hida $p$-adic Rankin $L$-series, an anticyclotomic Katz $p$-adic $L$-function, and the class number of $K$. As a result, by the same calculation as in [CGS23, Prop. 1.4.5], letting $L_p^{Gr,\Sigma}(g)_{ac}$ denote the image of the $\Sigma$-imprimitive $L_p^{Gr}(g)$ (defined in the same manner as in [Cas18, (3.1)]) under the natural projection $\tilde{\Lambda}_\rho \to \Lambda_\rho$, we have

$$
L_p^{Gr,\Sigma}(g)_{ac} = L_p^{\Sigma}(g)
$$

up to a $p$-adic unit. Taking a $\Sigma$ that contains all primes dividing $M$, by [JSW17, Cor. 3.4.2] it follows that (2.4) yields the divisibility

$$
Ch_{\Lambda_\rho}(X_{ac}^\Sigma(A_g))\Lambda_{\rho^{ur}} \subset (L_p^{\Sigma}(g))
$$

in $\Lambda_{\rho^{ur}}$. Since we have seen that $X_{ac}^\Sigma(A_g)$ is $\Lambda_\rho$-torsion, by the same argument as in [JSW17, Thm. 6.1.6] the divisibility (2.5) for $\Sigma$ containing all the bad primes implies the same divisibility for any $\Sigma$. Together with (2.3), this concludes the proof. \qed

**Proof of Theorem 1.1.** The result can now be deduced following the approach in [Ski16, §3.1]. Put $M = N$ if $p \nmid N$, and $M = N/p$ if $p \mid N$. As in the proof of Theorem 2.3, it suffices to prove the result for $X_{ac}^\Sigma(E[p^{\infty}])$ and $L_p^\Sigma(f)$ for $\Sigma$ a finite set of primes $v \nmid p$ of $K$ containing the primes dividing $M$. We claim that, after possibly enlarging $\Theta$, for each $m \geq 1$ there exists

(a) a $p$-ordinary newform $g_m \in S_k(\Gamma_0(M))$ defined over $\Theta$ of weight $k_m > 2$ with $k_m \equiv 2 \pmod{p - 1}$;

(b) a $G_{Q}$-stable lattice $T_{g_m} \subset V_{g_m}$ and an isomorphism $T_{g_m}/p^m T_{g_m} \simeq T/p^m T$ as $\Theta[G_{Q}]$-modules;

(c) an equality $(L_p^\Sigma(g_m), p^m) = (L_p^\Sigma(f), p^m) \subset \Lambda_{\rho^{ur}}$.

Indeed, (a) and (b) follow from Hida theory (see the discussion in [Ski16, §2.6]), and (c) follows from [Cas20, Thm. 2.11]. By Theorem 2.3\(^1\), the module $X_{ac}^\Sigma(A_{g_m})$ is $\Lambda_\rho$-torsion, with

$$
Ch_{\Lambda_\rho}(X_{ac}^\Sigma(A_{g_m}))\Lambda_{\rho^{ur}} = (L_p^\Sigma(g_m)).
$$

Moreover, by Lemma 2.2 we know that $Ch_{\Lambda_\rho}(X_{ac}^\Sigma(A_{g_m})) = \text{Fitt}_{\Lambda_\rho}(X_{ac}^\Sigma(A_{g_m}))$, and so from (b), Lemma 2.1, and basic properties of Fitting ideals we deduce the equality

$$
(\text{Fitt}_{\Lambda_\rho}(X_{ac}^\Sigma(E[p^{\infty}])), p^m)\Lambda_{\rho^{ur}} = (L_p^\Sigma(f), p^m).
$$

From this, the argument in [Ski16, p. 192] applies *verbatim*, using the nonvanishing of $L_p(f)$ that follows from the work of Cornut–Vatsal [CV07] and the explicit reciprocity law in [Cas20, Thm. 5.3] specialized to $f$ (see also [BCK21, Cor. 4.5]). \qed

\(^1\)Note that the hypothesis that $p_{g_m} \simeq E[p]$ is irreducible as a $G_{Q}$-module and ramified at some prime $q\nmid N$ nonsplit in $K$ implies that $\rho_{g_m}|_{G_K}$ is irreducible, see [Ski20, Lem. 2.8.1]. Moreover, by “rigidity of automorphic types” [FO12, Lem. 2.14], condition (iii) in Theorem 1.1 implies conditions (iii) and (iv) in Theorem 2.3.
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REFERENCES


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