ADDENDUM TO: IWASAWA–GREENBERG MAIN CONJECTURE FOR NONORDINARY MODULAR FORMS AND EISENSTEIN CONGRUCENCES ON $\mathrm{GU}(3,1)$

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ABSTRACT. We explain how to adapt the proof of the main result in [CLW22] under weaker mod p nonvanishing results than previously used.

1. INTRODUCTION

The proof in [CLW22] uses $\mathfrak{l}$-tower mod p nonvanishing results proved in [Hsi12, Hsi14b] for Hecke L-values for imaginary quadratic fields and for central L-values for base change to imaginary quadratic fields of automorphic representations of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ twisted by anticyclotomic characters. The proof in [Hsi12, Hsi14b] are based on the ideas of Hida in [Hid10, Hid04], and a key ingredient is [Hid04, Theorem 3.2]. A few years ago, a gap in the proof of [Hid04, Theorem 3.2] was pointed out by Venkatesh. To address it, an argument has been given in [Hid24] to prove a weakened version of that theorem, which is not sufficient for proving the assertions about nonvanishing modulo p for almost all anticyclotomic twists in an $\mathfrak{l}$-tower in [Hid04, Hsi12, Hsi14b], but can be used to prove a weakened statement with "almost all" replaced by "infinitely many".

More precisely, let $\mathcal{K}$ be an imaginary quadratic extension of $\mathbb{Q}$ in which p splits and $\ell \neq p$ be an odd prime. Let $\mathcal{K}^-_{\ell^\infty}$ denote the maximal pro-ℓ anticyclotomic extension of $\mathcal{K}$ and denote by $\mathfrak{X}^-_\ell$ the set of finite order characters of $\mathrm{Gal}(\mathcal{K}^-_{\ell^\infty}/\mathcal{K})$. By replacing the use of [Hid04, Theorem 3.2] in [Hsi12, Hsi14b] by [Hid04, Theorem 0.1], one gets the following weakened versions of [Hsi12, Theorem B] and [Hsi14b, Theorem C]. (Note that [Hsi12, Hsi14b] treat general CM fields, but we will only state the results in the case of imaginary quadratic fields, which is all that we need.)

Theorem 1.0.1. *Let $\chi : \mathcal{K}^\times\backslash\mathbb{A}^\times_\mathcal{K} \to \mathbb{C}^\times$ be a Hecke character of ∞-type $(k+m, -m)$ with k, m positive integers satisfying the conditions in [Hsi12, Theorem B]. Then there are infinitely many $\phi \in \mathfrak{X}^-_\mathfrak{l}$ such that*

$$\left(\frac{2\pi i\cdot\Omega_p}{\Omega_\infty}\right)^{k+2m}\frac{\Gamma(k+m)}{(2\pi i)^{k+m}}\,L^{\ell\infty}(0,\chi\phi)\not\equiv 0 \mod \mathfrak{m}_p,$$

with $\Omega_\infty \in \mathbb{C}^\times$ (resp. $\Omega_p \in \hat{\mathbb{Z}}_p^{\mathrm{ur},\times}$) the complex (resp. p-adic) CM period in [Hsi14a, Section 2.8].

Theorem 1.0.2. *Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with unitary central character generated by a modular form of weight t and $\chi : \mathcal{K}^\times\backslash\mathbb{A}^\times_\mathcal{K} \to \mathbb{C}^\times$ be a Hecke character of ∞-type $(\frac{t}{2}+m, -\frac{t}{2}-m)$ with $m \in \mathbb{Z}_{\geq 0}$ such that π and χ satisfy the conditions in [Hsi14b, Theorem C]. Then there are infinitely many $\phi \in \mathfrak{X}^-_\mathfrak{l}$ such that*

$$\left(\frac{2\pi i\cdot\Omega_p}{\Omega_\infty}\right)^{2t+4m}\frac{\Gamma(t+m)\Gamma(m+1)}{(2\pi i)^{t+2m+1}}\,L\left(\frac{1}{2},\mathrm{BC}(\pi)\times\lambda^2\chi_{h,2}\chi_{\theta,2}\right)\not\equiv 0 \mod \mathfrak{m}_p.$$

In [CLW22], [Hsi12, Theorem B] and [Hsi14b, Theorem C] are used in §5.6 to guarantee the existence of auxiliary Hecke characters χ_θ, χ_h such that three normalized L-values are simultaneously p-adic units. The above two theorems (weaker than [Hsi12, Theorem B], [Hsi14b, Theorem C]) can only guarantee the mod p simultaneous nonvanishing of two among the three.

In the following, we explain how to modify the argument in [CLW22] to prove the main theorem there based on the above two theorems. The idea is to choose two sets of auxiliary data, each with two of those three normalized L-values p-adic units, and construct two Klingen Eisenstein families. The key observation is that the p-adic L-function interpolating the third L-value, appearing in the analysis of Fourier–Jacobi coefficients of each of the Klingen Eisenstein family, is one-variable, and the variable is different for the two families, so no height one prime can contain both of the two p-adic L-functions.

2. Recall the setting

Let π be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ generated by a newform f of weight 2. Let $\mathcal{K}$ be an imaginary quadratic field. Take a finite extension L of $\mathbb{Q}_p$ containing all the Hecke eigenvalues of f. We assume the following conditions on π:

- for all finite places v of $\mathbb{Q}$, π_v is either unramified or Steinberg or Steinberg twisted by an unramified quadratic character of $\mathbb{Q}_v^\times$,
- π_p is unramified,
- there exists a prime q not split in $\mathcal{K}$ such that π is ramified at q, and if 2 does not split in $\mathcal{K}$, then π is ramified at 2,
- $\bar{\rho}_\pi|_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathcal{K})}$ is irreducible, (which is automatically true if π is not ordinary at p because in this case $\bar{\rho}_\pi|_{G_{\mathcal{K},\mathfrak{p}}} \cong \bar{\rho}_\pi|_{G_{\mathbb{Q},p}}$ is irreducible by [Edi92]), where $\bar{\rho}_\pi$ denotes the residual representation of the p-adic Galois representation $\rho_\pi : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(L)$.

Also, we fix

- an algebraic Hecke character $\xi : \mathcal{K}^\times \backslash \mathbb{A}_{\mathcal{K}}^\times \to \mathbb{C}^\times$ of ∞-type $(0, k_0)$ with k_0 an even integer.

We also recall some notation from [CLW22]. Given a Hecke character, we use the subscript $_0$ to denote its twist by a power of the norm character which is unitary. For example, $\xi_0 = \xi|\cdot|_{\mathbb{A}_{\mathcal{K}}}^{-\frac{k_0}{2}}$.

The definite unitary groups $\mathrm{U}(2), \mathrm{GU}(2)$ and the unitary group $\mathrm{GU}(3,1)$ are defined as in §5.4 in *op. cit.*

The character $\lambda : \mathcal{K}^\times \backslash \mathbb{A}_{\mathcal{K}}^\times \to \mathbb{C}^\times$ is a chosen character for theta correspondence for unitary groups satisfying

$$\lambda|_{\mathbb{A}_{\mathbb{Q}}^\times} = \eta_{\mathcal{K}/\mathbb{Q}}, \qquad \lambda_\infty(z) = \frac{|z\bar{z}|^{1/2}}{z}.$$

Let $\mathcal{K}_\infty$ be the maximal abelian pro-p extension of $\mathcal{K}$ unramified outside p and $\Gamma_{\mathcal{K}} = \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K}) (\cong \mathbb{Z}_p^2)$. Denote by $\hat{\mathcal{O}}_L^{\mathrm{ur}}$ the ring of integers of the completion of the maximal unramified extension of L.

3. The auxiliary data for two Klingen Eisenstein families

We fix the following two sets of auxiliary data for constructing Klinegn Eisenstein families:

- primes $\ell, \ell' \neq 2, p$ such that ℓ splits in $\mathcal{K}/\mathbb{Q}$, ℓ' is inert in $\mathcal{K}/\mathbb{Q}$, and $\pi_\ell, \pi_{\ell'}$ are unramified,
- positive integers $c_{v,1}, c_{v,2}$ for each place $v \in \Sigma \cup \{\ell, \ell'\}$,
- auxiliary Hecke characters $\chi_{\theta,1}, \chi_{h,1}, \chi_{\theta,2}, \chi_{h,2} : \mathcal{K}^\times \backslash \mathbb{A}_{\mathcal{K}}^\times \to \mathbb{C}^\times$ of ∞-type $(0,0)$ with

$$\chi_{h,1}\chi_{\theta,1}^c|_{\mathbb{A}_{\mathbb{Q}}^\times} = \chi_{h,2}\chi_{\theta,2}^c|_{\mathbb{A}_{\mathbb{Q}}^\times} = \mathrm{triv}$$

such that for each $i = 1, 2$, the triple $(c_{v,i}, \chi_{\theta,i}, \chi_{h,i})$ satisfies properties (1)-(5) listed in [CLW22, §5.6], and furthermore,

$$\left(\frac{2\pi i \cdot \Omega_p}{\Omega_\infty}\right)^k \frac{\Gamma(k-1)}{(2\pi i)^{k-1}} \cdot \gamma_{\bar{\mathfrak{p}}}\left(\frac{k-2}{2}, \chi_{h,1}\chi_{\theta,1}^c\xi_0\tau_0\right)^{-1} L^{p\infty}\left(\frac{k-2}{2}, \chi_{h,1}\chi_{\theta,1}^c\xi_0\tau_0\right),$$
$$\left(\frac{2\pi i\Omega_p}{\Omega_\infty}\right)^{k-2} \frac{\Gamma(k-2)}{(2\pi i)^{k-2}} \cdot L_{\mathfrak{p}}\left(\frac{k-2}{2}, \lambda^2\chi_{h,1}\chi_{\theta,1}\xi_0^c\tau_0^c\right) L^{p\infty}\left(\frac{k-2}{2}, \lambda^2\chi_{h,1}\chi_{\theta,1}\xi_0^c\tau_0^c\right),$$
$$1 - (\chi_{h,1}\chi_{\theta,1}^c\xi_0\tau_0)_q(q)q^{-\frac{k-2}{2}},$$

and

$$\left(\frac{2\pi i \cdot \Omega_p}{\Omega_\infty}\right)^k \frac{\Gamma(k-1)}{(2\pi i)^{k-1}} \cdot \gamma_{\bar{\mathfrak{p}}}\left(\frac{k-2}{2}, \chi_{h,2}\chi_{\theta,2}^c\xi_0\tau_0\right)^{-1} L^{p\infty}\left(\frac{k-2}{2}, \chi_{h,2}\chi_{\theta,2}^c\xi_0\tau_0\right),$$
$$\left(\frac{2\pi i \cdot \Omega_p}{\Omega_\infty}\right)^4 \cdot \frac{\Gamma(2)\Gamma(1)}{(2\pi i)^3} \cdot \gamma_p\left(\frac{1}{2}, \pi_p \times \left(\lambda^2\chi_{h,2}\chi_{\theta,2}\right)_{\bar{\mathfrak{p}}}\right)^{-1} L^{p\infty}\left(\frac{1}{2}, \mathrm{BC}(\pi) \times \lambda^2\chi_{h,2}\chi_{\theta,2}\right),$$
$$1 - (\chi_{h,2}\chi_{\theta,2}^c\xi_0\tau_0)_q(q)q^{-\frac{k-2}{2}}$$

are all p-adic units, and

$$L^q\left(\frac{1}{2}, \mathrm{BC}(\pi) \times \chi_{h,1}\chi_{\theta,1}^c\right) \neq 0, \qquad L^q\left(\frac{1}{2}, \mathrm{BC}(\pi) \times \chi_{h,2}\chi_{\theta,2}^c\right) \neq 0.$$

Here $\tau : \mathcal{K}^\times\backslash\mathbb{A}_\mathcal{K}^\times \to \mathbb{C}^\times$ is an algraic Hecke character such that $\tau_{p\text{-adic}}$ factors through $\Gamma_\mathcal{K}$ and $\xi\tau$ has ∞-type $(0, k)$, $k \geq 6$ even.

To see that the desired $(c_{v,i}, \chi_{\theta,i}, \chi_{h,i})$, $i = 1, 2$, exists, we can first fix places ℓ, ℓ' and positive integer c_v for $v \in \Sigma_{\mathrm{ns}} \cup \{\ell'\}$, and choose $\chi_{\theta,0}, \chi_{h,0}$ satisfying the properties (1)-(5) listed in [CLW22, §5.6]. (We don't require the last condition in (4) for $c_{1,v}, c_{2,v} \in \Sigma_{\mathrm{s}} \cup \{\ell\}$. We can choose these $c_{1,v}, c_{2,v}$'s after choosing $\chi_{\theta,1}, \chi_{h,1}, \chi_{\theta,2}, \chi_{h,2}$.) Then by Theorem 1.0.1 and Theorem 1.0.2 we know that there are infinitely many $\eta_1 \in \mathfrak{X}_\ell^-$, infinitely many $\eta_2 \in \mathfrak{X}_\ell^-$ and infinitely many $\eta_3 \in \mathfrak{X}_\ell^-$ such that

$$\left(\frac{2\pi i\Omega_p}{\Omega_\infty}\right)^k \frac{\Gamma(k-1)}{(2\pi i)^{k-1}} \cdot \gamma_{\bar{\mathfrak{p}}}\left(\frac{k-2}{2}, \eta_1\chi_{h,0}\chi_{\theta,0}^c\xi_0\tau_0\right)^{-1} L^{p\infty}\left(\frac{k-2}{2}, \eta_1\chi_{h,0}\chi_{\theta,0}^c\xi_0\tau_0\right),$$
$$\left(\frac{2\pi i \cdot \Omega_p}{\Omega_\infty}\right)^{k-2} \frac{\Gamma(k-2)}{(2\pi i)^{k-2}} \cdot L_{\mathfrak{p}}\left(\frac{k-2}{2}, \lambda^2\eta_2\chi_{h,0}\chi_{\theta,0}\xi_0^c\tau_0^c\right) L^{p\infty}\left(\frac{k-2}{2}, \lambda^2\eta_2\chi_{h,0}\chi_{\theta,0}\xi_0^c\tau_0^c\right), \tag{3.0.1}$$
$$\left(\frac{ii \cdot \Omega_p}{\Omega_\infty}\right)^4 \frac{\Gamma(2)\Gamma(1)}{(2\pi i)^3} \cdot \gamma_p\left(\frac{1}{2}, \pi_p \times \left(\lambda^2\eta_3\chi_{h,0}\chi_{\theta,0}\right)_{\bar{\mathfrak{p}}}\right)^{-1} L^{p\infty}\left(\frac{1}{2}, \mathrm{BC}(\pi) \times \lambda^2\eta_3\chi_{h,0}\chi_{\theta,0}\right)$$

are all p-adic units. Also, by [Hun17], we know that for all but finitely many $\eta_1, \eta_2, \eta_3 \in \mathfrak{X}_\ell^-$,

$$1 - \eta_1(\chi_{h,0}\chi_{\theta,0}^c\xi_0\tau_0)_q(q)q^{-\frac{k-2}{2}} \tag{3.0.2}$$

is a p-adic unit, and

$$L^q\left(\frac{1}{2}, \mathrm{BC}(\pi) \times \eta_1\chi_{h,0}\chi_{\theta,0}^c\right) \neq 0. \tag{3.0.3}$$

(For this L-value, we can choose $p' \neq \ell$, different from p, such that the results in [Hun17] apply.) Therefore, we can pick $\eta_1, \eta_2, \eta_3 \in \mathfrak{X}_\ell^-$ such that the values in (3.0.1)(3.0.2) are p-adic units and (3.0.3) holds. Take $\nu_1, \mu_1, \nu_2, \mu_2 \in \mathfrak{X}_\ell^-$ such that

$$\nu_1\mu_1^c = \nu_2\mu_2^c = \eta_1, \qquad \nu_1\mu_1 = \eta_2, \qquad \nu_2\mu_2 = \eta_3.$$

Then

$$\chi_{\theta,i} = \nu_i \chi_{\theta,0}, \qquad \chi_{h,i} = \mu_i \chi_{h,0}, \qquad i = 1,2$$

are the desired characters.

4. Klingen Eisenstein families and their nonvanishing property

Theorem 4.0.1. *There exist $\varphi_1, \varphi_2 \in \pi^{\mathrm{GU}(2)}$ such that from the given auxiliary data $\chi_{\theta,i}, \chi_{h,i}, c_{v,i}$ fixed in §3 and φ_i, $i = 1,2$, we can construct semi-ordinary Klingen Eisenstein families*

$$\boldsymbol{E}^{\mathrm{Kling}}_{\varphi_1,1}, \boldsymbol{E}^{\mathrm{Kling}}_{\varphi_2,2} \in \mathcal{M}eas\left(\Gamma_{\mathcal{K}}, V_{\mathrm{GU}(3,1),\xi} \widehat{\otimes}\, \hat{\mathcal{O}}^{\mathrm{ur}}_L\right)^{\natural} \otimes_{\mathbb{Z}} \mathbb{Q}.$$

satisfying the following properties:

(i) For all algebraic Hecke characters $\tau : \mathcal{K}^\times \backslash \mathbb{A}^\times_{\mathcal{K}} \to \mathbb{C}^\times$ such that its p-adic avatar $\tau_{p\text{-adic}}$ factors through $\Gamma_{\mathcal{K}}$ and $\xi\tau$ has ∞-type $(0,k)$ with $k \geq 6$ even, $\boldsymbol{E}^{\mathrm{Kling}}_{\varphi_1,1}(\tau_{p\text{-adic}}), \boldsymbol{E}^{\mathrm{Kling}}_{\varphi_2,2}(\tau_{p\text{-adic}})$ are Klingen Eisenstein series on $\mathrm{GU}(3,1)$ inducing $\xi_0\tau_0|\cdot|^{\frac{k-3}{2}} \boxtimes \pi^{\mathrm{GU}(2)}$.

(ii) For all cusp labels $g \in C(K^p_f)$ (defined as in [CLW22, §4.1]) and $g' \in \mathrm{GU}(2)(\mathbb{A}_{\mathbb{Q},f})$,

$$\left(\Phi_g\left(\boldsymbol{E}^{\mathrm{Kling}}_{\varphi_i,i}\right)(g')\right) \subset \left(\mathcal{L}^{\Sigma\cup\{\ell,\ell'\}}_{\xi,\mathbb{Q}}\mathcal{L}^{\Sigma\cup\{\ell,\ell'\}}_{\pi,\mathcal{K},\xi}\right), \qquad i = 1,2,$$

as ideals in $\hat{\mathcal{O}}^{\mathrm{ur}}_L[\![\Gamma_{\mathcal{K}}]\!] \otimes_{\mathbb{Z}} \mathbb{Q}$, where Φ_g is the restriction to the stratum indexed by g (cf. Proposition 4.4.1 in loc. cit, $\mathcal{L}^{\Sigma\cup\{\ell,\ell'\}}_{\xi,\mathbb{Q}}, \mathcal{L}^{\Sigma\cup\{\ell,\ell'\}}_{\pi,\mathcal{K},\xi} \in \hat{\mathcal{O}}^{\mathrm{ur}}_L[\![\Gamma_{\mathcal{K}}]\!] \otimes_{\mathbb{Z}} \mathbb{Q}$ are the p-adic L-functions introduced at the beginning of §6.1 in loc. cit. ($\mathcal{L}^{\Sigma\cup\{\ell,\ell'\}}_{\pi,\mathcal{K},\xi}$ is the p-adic L-function appearing in the main conjecture studied in loc. cit.)

(iii) Let $\beta = 1$ and $\boldsymbol{E}^{\mathrm{Kling}}_{\varphi_i,i,\beta,u} \in \mathcal{M}eas\left(\Gamma_{\mathcal{K}}, V^{J,\beta}_{\mathrm{GU}(2)} \widehat{\otimes}\, \hat{\mathcal{O}}^{\mathrm{ur}}_L\right)$ is the β-th Fourier–Jacobi coefficient of $\begin{pmatrix} u & & \\ & \mathbf{1}_2 & \\ & & u \end{pmatrix} \boldsymbol{E}^{\mathrm{Kling}}_{\varphi,i}$ along the boundary stratum indexed by the cusp label $\mathbf{1}_4 \in C(K^p_f K^1_{p,n})_{\mathrm{ord}}$ as defined in Equation (7.0.1) in loc. cit. There exist linear functionals

$$l_{\theta^J_{1,1}}, l_{\theta^J_{1,2}} : V^{J,\beta}_{\mathrm{GU}(2)} \longrightarrow V_{\mathrm{U}(2)}$$

such that the ideal generated by

$$l_{\theta^J_{1,1}}\left(\boldsymbol{E}^{\mathrm{Kling}}_{\varphi_1,1,\beta,u}\right)(g),\ l_{\theta^J_{1,2}}\left(\boldsymbol{E}^{\mathrm{Kling}}_{\varphi_2,2,\beta,u}\right)(g),\quad g \in \mathrm{U}(2)(\mathbb{A}_{\mathbb{Q},f}),\quad u \in \bigotimes_{v\in\Sigma_{\mathrm{ns}}\cup\{\ell'\}} \mathrm{U}(1)(\mathbb{Q}_v), \tag{4.0.1}$$

does not belong to any height one prime in $\hat{\mathcal{O}}^{\mathrm{ur}}_L[\![\Gamma_{\mathcal{K}}]\!] \otimes_{\mathbb{Z}} \mathbb{Q}$.

See Equation (5.8.6), §5.4, Equation (4.7.2), §5.8.3 in [CLW22] for the notations $\mathcal{M}eas(\Gamma_{\mathcal{K}}, V_{\mathrm{GU}(3,1),\xi})^{\natural}$, $\pi^{\mathrm{GU}(2)}$, $V^{J,\beta}_{\mathrm{GU}(2)}$, $V_{\mathrm{U}(2)}$.

Proof. Let $U_{\mathcal{K},p} = 1 + p\mathcal{O}_{\mathcal{K},p}$ and pick $\mathtt{N}$, a non-negative power of p, such that we have the embedding

$$\mathscr{P}_{\mathtt{N}} : \Gamma_{\mathcal{K}} \xhookrightarrow{\mathtt{N}\text{-th power}} U_{\mathcal{K},p}.$$

The embedding $\mathscr{P}_{\mathtt{N}}$ induces maps $\mathscr{P}_{\mathtt{N},*}$ from p-adic measures on $\Gamma_{\mathcal{K}}$ to p-adic measures on $U_{\mathcal{K},p}$. For $i = 1,2$, let

$$\mathcal{L}_{1,i}, \mathcal{L}_{5,i}, \mathcal{L}_{6,i} \in \hat{\mathcal{O}}^{\mathrm{ur}}_L[\![U_{\mathcal{K},p}]\!] \otimes_{\mathbb{Z}} \mathbb{Q}, \qquad \mathcal{L}_{2,i} \in L^{\mathrm{ur}}$$

be as in [CLW22, page 76, page 74] with χ_h, χ_θ in *loc.cit* replaces by $\chi_{h,i}, \chi_{\theta,i}$, with $\mathcal{L}_{1,i}, \mathcal{L}_{2,i}$ interpolating

$$L^{p\infty}\left(\frac{1}{2}, \mathrm{BC}(\pi) \times \lambda^2 \chi_{h,i}\chi_{\theta,i}\tau_{\mathfrak{p},\mathscr{P}_{\mathtt{N}}}\tau^{-c}_{\mathfrak{p},\mathscr{P}_{\mathtt{N}}}\right), \qquad L^{\infty}\left(\frac{1}{2}, \mathrm{BC}(\pi) \times \chi_{h,i}\chi^c_{\theta,i}\right),$$

and $\mathcal{L}_{5,i}, \mathcal{L}_{6,i}$ interpolating

$$L^{p\infty}\left(\frac{\mathtt{N}k-2}{2}, \chi_h\chi_\theta^c\xi_0\tau_0^{\mathtt{N}}\right), \qquad L^{p\infty}\left(\frac{\mathtt{N}k-2}{2}, \lambda^2\chi_h\chi_\theta\tau_{\mathfrak{p},\mathcal{P}_{\mathtt{N}}}\tau_{\mathfrak{p},\mathcal{P}_{\mathtt{N}}}^{-c}(\xi_0\tau_0^{\mathtt{N}})^c\right),$$

where the character $\tau_{\mathfrak{p},\mathcal{P}_{\mathtt{N}}}$ of $\Gamma_{\mathcal{K}}$ is defined as $\tau_{\mathfrak{p},\mathcal{P}_{\mathtt{N}}} = \tau_{p\text{-adic}}|_{U_{\mathcal{K},\mathfrak{p}}} \circ \mathcal{P}_{\mathtt{N}}$. Since $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits in $\mathcal{K}$, we have

$$U_{\mathcal{K},p} = U_{\mathcal{K},\mathfrak{p}} \times U_{\mathcal{K},p}^+$$

with

$$U_{\mathcal{K},\mathfrak{p}} = 1 + p\mathcal{O}_{\mathcal{K},\mathfrak{p}}, \qquad U_{\mathcal{K},p}^+ = \{(a,a) \in (1+\mathcal{O}_{\mathcal{K},\mathfrak{p}}) \times (1+\mathcal{O}_{\mathcal{K},\bar{\mathfrak{p}}}) : a \in 1 + p\mathbb{Z}_p\}.$$

A key observation here is that, by looking at the interpolation properties of $\mathcal{L}_{1,i}, \mathcal{L}_{6,i}$, we see that

$$\mathcal{L}_{1,i} \in \hat{\mathcal{O}}_L^{\mathrm{ur}}[\![U_{\mathcal{K},\mathfrak{p}}]\!] \otimes_{\mathbb{Z}} \mathbb{Q}, \qquad \mathcal{L}_{6,i} \in \hat{\mathcal{O}}_L^{\mathrm{ur}}[\![U_{\mathcal{K},p}^+]\!] \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{4.0.2}$$

It follows from the choice of $\chi_{\theta,i}, \chi_{h,i}$ in §3 that

$$\begin{aligned} \mathcal{L}_{1,1}\mathcal{L}_{2,1}\mathcal{L}_{5,1}\mathcal{L}_{6,1} &\in \mathcal{L}_{1,1} \cdot \left(\hat{\mathcal{O}}_L^{\mathrm{ur}}[\![U_{\mathcal{K},p}]\!] \otimes_{\mathbb{Z}} \mathbb{Q}\right)^\times, \\ \mathcal{L}_{1,2}\mathcal{L}_{2,2}\mathcal{L}_{5,2}\mathcal{L}_{6,2} &\in \mathcal{L}_{6,2} \cdot \left(\hat{\mathcal{O}}_L^{\mathrm{ur}}[\![U_{\mathcal{K},p}]\!] \otimes_{\mathbb{Z}} \mathbb{Q}\right)^\times \end{aligned} \tag{4.0.3}$$

We also know that

$$\mathcal{L}_{1,1} \neq 0, \qquad \mathcal{L}_{6,2} \neq 0. \tag{4.0.4}$$

The nonvanishing of $\mathcal{L}_{6,2}$ follows immediately from the nonvanishing of the values with $k \gg 0$ it interpolates. The nonvanishing of $\mathcal{L}_{1,1}$ follows from the results on the μ-invariants in [Hsi14b]. (Note that the proof Theorem B in *op. cit.* shows that the μ-invariant of $\mathcal{L}_{1,1}$ is a positive integer without the need to assume the assumptions (2)(3) there.)

For $i = 1, 2$, we choose Schwartz function $\phi_{1,i}$ on $\mathbb{A}_{\mathbb{Q}}^2$ as in [CLW22, Proposition 7.6.1] (noting that the choice in *loc.cit* works for all forms inside the space (5.6.1) in *loc.cit*, so we can choose $\phi_{1,i}$ only depending on the auxiliary data fixed in §3). Then in the same way as in §7.7 in *op. cit.*, we construct integral ordinary CM families $\boldsymbol{h}_i, \tilde{\boldsymbol{h}}_{3,i}, \boldsymbol{\theta}_i, \tilde{\boldsymbol{\theta}}_{3,i}$ on $\mathrm{U}(2)$ (as p-adic measures on $U_{\mathcal{K},p}$), and choose φ_i as in Proposition 7.11.2 in *loc.cit.* The same construction as in §§5.7-5.10 in *loc.cit* applied to the auxiliary data fixed in §3 gives semi-ordinary p-adic Klingen Eisenstein families

$$\boldsymbol{E}_{\varphi_1,1}^{\mathrm{Kling}}, \boldsymbol{E}_{\varphi_2,2}^{\mathrm{Kling}} \in \mathcal{M}eas\left(\Gamma_{\mathcal{K}}, V_{\mathrm{GU}(3,1),\xi}\widehat{\otimes}\,\hat{\mathcal{O}}_L^{\mathrm{ur}}\right)^\natural \otimes_{\mathbb{Z}} \mathbb{Q}$$

satisfying (i), and the same computation as in §6 in *op. cit.* shows (ii).

To prove (iii), the same proof of Propositions 7.9.1, 7.11.1, 7.11.2 in *loc.cit* shows

$$\begin{aligned} &\left\langle \left(\mathscr{T}_{\mathrm{ns}}\breve{\boldsymbol{h}}\right)|_{\mathrm{U}(2)}, \mathcal{P}_{N,*}\left(l_{\theta_1^J}\left(\sum\nolimits_i b_i \boldsymbol{E}_{\varphi,\beta,u_i}^{\mathrm{Kling}}\right)\right)\right\rangle_{p\text{-adic}} \\ \underset{\hat{\mathcal{O}}_L^{\mathrm{ur}}[\![U_{\mathcal{K},p}]\!]\otimes_{\mathbb{Z}}\mathbb{Q}}{\overset{\text{up to a unit in}}{=}} &\frac{\mathcal{L}_{1,i}\mathcal{L}_{2,i}\mathcal{L}_{5,i}\mathcal{L}_{6,i}}{\left\langle\left(\mathscr{T}'_{\mathrm{ns}}\breve{\tilde{\boldsymbol{h}}}_3\right)|_{\mathrm{U}(2)}, \tilde{\boldsymbol{\theta}}_3^\lambda\varphi'\right\rangle_{p\text{-adic}}}. \end{aligned} \tag{4.0.5}$$

Denote by J the ideal in $\hat{\mathcal{O}}_L^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!] \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by elements in (4.0.1). Combining (4.0.3) and (4.0.5), we see that $\mathcal{P}_{\mathtt{N}}(J)$ contains $(\mathcal{L}_{1,1}, \mathcal{L}_{6,2}) \subset \hat{\mathcal{O}}_L^{\mathrm{ur}}[\![U_{\mathcal{K},p}]\!] \otimes_{\mathbb{Z}} \mathbb{Q}$. Thanks to (4.0.2), we deduce that $\mathcal{P}_{\mathtt{N}}(J)$ does not belong to any height one prime in $\hat{\mathcal{O}}_L^{\mathrm{ur}}[\![U_{\mathcal{K},p}]\!] \otimes_{\mathbb{Z}} \mathbb{Q}$. Because $\mathcal{P}_{\mathtt{N}}$ embeds $\hat{\mathcal{O}}_L^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!] \otimes_{\mathbb{Z}} \mathbb{Q}$ into $\hat{\mathcal{O}}_L^{\mathrm{ur}}[\![U_{\mathcal{K},p}]\!] \otimes_{\mathbb{Z}} \mathbb{Q}$, which is integral over $\mathcal{P}_{\mathtt{N}}(\hat{\mathcal{O}}_L^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!] \otimes_{\mathbb{Z}} \mathbb{Q})$, it follows that J does not belong to any height one prime in $\hat{\mathcal{O}}_L^{\mathrm{ur}}[\![\Gamma_{\mathcal{K}}]\!] \otimes_{\mathbb{Z}} \mathbb{Q}$. $\square$

5. The Klingen Eisenstein ideal and Greenberg–Iwasawa main conjecture

Theorem 5.0.1. *Theorem 8.1.1 in [CLW22] holds.*

Proof. The Eisenstein families $\boldsymbol{E}^{\mathrm{Kling}}_{\varphi_1,1}, \boldsymbol{E}^{\mathrm{Kling}}_{\varphi_2,2}$ in Theorem 4.0.1 belongs to the Hecke eigensystem $\lambda_{\mathrm{Eis},\pi,\xi}$ in §8.1 in *op. cit.* Let P be a height one prime of $\hat{\mathcal{O}}^{\mathrm{ur}}_L[\![\Gamma_{\mathcal{K}}]\!]\otimes_{\mathbb{Z}}\mathbb{Q}$ considered in the proof of Theorem 8.1.1 in *op. cit.* Then Theorem 4.0.1 implies that for $i=1$ or 2, $\beta=1$, there exist $g\in \mathrm{U}(2)(\mathbb{A}_{\mathbb{Q}})$ and $u\in\bigotimes_{v\in\Sigma_{\mathrm{ns}}\cup\{\ell'\}}\mathrm{U}(1)(\mathbb{Q}_v)$ such that $l_{\theta^J_{1,i}}\left(\boldsymbol{E}^{\mathrm{Kling}}_{\varphi_i,i,\beta,u}\right)(g)\notin P$. The remaining argument in Theorem 8.1.1 in *op. cit.* goes through with $l_{\theta^J_1}\left(\boldsymbol{E}^{\mathrm{Kling}}_{\varphi,\beta,u}\right)(g)$ replaced by $l_{\theta^J_{1,i}}\left(\boldsymbol{E}^{\mathrm{Kling}}_{\varphi_i,i,\beta,u}\right)(g)$. ☐

Once Theorem 8.1.1 in [CLW22] on the Klingen Eisenstein ideal is proved, the main results on the Greenberg–Iwasawa main conjecture in *op. cit.* follow without change from the lattice construction there.

Theorem 5.0.2. *Theorems 8.2.1 and 8.2.3 in [CLW22] hold.*

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