\textbf{p^{\infty}-SELMER GROUPS AND RATIONAL POINTS ON CM ELLIPTIC CURVES}

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\textit{To Bernadette Perrin-Riou, with admiration}

\begin{abstract}
Let \( E/\mathbb{Q} \) be a CM elliptic curve and \( p \) a prime of good ordinary reduction. We show that if \( \text{Sel}_p(\mathbb{Q}) \) has \( \mathbb{Z}_p \)-corank one, then \( E(\mathbb{Q}) \) has a point of infinite order. The non-torsion point arises from a Heegner point, and thus \( \text{ord}_{s=1} L(E, s) = 1 \), yielding a \( p \)-converse to a theorem of Gross–Zagier, Kolyvagin, and Rubin in the spirit of [Ski20].

For \( p > 3 \), this gives a new proof of the main result of [BT20], which our approach extends to small primes. The approach generalizes to CM elliptic curves over totally real fields [BCST21].
\end{abstract}

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\section*{Introduction}

Starting with [Ski20], several works have been devoted to \( p \)-converses to a celebrated theorem of Gross–Zagier, Kolyvagin, and Rubin: If the \( p^{\infty} \)-Selmer group \( \text{Sel}_p(\mathbb{Q}) \) has \( \mathbb{Z}_p \)-corank 1 for an elliptic curve \( E/\mathbb{Q} \), then \( \text{ord}_{s=1} L(E, s) = 1 \) (see [BST21] for an overview of some of these works). Besides being an evidence for the Birch and Swinnerton-Dyer conjecture, an important impetus for the \( p \)-converse theorems has come from recent developments in arithmetic statistics.

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For instance, such $p$-converse theorems\footnote{Along with an analogous result in rank zero, typically deduced from the cyclotomic Iwasawa main conjecture for modular forms [Kat04,SU14,Wan20].} have led to the proof [BSZ15] that a large proportion of elliptic curves over $\mathbb{Q}$—and conditionally, 100% of them—satisfy the Birch and Swinnerton-Dyer conjecture.

The main result of [Ski20] proceeds via exhibiting a certain Heegner point on $E$ with infinite order, and holds for primes $p > 3$ of good ordinary reduction of $E$, and under certain hypotheses that excluded the CM curves.

Our main result is a CM $p$-converse theorem. For $p > 3$, the result was first obtained in [BT20].

**Theorem A.** Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication by an order of an imaginary quadratic field $K$ of discriminant $-D_K < 0$. Assume that the Hecke character associated to $E$ has conductor exactly divisible by $\mathfrak{o}_K := (\sqrt{-D_K})$. Let $p$ be a prime of good ordinary reduction for $E$. Then

$$\text{corank}_{\mathbb{Z}_p}\text{Sel}_p(E/\mathbb{Q}) = 1 \implies \text{ord}_{s=1}L(E, s) = 1.$$ 

In particular, if $\text{corank}_{\mathbb{Z}_p}\text{Sel}_p(E/\mathbb{Q}) = 1$ then $\text{rank}_{\mathbb{Z}}E(\mathbb{Q}) = 1$ and $\#\mathfrak{I}(E/\mathbb{Q}) < \infty$.

Of course, the ‘in particular’ clause in Theorem A follows from combining its conclusion with the fundamental work of Gross–Zagier, Kolyvagin, and Rubin. In turn, this consequence yields the following mod $p$ criterion for analytic rank one.

**Corollary B.** Let $(E, K)$ be as in Theorem A. Let $p$ be a prime of good ordinary reduction for $E$ such that:

(i) $E(\mathbb{Q})[p] = 0$;

(ii) $\text{Sel}_p(E/\mathbb{Q}) \cong \mathbb{Z}/p\mathbb{Z}$, where $\text{Sel}_p(E/\mathbb{Q}) \subset H^1(\mathbb{Q}, E[p])$ is the $p$-Selmer group of $E$.

Then $\text{ord}_{s=1}L(E, s) = 1$ and $\mathfrak{I}(E/\mathbb{Q})[p \infty] = 0$.

More generally, we prove a $p$-converse for CM abelian varieties $B_\lambda/K$ associated with Hecke characters $\lambda$ over $K$ of infinity type $(-1, 0)$ (cf. Theorem 7.0.2).

A salient feature of the approach in this note is that it generalizes to CM elliptic curves over totally real fields: It sidesteps the inherent use of elliptic units in [BT20], and leads to the first $p$-converse theorems over general totally real fields, [BCST21]. The present note might thus be viewed as a prelude to [BCST21].

The conductor hypothesis in Theorem A arises from an appeal to [CH18] which supposes a classical Heegner hypothesis. This hypothesis may be removed (cf. [BCK20], [BCST21]) via the $p$-adic Waldspurger formula of Liu–Zhang–Zhang [LZZ18], whose habitat is the general Yuan–Zhang framework [YZZ13]. (The hypothesis is not present in [BT20], thanks to Disegni’s $p$-adic Gross–Zagier formula [Dis17], also in the framework of [YZZ13].)

**Remark C.** Independently, the $p$-converse as in Theorem A is due to Ressler and Yu [RY21], [Yu21]. Complementing the present note, their approach builds on [BT20]. (A key new element in their work is a counterpart of the main results of Agboola–Howard [AH06] for small primes.)

**Remark D.** A recent spectacular result of Smith [Smi17] reduces Goldfeld’s conjecture [Gol79] for CM elliptic curves $E/Q$ with $E(\mathbb{Q})[2] \cong (\mathbb{Z}/2\mathbb{Z})^2$ and admitting no rational cyclic 4-isogeny to the proof of the implication in Theorem A (and its analogue in rank zero) for $p = 2$. (Indeed, the unconditional rank zero CM $p$-converse is proved in [BT19], [BT21].) Unfortunately, Theorem A falls short of providing the desired implication in rank one: First, $E$ should be allowed to just have potentially good ordinary reduction at $p$ (although this might be approachable by similar methods); the second—and the main—hindrance is that $\mathbb{Q}(\sqrt{-7})$ is the only imaginary quadratic field of class number 1 where 2 splits, and $E(\mathbb{Q})[2] \cong \mathbb{Z}/2\mathbb{Z}$ for all elliptic curves $E/Q$ with CM by $\mathbb{Q}(\sqrt{-7})$ (see e.g. the table in [Ols74, p. 2]).
The approach. Assuming \#III(E/\mathbb{Q})[p^\infty] < \infty and \( p \nmid \#\mathcal{O}_K^\times \), the \( p \)-converse (cf. Theorem A) goes back to Rubin [Rub94, Thm. 4]. Around the same time, Rubin proved a striking formula [Rub92] expressing the \( p \)-adic formal group logarithm of a point \( P \in E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Z}_p \) in terms of the value of a Katz \( p \)-adic \( L \)-function outside its range of interpolation. Our approach to Theorem A is inspired by the \( p \)-adic Waldspurger formula of Bertolini–Darmon–Prasanna [BDP13], which is a remarkable generalization of Rubin’s formula, and sheds new light on [Rub92] (cf. [BDP12]).

Let \( E/\mathbb{Q} \) be an elliptic curve with CM by an imaginary quadratic field \( K \), and \( p \) a prime of good ordinary reduction. Let \( \lambda \) be the Hecke character over \( K \) associated with \( E \), so that

\[
L(E, s) = L(\lambda, s).
\]

In view of certain non-vanishing results by Rohrlich recalled in the body of the paper, we pick a pair \( (\psi, \chi) \) of Hecke characters over \( K \) with \( \chi \) of finite order such that

\[
\psi\chi = \lambda, \quad L(\psi^*\chi, 1) \neq 0,
\]

where \( \psi^* := \psi \circ c \) is the composition of \( \psi \) with the non-trivial element \( c \in \text{Gal}(K/\mathbb{Q}) \). For \( f = \theta_\psi \) the theta series associated to \( \psi \), the main result of [BDP13] relates the \( p \)-adic formal group logarithm of a Heegner point \( P_{\psi, \chi} \in B_{\psi, \chi}(K) \) to a value (outside the range of interpolation) of a \( p \)-adic Rankin \( L \)-series \( \mathcal{L}_v(f, \chi) \). Here \( B_{\psi, \chi} \) is a CM abelian variety over \( K \) endowed with a \( K \)-rational map \( i_\chi : B_{\psi, \chi} \to E \). In view of the Gross–Zagier formula [YZZ13], \( P_{\psi, \chi} \) is non-torsion if and only if \( L'(f, \chi, 1) \neq 0 \). Setting

\[
P_K := i_\chi(P_{\psi, \chi}) \in E(K),
\]

we thus obtains a point on \( E \) which, in light of (0.1) and the factorization

\[
L(f, \chi, s) = L(\lambda, s) \cdot L(\psi^*\chi, s),
\]

is non-torsion if and only if \( \text{ord}_{s=1} L(E, s) = 1 \).

Theorem A is thus shown to be equivalent to: If \( \text{Sel}_{p^\infty}(E/\mathbb{Q}) \) has \( \mathbb{Z}_p \)-corank 1, then the point \( P_K \) is non-torsion. In [BT20] the non-triviality is shown via the following.

1. The anticyclotomic Iwasawa main conjecture (IMC) for (Hecke characters over) \( K \) in the root number \( +1 \) case [Rub91];
2. The anticyclotomic IMC for \( K \) in the root number \( -1 \) case [AH06], [Arn07];
3. The non-vanishing of the \( \Lambda \)-adic regulator appearing in (2) [Bur15];
4. The \( \Lambda \)-adic Gross–Zagier formula [Dis17].

Here \( \Lambda \) denotes the anticyclotomic Iwasawa algebra over \( K \) (with certain coefficients).

Bypassing (2), (3), and (4), our approach builds on the explicit reciprocity law [CH18], which realizes \( \mathcal{L}_v(f, \chi) \) as the image of a \( \Lambda \)-adic Heegner class \( z_{f, \chi} \) under a Perrin-Riou big logarithm map. Similarly as in [BDP12], we establish a factorization

\[
\mathcal{L}_v(\theta_\psi, \chi)^2 = \mathcal{L}_v(\psi\chi) \cdot \mathcal{L}_v(\psi^*\chi)
\]

relating \( \mathcal{L}_v(\theta_\psi, \chi)^2 \) to the product of two Katz \( p \)-adic \( L \)-functions (cf. §3), mirroring (0.2). In particular, this leads to an expression relating the formal group logarithm of \( P_K \) to a value of a Katz \( p \)-adic \( L \)-function, as in Rubin’s formula.

Along with an analogous decomposition for Selmer groups (cf. §2), the Iwasawa–Greenberg main conjecture for \( \mathcal{L}_v(\theta_\psi, \chi)^2 \) is readily seen to be a consequence of [Rub91], [JLK11]. In light of the Iwasawa explicit reciprocity law, this main conjecture is equivalent to another IMC involving the zeta element \( z_{f, \chi} \) (cf. §4). Finally, the latter yields the implication

\[
\text{corank}_{\mathbb{Z}_p} \text{Sel}_{p^\infty}(E/\mathbb{Q}) = 1 \implies P_K \neq 0 \in E(K) \otimes_{\mathbb{Z}} \mathbb{Q}
\]

via a variant of Mazur’s control theorem.
**Dedication.** The $p$-converse for CM elliptic curves $E/Q$ is due to Rubin if $\#\Im(E/Q)[p^\infty] < \infty$. The essence of our removal of this hypothesis is Iwasawa theory of Heegner points, as pioneered by Perrin-Riou [PR87]. The theory of big logarithm maps [PR94], another major contribution of Perrin-Riou, is also elemental to our approach. It is a great pleasure to dedicate this note to Bernadette Perrin-Riou as a humble gift on the occasion of her 65th birthday.

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1. Preliminaries

Fix throughout a prime $p$, an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and embeddings $\mathbb{C} \overset{\iota \sim}{\rightarrow} \overline{\mathbb{Q}}_{p} \rightarrow \mathbb{C}$. Fix also an imaginary quadratic field $\mathcal{K}$ of discriminant $-D_{\mathcal{K}} < 0$ and ring of integers $\mathcal{O}_{\mathcal{K}}$.

1.1. CM abelian varieties. We say that a Hecke character $\psi : \mathcal{K}^\times \backslash \mathbb{A}_{\mathcal{K}}^\times \rightarrow \mathbb{C}^\times$ has [infinity type](#) $(a, b) \in \mathbb{Z}^2$ if, writing $\psi = (\psi_v)_v$ with $v$ running over the places of $\mathcal{K}$, the component $\psi_v(\infty) = z^a \bar{z}^b$ for all $z \in (\mathcal{K} \otimes \mathbb{Q} \mathbb{R})^\times \simeq \mathbb{C}^\times$, where the identification is made via $\iota_{\infty}$. Hence in particular the norm character $\mathcal{N}_{\mathcal{K}}$, given by $q \mapsto \#(\mathcal{O}_{\mathcal{K}}/q)$ on ideals of $\mathcal{O}_{\mathcal{K}}$, has infinity type $(-1, -1)$. The [central character](#) of such $\psi$ is the character $\omega_{\psi}$ on $\mathbb{A}_{\mathcal{K}}^\times$ defined by

$$\psi |_{\mathbb{A}_{\mathcal{K}}^\times} = \omega_{\psi} \cdot \mathcal{N}^{-(a+b)},$$

where $\mathcal{N}$ is the norm on $\mathbb{A}_{\mathcal{K}}^\times$.

Our fixed embedding $\iota_{p}$ defines a natural map $\sigma : \mathcal{K} \otimes \mathbb{Q}_{p} \rightarrow \mathbb{C}_{p}$, and we let $\overline{\sigma} : \mathcal{K} \otimes \mathbb{Q}_{p} \rightarrow \mathbb{C}_{p}$ be composition of $\sigma$ with the non-trivial automorphism of $\mathcal{K}$. The $p$-adic avatar of a Hecke character $\psi$ of infinity type $(a, b)$ is the character $\hat{\psi} : \mathcal{K}^\times \backslash \mathbb{A}_{\mathcal{K}^p}^\times \rightarrow \mathbb{C}_{p}^\times$ given by

$$\hat{\psi}(x) = \iota_{p} \circ \iota_{p}^{-1}(\psi(x)) \sigma(x_{p})^a \overline{\sigma}(x_{p})^b$$

for all $x \in \mathbb{A}_{\mathcal{K}^p}^\times$, where $x_{p} \in (\mathcal{K} \otimes \mathbb{Q}_{p})^\times$ is the $p$-component of $x$.

Thoughout the following, we shall often omit the notational distinction between an algebraic Hecke character and its $p$-adic avatar, as it will be clear from the context which one is meant.

Let $\hat{\psi}$ be an algebraic Hecke character of $\mathcal{K}$ infinity type $(-1, 0)$ with values in a number field $F_{\psi} \subset \mathbb{Q}$ with ring of integer $\mathcal{O}_{\psi}$. Let $\mathfrak{P}$ be the prime of $F_{\psi}$ above $p$ induced by $\iota_{p}$, and denote by $\Phi_{\psi}$ the completion of $F_{\psi}$ at $\mathfrak{P}$ and by $\mathcal{O}_{\psi}$ the ring of integers of $\Phi_{\psi}$. By a well-known theorem of Casselman’s (see [BDP12, Thm. 2.5] and the reference [Shi71, Thm. 6] therein), attached to $\psi$ there is a CM abelian variety $B_{\psi/\mathcal{K}}$, unique up to isogeny over $\mathcal{K}$, with the property that

$$V_{\mathfrak{P}}B_{\psi} \simeq \hat{\psi}$$

as one-dimensional $\Phi_{\psi}$-representations of $G_{\mathcal{K}}$, where $V_{\mathfrak{P}}B_{\psi} = (\varprojlim B_{\psi}[\mathfrak{P}]) \otimes_{\mathcal{O}_{\psi}} \Phi_{\psi}$ is the rational $\mathfrak{P}$-adic Tate module of $B_{\psi}$.

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[More precisely, the proof of the “Heegner point main conjecture” (cf. §1.3); one of the divisibilities suffices for the $p$-converse.]
1.2. **Heegner points.** Let \( f \in S_2(\Gamma_1(N)) \) be a normalized eigenform of weight 2, level \( N \) prime to \( p \), and nebentypus \( \varepsilon_f \). We assume that \( \mathcal{K} \) satisfies the *Heegner hypothesis* relative to \( N \):

\[
\text{there is an ideal } \mathfrak{R} \subset \mathcal{O}_K \text{ with } \mathcal{O}_K/\mathfrak{R} \simeq \mathbb{Z}/NZ, \tag{Heeg}
\]

and fix once and for all an ideal \( \mathfrak{R} \) as above. We assume also that

\[
p\mathcal{O}_K = v\mathfrak{R} \text{ splits in } \mathcal{K}, \tag{spl}
\]

with \( v \) the prime of \( \mathcal{K} \) above \( p \) induced by our fixed embedding \( \iota_p \). Let \( F \subset \overline{\mathbb{Q}} \) be the number field generated by the Fourier coefficients of \( f \). Denote by \( \mathfrak{P} \) the prime of \( F \) above \( p \) induced by \( \iota_p \), and assume that \( f \) is \( \mathfrak{P} \)-ordinary, i.e. \( \mathfrak{P}(a_p(f)) = 0 \), where \( \mathfrak{P} \) is the \( \mathfrak{P} \)-adic valuation on \( F \).

Let \( A_f/\mathbb{Q} \) be the abelian variety of \( \text{GL}_2 \)-type associated to \( f \), determined up to isogeny over \( \mathbb{Q} \) by the equality of \( L \)-functions

\[
L(A_f, s) = \prod_{\tau:F \to \mathbb{C}} L(f_{\tau}, s),
\]

where \( f_{\tau} \) runs over all the conjugates of \( f \). Denote by \( \Phi \) the completion of \( F \) at \( \mathfrak{P} \), and let \( \mathcal{O} \) be the ring of integers of \( \Phi \). Let \( T_{\mathfrak{P}} A_f := \varprojlim A_f[\mathfrak{P}] \) be the \( \mathfrak{P} \)-adic Tate module of \( A_f \), which is free of rank two over \( \mathcal{O} \).

For every positive integer \( c \), let \( \mathcal{K}_c \) be the ring class field of \( \mathcal{K} \) of conductor \( c \), so \( \text{Gal}(\mathcal{K}_c/\mathcal{K}) \simeq \text{Pic}(\mathcal{O}_c) \) by class field theory, where \( \mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_K \) is the order of \( \mathcal{K} \) of conductor \( c \). For every \( c > 0 \) prime to \( N \) and every ideal \( a \) of \( \mathcal{O}_c \), we consider the CM point \( x_a \in X_1(N)(\mathcal{K}_c) \) constructed in [CH18, §2.3], where \( \mathcal{K}_c \) is the compositum of \( \mathcal{K}_c \) and the ray class field of \( \mathcal{K} \) of conductor \( \mathfrak{R} \). Let \( \Delta_a \) be the class of the degree 0 divisor \( (x_a) - (\infty) \) in \( J_1(N) = \text{Jac}(X_1(N)) \), and denote by \( z_a = \delta(\Delta_a) \) its image under the Kummer map

\[
\delta : J_1(N)(\mathcal{K}_c) \to H^1(\mathcal{K}_c, T_p J_1(N)).
\]

Fix a parametrization \( \pi : J_1(N) \to A_f \), and let \( y_{a, \pi} \in H^1(\mathcal{K}_c, T_{\mathfrak{P}} A_f) \) be the image of \( y_a \) under the natural projection

\[
H^1(\mathcal{K}_c, T_p J_1(N)) \xrightarrow{\pi_*} H^1(\mathcal{K}_c, T_p A_f) \to H^1(\mathcal{K}_c, T_{\mathfrak{P}} A_f).
\]

For the ease of notation, we set \( y_{f, c} = y_{a, \pi} \) for \( a = \mathcal{O}_c \). A standard calculation shows that if \( p \nmid c \), then for every \( n > 0 \) we have

\[
\text{Cor}_{\mathcal{K}_c^{p^n}/\mathcal{K}_c^{p^{n-1}}}(y_{f, c^{p^n}}) = \begin{cases} a_p(f) \cdot y_{f, c^{p^{n-1}}} - \varepsilon_f(p) \cdot y_{f, c^{p^{n-2}}} & \text{if } n > 1, \\
\nu_{c^{-1}}(a_p(f) - \sigma_v - \sigma_{\mathfrak{P}}) \cdot y_{f, c} & \text{if } n = 1,
\end{cases}
\]

where \( \nu_c := [\mathcal{O}_c^x : \mathcal{O}_{c_p}^x] \) and \( \sigma_v, \sigma_{\mathfrak{P}} \in \text{Gal}(\mathcal{K}_c/K) \) are Frobenius elements at the primes of \( \mathcal{K} \) above \( p \) (cf. [CH18, Prop. 4.4]).

Let \( \alpha \) be the \( \mathfrak{P} \)-adic unit root of \( x^2 - a_p(f)x + \varepsilon_f(p)p \), and for any positive integer \( c \) prime to \( N \) define the *\( \alpha \)-stabilized Heegner class* \( y_{f, c, \alpha} \) by

\[
y_{f, c, \alpha} := \begin{cases} y_{f, c} - \varepsilon_f(p)\alpha^{-1} \cdot y_{f, c/p} & \text{if } p \mid c, \\
\nu_{c^{-1}}(1 - \sigma_v\alpha^{-1} - \sigma_{\mathfrak{P}}\alpha^{-1}) \cdot y_{f, c} & \text{if } p \nmid c.
\end{cases}
\]

This definition is motivated by the following result.

**Lemma 1.2.1.** For all positive integers \( c \) prime to \( N \), we have

\[
\text{Cor}_{\mathcal{K}_c/p/\mathcal{K}_c}(y_{f, c^{p^n}, \alpha}) = \alpha \cdot y_{f, c, \alpha}.
\]

**Proof.** This follows immediately from (1.1). 
\( \square \)
1.3. **Heegner point main conjecture.** Let $\mathcal{K}_\infty$ be the anticyclotomic $\mathbb{Z}_p$-extension of $\mathcal{K}$, with Galois group

$$
\Gamma = \text{Gal}(\mathcal{K}_\infty/\mathcal{K}) \simeq \mathbb{Z}_p,
$$

and for every $n$ denote by $\mathcal{K}_n$ the subextension of $\mathcal{K}_\infty$ with $[\mathcal{K}_n : \mathcal{K}] = p^n$. Let $\chi$ be a finite order Hecke character of $\mathcal{K}$ with $\chi|_{\mathcal{K}^\times} = \varepsilon_f^{-1}$. Upon enlarging $F$ is necessary, assume that $\Phi$ contains the values of $\chi$. For each $n$, take $m \gg 0$ so that $\mathcal{K}_{cp^m} \supset \mathcal{K}_n$, and set

$$
z_{f,\chi,n} := \alpha^{-m} \cdot \text{Cor}_{\mathcal{K}_{cp^m}/\mathcal{K}_n} \left( \sum_{\sigma \in \text{Gal}(\mathcal{K}_{cp^m}/\mathcal{K})} \chi(\sigma) \cdot y_{f,\chi,cp^m,\alpha} \right).
$$

(1.2)

In view of Lemma 1.2.1, the definition of $z_{f,\chi,n}$ does not depend on the choice of $m$. Moreover, letting $A_{f,\chi}$ be the Serre tensor $A_f \otimes \chi$, we see that $z_{f,\chi,n}$ defines a class $z_{f,\chi,n} \in \text{H}^1(\mathcal{K}_n, T_{\mathfrak{p}}A_{f,\chi})$. Let

$$
\Lambda_0 = \Theta[\Gamma], \quad \Lambda = \Lambda_0 \otimes \Theta \Phi
$$

(1.3)

be the anticyclotomic Iwasawa algebras. From their construction, the classes $z_{f,\chi,n}$ are contained in the pro-$\mathfrak{p}$ Selmer group $S_\mathfrak{p}(A_{f,\chi}/\mathcal{K}_n) \subset \text{H}^1(\mathcal{K}_n, T_{\mathfrak{p}}A_{f,\chi})$, and by Lemma 1.2.1 they are norm-compatible, hence defining a class $z_{f,\chi} = \{z_{f,\chi,n}\}_n$ in the compact $\Lambda_0$-adic Selmer group

$$
\mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty) := \lim_n S_\mathfrak{p}(A_{f,\chi}/\mathcal{K}_n).
$$

Denote by $\text{Sel}_{\mathfrak{p}^\infty}(A_{f,\chi}/\mathcal{K}_n) \subset \text{H}^1(\mathcal{K}_n, A_{f,\chi}[\mathfrak{p}^\infty])$ the $\mathfrak{p}^\infty$-Selmer groups of $A_{f,\chi}$, and set

$$
\mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty) := \text{Hom}_{\mathbb{Z}_p} \left( \lim_n \text{Sel}_{\mathfrak{p}^\infty}(A_{f,\chi}/\mathcal{K}_n), \Phi/\Theta \right).
$$

Set also

$$
\mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty) = \mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty) \otimes \Theta \Phi, \quad \mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty) = \mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty) \otimes \Theta \Phi,
$$

which are finitely generated $\Lambda$-modules.

The following conjecture in a natural extension of Perrin-Riou’s Heegner point main conjecture, [PR87, Conj. B].

**Conjecture 1.3.1.** The modules $\mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty)$ and $\mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty)$ have both $\Lambda$-rank one, and

$$
\text{char}_\Lambda(\mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty)_{\Lambda\text{-tors}}) = \text{char}_\Lambda(\mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty)/\Lambda \cdot z_{f,\chi})^2,
$$

where the subscript $\Lambda$-tors denotes the maximal $\Lambda$-torsion submodule.

In [BT20] a conjecture similar to Conjecture 1.3.1 is formulated in terms of a $\Lambda$-adic Heegner class deduced from work of Disegni [Dis17] (see [BT20, Conj. 2.2]). Similarly as in [BT20]\(^3\), our proof of Theorem A is based on a study of Conjecture 1.3.1. The novelty in our approach is in the proof of cases of this conjecture.

2. **Selmer groups**

In this section we introduce the different Selmer groups entering in our arguments. In particular, the decomposition in Proposition 2.3.1 will play a key role.

\(^3\)Also in other results on the $p$-converse theorem in rank 1 without a finiteness condition on the Tate–Shafarevich group that appeared after [Ski20]: [Wan14], [CW16], [CGLS20], etc.
2.1. Selmer groups of certain Rankin–Selberg convolutions. As in §1.2, let \( f \in S_2(\Gamma_1(N)) \) be a \( \Psi \)-ordinary newform with nebentypus \( \varepsilon_f \), and let \( K \) be an imaginary quadratic field satisfying (Heeg) and (spl).

Let \( c > 0 \) be a positive integer prime to \( N \). Similarly as in [BDP12, Def. 3.10], we say that a Hecke character \( \xi \) of infinity type \((2 + j, -j)\), with \( j \in \mathbb{Z} \), has finite type \((c, \mathfrak{N}, \varepsilon_f)\) if it satisfies:

(a) \( \xi|_{\mathbb{A}_\mathbb{K}} = \varepsilon_f^{-1} \cdot N^{-\frac{c}{2}} \), i.e., \( \omega_{c, \varepsilon_f} = 1 \);

(b) \( \Phi = c \cdot \Phi' \), where \( \Phi' \) is the unique divisor of \( \mathfrak{N} \) with norm equal to the conductor of \( \varepsilon_f \);

(c) the local sign \( \varepsilon_q(f, \chi) = +1 \) for all finite prime \( q \).

These conditions imply that the Rankin–Selberg \( L \)-function \( L(f, \chi, s) \) is self-dual, with \( s = 0 \) as the central critical point. The sign in the functional equation is \( +1 \) (resp. \( -1 \)) when \( j \geq 0 \) (resp. \( j < 0 \)). Denote by \( \Sigma_{cc}(c, \mathfrak{N}, \varepsilon_f) \) the set of such characters \( \xi \), and put

\[
\Sigma_{cc}^{(1)}(c, \mathfrak{N}, \varepsilon_f) = \{ \xi \in \Sigma_{cc}(c, \mathfrak{N}, \varepsilon_f) \mid j < 0 \}, \quad \Sigma_{cc}^{(2)}(c, \mathfrak{N}, \varepsilon_f) = \{ \xi \in \Sigma_{cc}(c, \mathfrak{N}, \varepsilon_f) \mid j \geq 0 \}.
\]

Let \( \chi \) be a finite order character of \( K \) such that \( \chi \mathfrak{N}^{-1} \in \Sigma_{cc}^{(1)}(c, \mathfrak{N}, \varepsilon_f) \). Denote by \( \rho_f : G_\mathbb{Q} \to \text{Aut}_\mathbb{Q}(V_f) \) the \( \Psi \)-adic Galois representation associated to \( f \), where \( V_f = \Phi \otimes \phi T \Phi A_f \), and consider the conjugate self-dual \( G_K \)-representation

\[
V_{f, \chi} := V_f|_{G_K} \otimes \chi. \quad (2.1)
\]

For any \( \Lambda_0 \)-module \( M \), let \( M^\vee = \text{Hom}_{cts}(M, \mathbb{Q}_p/\mathbb{Z}_p) \) be the Pontryagin dual. Fix a \( G_K \)-stable lattice \( T_{f, \chi} \subset V_{f, \chi} \), and define the \( G_K \)-modules

\[
W_{f, \chi} := T_{f, \chi} \otimes_\theta \Lambda_0', \quad T_{f, \chi} := W_{f, \chi}^\vee(1) \simeq T_{f, \chi} \otimes_\theta \Lambda_0,
\]

where the tensor products are endowed with the diagonal Galois actions, with \( G_K \) acting on \( \Lambda_0 \) (resp. \( \Lambda_0' \)) via the tautological character \( \Psi : G_K \to \text{Gal}(K_\infty/K) \hookrightarrow \Lambda_0^\vee \) (resp. \( \Psi^{-1} \)).

**Definition 2.1.1.** Fix a finite set \( \Sigma \) of places of \( K \) containing \( \infty \) and the primes dividing \( N_p \), and denote by \( K_\Sigma \subset \mathbb{Q}_\infty \) the maximal extension of \( K \) unramified outside \( \Sigma \). The **BDP Selmer group** of \( T_{f, \chi} \) over \( K_\infty \) is defined by

\[
\mathcal{X}_w(T_{f, \chi}/K_\infty) := \ker \left\{ H^1(K_\Sigma/K, W_{f, \chi}) \to H^1(K_w, W_{f, \chi}) \times \prod_{w \in \Sigma, w \nmid p} H^1(K_w, W_{f, \chi}) \right\}.
\]

We also set

\[
\mathcal{X}_w(f, \chi) = \mathcal{X}_w(T_{f, \chi}/K_\infty) \otimes_\theta \Phi,
\]

which is independent of the lattice \( T_{f, \chi} \). We define the compact version \( \mathcal{S}_w(f, \chi) \subset H^1(K_\Sigma/K, T_{f, \chi}) \) in the same manner, replacing \( W_{f, \chi} \) by \( T_{f, \chi} \).

The BDP Selmer groups of Definition 2.1.1 differs from the Selmer group \( \mathcal{X}(A_{f, \chi}/K_\infty) \) and \( \mathcal{S}(A_{f, \chi}/K_\infty) \) in §1.3 in their defining local conditions at the primes above \( p \). More precisely, by \( \Psi \)-ordinarity, for every \( w \nmid p \) there is a \( G_{K_w} \)-module exact sequence

\[
\mathcal{X}_w^+(T_{f, \chi}) \to \mathcal{X}_w(T_{f, \chi}) \to \mathcal{X}_w^-T_{f, \chi} \to 0 \quad (2.2)
\]

with \( \mathcal{X}_w^+T_{f, \chi} \) free of rank one over \( \mathcal{O}_w \), and the \( G_{K_w} \)-action on \( \mathcal{X}_w^-T_{f, \chi} \) being unramified. Then the Selmer group defined by

\[
\mathcal{X}_{\text{ord}}(T_{f, \chi}/K_\infty) := \ker \left\{ H^1(K_\Sigma/K, W_{f, \chi}) \to \prod_{w \mid p} \frac{H^1(K_w, W_{f, \chi})}{H^1(K_w, \mathcal{X}_w^-T_{f, \chi})} \times \prod_{w \in \Sigma, w \nmid p} H^1(K_w, W_{f, \chi}) \right\},
\]

where \( \mathcal{X}_wW_{f, \chi} = (\mathcal{X}_wT_{f, \chi}) \otimes_\theta \Lambda_0' \), satisfies

\[
\mathcal{X}_{\text{ord}}(f, \chi) := \mathcal{X}_{\text{ord}}(T_{f, \chi}/K_\infty) \otimes_\theta \Phi' \cong \mathcal{X}(A_{f, \chi}/K_\infty). \quad (2.3)
\]
Defining \( S_{\text{ord}}(f, \chi) \subset H^1(K^\Sigma/K, \mathcal{T}_{f,\chi}) \) in the same manner, we similarly have
\[
S_{\text{ord}}(f, \chi) \cong S(A_{f,\chi}/K_\infty) \tag{2.4}
\]
(see e.g. [CG96, §4]).

2.2. Selmer groups of characters. We keep the hypothesis that the imaginary quadratic field \( K \) satisfies (spl), and let \( \xi \) be a Hecke character of \( K \) of conductor \( f_\xi \). Let \( F \) be a number field containing the values of \( \xi \). Let \( \Phi \) be the completion of \( F \) at the prime \( \mathfrak{P} \) of \( F \) above \( p \) induced by \( \mathfrak{p} \), and let \( \mathcal{O} \) be the ring of integers of \( \Phi \). Denote by \( T_\xi \) the free \( \mathcal{O} \)-module of rank one on which \( G_K \) acts via \( \xi \), and consider the \( G_K \)-module
\[
\mathcal{W}_\xi := T_\xi \otimes _\mathcal{O} \Lambda^\vee_0,
\]
where as before the Galois action on \( \Lambda^\vee_0 \) is given by the character \( \Psi^{-1} \).

**Definition 2.2.1.** Let \( \Sigma \) be a finite set of places of \( K \) containing \( \infty \) and the primes dividing \( p \) or \( f_\psi \). The \( v \)-Selmer group of \( \psi \) over \( K_\infty \) is defined by
\[
\mathcal{X}_v(T_\xi/K_\infty) := \ker \left\{ H^1(K^\Sigma/K, \mathcal{W}_\xi) \to H^1(K_\Sigma, \mathcal{W}_\xi) \times \prod_{w \in \Sigma \cup \{\psi\}} H^1(K_w, \mathcal{W}_\xi) \right\}.
\]
We also set \( \mathcal{X}_v(\xi) = \mathcal{X}_v(T_\xi/K_\infty) \otimes _\mathcal{O} \Phi \).

**Remark 2.2.2.** Suppose \( \xi \) has infinity type \((-1, 0)\), and denote by \( \xi^* \) the composition of \( \xi \) with the non-trivial automorphism of \( K \), so \( \xi^* \) has infinity type \((0, -1)\). Then from e.g. [AH06, §1.1] we see that \( \mathcal{X}_v(\xi^*) \) corresponds to the Bloch–Kato Selmer group of \( \xi^* \) over the tower \( K_\infty/K \), whereas \( \mathcal{X}_v(\xi) \) corresponds to the Bloch–Kato Selmer group for \( \xi \) with the reversed local conditions at the primes above \( p \).

2.3. Decomposition. We now specialize the set-up in §2.1 to the case where \( f = \theta_\psi \) is the theta series of a Hecke character \( \psi \) of \( K \) of infinity type \((-1, 0)\). Then \( f \) has level \( N = D_K \cdot N_K(f_\psi) \) and nebentypus \( \varepsilon_f = \eta_K \cdot \omega_\psi \), where \( \eta_K \) is the quadratic character associated to \( K \).

One easily checks (see [BDP12, Lem. 3.14]) that if \( f_\psi \) is a cyclic ideal of norm \( N_K(f_\psi) \) prime to \( D_K \), then \( K \) satisfies the Heegner hypothesis (Heeg) relative to \( N \), and one may take
\[
\mathcal{M} = \delta_K \cdot f_\psi, \quad \text{where} \quad \delta_K := (\sqrt{-D_K}). \tag{2.5}
\]

In the following, we assume that \( f_\psi \) satisfies the above condition, and take \( \mathcal{M} \) as in (2.5). On the other hand, since we assume (spl), the CM form \( f \) is \( \mathcal{P} \)-ordinary. Finally, fix a positive integer \( c \) prime to \( np \), and let \( \chi \) be a finite order character such that \( \chi N_K^{-1} \in \Sigma^{(1)}_{cc}(c, \mathcal{M}, \varepsilon_f) \).

The following decomposition will play an important role later.

**Proposition 2.3.1.** Let \( \psi \) and \( \chi \) be as above. There is a \( \Lambda \)-module isomorphism
\[
\mathcal{X}_v(\theta_\psi, \chi) \cong \mathcal{X}_v(\psi \chi) \oplus \mathcal{X}_v(\psi^* \chi).
\]

**Proof.** Put \( f = \theta_\psi \), and note that there is a \( G_K \)-module decomposition
\[
V_{f,\chi} \cong V_{\psi \chi} \oplus V_{\psi^* \chi}. \tag{2.6}
\]
Since the module \( \mathcal{X}_v(f, \chi) \subset H^1(K^\Sigma/K, \mathcal{W}_{f,\chi}) \otimes _\mathcal{O} \Phi \) does not depend on the lattice \( T_{f,\chi} \subset V_{f,\chi} \) chosen to define \( \mathcal{W}_{f,\chi} \), by (2.6) we may assume that \( T_{f,\chi} \cong T_{\psi \chi} \oplus T_{\psi^* \chi} \) as \( G_K \)-modules, and so
\[
\mathcal{W}_{f,\chi} \cong \mathcal{W}_{\psi \chi} \oplus \mathcal{W}_{\psi^* \chi}
\]
as \( G_K \)-modules. The result thus follows immediately by comparing the defining local conditions of the three Selmer groups involved at all places. \( \square \)
3. \(p\)-adic \(L\)-functions

In this section we introduce the two \(p\)-adic \(L\)-functions needed for our arguments, and prove Proposition 3.3.1 relating the two.

3.1. The BDP \(p\)-adic \(L\)-function. As in §2.1, let \(f \in S_2(\Gamma_1(N))\) be an eigenform with \(p \nmid N\) and nebentypus \(\varepsilon_f\), let \(\mathcal{K}\) be an imaginary quadratic field satisfying \((\text{Heeg})\) and \((\text{spl})\), and fix an ideal \(\mathfrak{R} \subset \mathcal{O}_\mathcal{K}\) with cyclic quotient of order \(N\). Let \(c\) be a positive integer prime to \(Np\), and let \(\chi\) be a finite order Hecke character of \(\mathcal{K}\) such that \(\chi\mathcal{N}_\mathcal{K}^{-1} \in \Sigma_{cc}^{(1)}(c, \mathfrak{R}, \varepsilon_f)\).

Let \(F \subset \overline{\mathbb{Q}}\) be a number field containing \(\mathcal{K}\), the Fourier coefficients of \(f\), and the values of \(\chi\), and \(\Phi\) be the completion of \(F\) at the prime of \(F\) above \(p\) induced by \(\iota_p\), with ring of integers \(\mathcal{O} \subset \Phi\). Let \(\Lambda_0 = \mathcal{O}[\Gamma]\), and \(\Lambda = \Lambda_0 \otimes_{\mathcal{O}} \Phi\) be the anticyclotomic Iwasawa algebras as in (1.3), and set

\[
\Lambda_0^{ur} := \Lambda_0 \otimes_{\mathcal{O}} \mathbb{Z}_p^{ur} \simeq \mathcal{O}^{ur}[\Gamma], \quad \Lambda^{ur} := \Lambda_0^{ur} \otimes_{\mathcal{O}} \Phi,
\]

where \(\mathbb{Z}_p^{ur}\) is the completion of the ring of integers of the maximal unramified extension of \(\mathbb{Q}_p\).

The \(p\)-adic \(L\)-function in the next theorem was first constructed in [BDP13] as a continuous function on characters of \(\Gamma\). Its realization as a measure \(\Lambda_0^{ur}\) was first given in [CH18] following an approach introduced in [Bra11]. As it will suffice for our purposes, we describe below a \(\Phi^\times\)-multiple of that \(p\)-adic \(L\)-function.

As in [CH18, §2.3], define \(\vartheta \in \mathcal{K}\) by

\[
\vartheta := \frac{D' + \sqrt{-D_K}}{2}, \quad \text{where} \quad D' = \begin{cases} D_K & \text{if } 2 \nmid D_K, \\ D_K/2 & \text{else,} \end{cases}
\]

and let \(\Omega_p\) and \(\Omega_K\) be CM periods attached to \(\mathcal{K}\) as in [op.cit., §2.5].

**Theorem 3.1.1.** There exists an element \(\mathcal{L}_v(f, \chi) \in \Lambda^{ur}\) such that for every character \(\xi\) of \(\Gamma\) crystalline at both \(v\) and \(\bar{v}\) and corresponding to a Hecke character of \(\mathcal{K}\) of infinity type \((n, -n)\) with\(^4\) \(n \geq 1\), we have

\[
\mathcal{L}_v(f, \chi)^2(\xi) = \frac{\Omega_p^{ln}}{\Omega_K^{ln}} \frac{\Gamma(n)\Gamma(n + 1)\xi(\mathfrak{N}_\mathcal{K}^{-1})}{(4\pi)^{2n+1}(\text{Im } \vartheta)^{2n}} \cdot (1 - a_p(f)(\chi \xi)(\overline{\vartheta})p^{-1} + \varepsilon_f(p)(\chi \xi)^2(\overline{\vartheta})p^{-1})^2 \cdot L(f, \chi \xi, 1).
\]

**Proof.** Let \(\eta\) be an anticyclotomic Hecke character of \(\mathcal{K}\) of infinity type \((1, -1)\) and conductor dividing \(\mathcal{C}\mathcal{O}_\mathcal{K}\), and define \(\mathcal{L}_{v, \eta}(f, \chi) \in \Lambda_0^{ur}\) by

\[
\mathcal{L}_{v, \eta}(f, \chi)(\phi) = \sum_{[a] \in \text{Pic}(\mathcal{O}_v)} (\eta \chi \mathcal{N}_\mathcal{K}^{-1})(a) \int_{\mathbb{Z}_p^\times} \eta_v(\phi[a]) \, d\mu_{f^\#},
\]

for all continuous characters \(\phi : \Gamma \to \overline{\mathbb{Q}}_p^\times\), where:

- \(f^\# = \sum_{\mathcal{O}_v} a_n(f)q^n\) is the \(p\)-depletion of \(f\),
- \(\mu_{f^\#}\) is the measure on \(\mathbb{Z}_p^\times\) corresponding (under the Amice transform) to the power series

\[
f^\#(t_a \mathcal{N}_\mathcal{K}(a)^{\sqrt{-D_K}^{-1}}) \in \mathcal{O}^{ur}[t_a - 1]
\]

with \(t_a\) the Serre–Tate coordinate of the reduction of the point \(x_a\) on the Igusa tower of tame level \(N\) constructed in [CH18, (2.5)],

- \(\eta_v(x) := \eta(\text{rec}_v(x))\) with \(\text{rec}_v : \overline{\mathbb{Q}}_p^\times \to \mathcal{C}\mathcal{O}_v^\times \to \Gamma\) the local reciprocity map at \(v\),

- \(\phi[a] : \mathbb{Z}_p^\times \to \overline{\mathbb{Q}}_p^\times\) is defined by \((\phi[a])(x) = \phi(\text{rec}_v(x)\sigma_a^{-1})\) with \(\sigma_a\) the Artin symbol of \(a\).

\(^4\)Therefore, \(\chi \xi \mathcal{N}_\mathcal{K}^{-1} \in \Sigma_{cc}^{(2)}(c, \mathfrak{R}, \varepsilon_f)\).
The same calculation as in [CH18, Prop. 3.8] then shows that the element $L_v(f, \chi) \in A^\ur$ defined by
\[ L_v(f, \chi)(\xi) = \mathcal{L}_v(f, \chi)(\eta^{-1}\xi) \]
has, in view of the explicit Waldspurger formula in [Hsi14b, Thm. 3.14], the stated interpolation property up to a fixed element in $\Phi^\times$. The result follows. \[\square\]

Remark 3.1.2. We our later use, we note that the complex period $\Omega_K \in \mathbb{C}^\times$ in Theorem 6.0.1 (which also agrees with that in [BDP13, (5.1.16)]) is different from the complex period $\Omega_\infty \in \mathbb{C}^\times$ defined in [dS87, p. 66] and [HT93, (4.4b)]. In fact, one has $\Omega_K = 2\pi i \cdot \Omega_\infty$.

In terms of $\Omega_\infty$, the interpolation formula in Theorem 6.0.1 reads
\[ L_v(f, \chi)^2(\xi) = \frac{\Omega_\infty^{4n}}{\Omega_v^{4n}} \cdot \frac{\Gamma(n)\Gamma(n+1)\xi(\mathfrak{N}^{-1})}{4\pi^{1-2n}(\Im \vartheta)^{2n}} \cdot (1 - a_p(f)\chi(\pi)p^{-1} + \varepsilon_f(p)\chi(\pi)^2p^{-1})^2 \cdot L(f, \chi, 1). \]

Specialized to the range of critical values for the representation $V_{f, \chi}$, the Iwasawa–Greenberg main conjecture [Gre94] predicts the following.

Conjecture 3.1.3. The module $\mathcal{X}_v(f, \chi)$ is $\Lambda$-torsion, and
\[ \text{char}_\Lambda(\mathcal{X}_v(f, \chi)) = (\mathcal{L}_v(f, \chi)^2). \]

In Theorem 4.0.2, we will explain the relation between Conjectures 1.3.1 and 3.1.3.

3.2. Katz $p$-adic $L$-functions. Let $K$ be an imaginary quadratic field satisfying (spl). Let $c \subset O_K$ be an ideal prime to $p$, and let $K(c/p^\infty)$ be the ray class field of $K$ of conductor $c/p^\infty$.

The $p$-adic $L$-function in the next theorem follows from the work of Katz [Kat78], as extended by Hida–Tilouine [HT93] (see also [dS87]). Here we shall use the construction in [Hsi14a], and similarly as in Theorem 3.1.1, it will suffice for our purposes to describe a fixed $\Phi^\times$-multiple of the integral measure constructed in op.cit.

For any Hecke character $\xi$ of $K$, we denote by $L_c(\xi, s)$ the Hecke $L$-function $L(\xi, s)$ with the Euler factors at the primes $\mathfrak{p}|c$ removed.

Theorem 3.2.1. Let $\phi$ be a character of $\text{Gal}(K(c/p^\infty)/K)$ corresponding to a Hecke character of $K$ of infinity type $(-1 - j, j)$, with $j \in \mathbb{Z}_{\geq 0}$ and conductor prime to $p$. There exists an element $\mathcal{L}_v(\phi) \in A^\ur$ such that for every character $\xi$ of $\Gamma$ crystalline at both $v$ and $\bar{v}$ and corresponding to a Hecke character of $K$ of infinity type $(-n, n)$ with $n + j \in \mathbb{Z}_{\geq 0}$, we have
\[ \mathcal{L}_v(\phi)(\xi) = \frac{\Omega_p^{2n+2j+1}}{\Omega_\infty^{2n+2j+1}} \cdot \Gamma(n + j + 1) \cdot \frac{(2\pi)^{n+j}}{(\Im \vartheta)^{n+j}} \cdot (1 - \phi^{-1}\xi^{-1}(\pi))^{2} \cdot L_c(\phi^{-1}\xi^{-1}, 0), \]
where $\Omega_p, \Omega_\infty = (2\pi i)^{-1}\Omega_K$, and $\vartheta$ are as in Theorem 6.0.1. Similarly, if $\phi$ as above has infinity type $(j, -1 - j)$, with $j \in \mathbb{Z}_{\geq 0}$, there exists an element $\mathcal{L}_v(\phi) \in A^\ur$ such that for every character $\xi$ of $\Gamma$ crystalline at both $v$ and $\bar{v}$ and corresponding to a Hecke character of $K$ of infinity type $(n, -n)$ with $n + j \in \mathbb{Z}_{\geq 0}$, we have
\[ \mathcal{L}_v(\phi)(\xi) = \frac{\Omega_p^{2n+2j+1}}{\Omega_\infty^{2n+2j+1}} \cdot \Gamma(n + j + 1) \cdot \frac{(2\pi)^{n+j}}{(\Im \vartheta)^{n+j}} \cdot (1 - \phi^{-1}\xi^{-1}(\pi))^{2} \cdot L_c(\phi^{-1}\xi^{-1}, 0). \]

Proof. Let $\mathcal{L}_v^\prime$ be the integral $p$-adic measure on $\text{Gal}(K(c/p^\infty)/K)$ constructed in [Hsi14a, §4.8], associated to the $p$-adic CM type corresponding to our fixed embedding $i_\infty, i_p$. Setting
\[ \mathcal{L}_v(\phi)(\xi) := \mathcal{L}_v(\phi^{-1} \cdot \pi^* \xi^{-1}) \]
Proposition 3.3.1. Suppose that on the analytic side of the Selmer group decomposition in Proposition 2.3.1 (see e.g. [BDP12, Rem. 3.7]).

where self-dual e.g. [BCG ∗ , Lem. 3.3.2(a)]. Similarly, we find $\mathcal{L}_\mathcal{V}(\psi^*\chi) = \mathcal{L}_\mathcal{V}(\psi\chi)(\xi)$, so the result follows from (3.2).

3.3. Factorization. As in §2.3, we now specialize to the case where $f = \theta_\psi$ for a Hecke character $\psi$ of $\mathcal{K}$ of infinity type $(-1,0)$ and conductor $f_\psi$ with cyclic quotient of norm prime to $D_\mathcal{K}$, so that $\mathcal{K}$ satisfies hypothesis (Heeg) relative to $N = D_\mathcal{K} \cdot N_\mathcal{K}(f_\psi)$.

Fix an integer $c > 0$ prime to $Np$, and let $\chi$ be a finite order Hecke character of $\mathcal{K}$ such that $\chi N_\mathcal{K}^{-1} \in \Sigma^{(1)}_{\mathcal{K}}(c, \mathfrak{N}, \varepsilon_f)$. Then we have a $G_\mathcal{K}$-module decomposition

$$V_{f,\chi} \cong V_{\psi\chi} \oplus V_{\psi^*\chi},$$

(3.1)

where $V_{f,\chi}$ is as in (2.1). Note that each of the characters appearing in the right-hand side are self-dual, in the sense that

$$\psi\chi \psi^*\chi^* = N_\mathcal{K}$$

(see e.g. [BDP12, Rem. 3.7]).

The following result is a manifestation of the Artin formalism arising from the decomposition (3.1). A similar result in shown in [BDP12, Thm. 3.17]. As we shall see in §6, this is a counterpart on the analytic side of the Selmer group decomposition in Proposition 2.3.1.

**Proposition 3.3.1.** Suppose that $f = \theta_\psi$ and $\chi$ are as above. Then

$$\mathcal{L}_\mathcal{V}(f, \chi)^2 = u \cdot \mathcal{L}_\mathcal{V}(\psi\chi) \cdot \mathcal{L}_\mathcal{V}(\psi^*\chi),$$

where $u$ is a unit in $(\Lambda^{ur})^\times$.

**Proof.** This will follow by comparing the values interpolated by each side of the desired equality, using that an element in $\Lambda^{ur}$ is uniquely determined by its values at infinitely many characters. Denote by $\iota : \Lambda^{ur} \to \Lambda^{ur}$ the involution given by $\gamma \mapsto \gamma^{-1}$ for $\gamma \in \Gamma$. We first claim that

$$\mathcal{L}_\mathcal{V}(f, \chi)^2 = u \cdot \mathcal{L}_\mathcal{V}(\psi^*\chi^{-1}N_\mathcal{K}) \cdot \mathcal{L}_\mathcal{V}((\psi^*)^{-1}\chi^{-1}N_\mathcal{K})^\iota$$

(3.2)

for some unit $u \in (\Lambda^{ur})^\times$. Indeed, let $\xi$ be a character of $\Gamma$ as in the statement of Theorem 3.1.1, of infinity type $(n, -n)$ with $n \geq 1$. The decomposition (2.6) yields

$$L(f, \chi, 1) = L(f, \chi N_\mathcal{K}^{-1}, 0) = L(\psi\chi N_\mathcal{K}^{-1}, 0) \cdot L(\psi^*\chi N_\mathcal{K}^{-1}, 0).$$

(3.3)

The factors in the right-hand side of (3.3) are interpolated by the values at $\xi^{-1}$ of $\mathcal{L}_\mathcal{V}(\psi^*\chi^{-1}N_\mathcal{K})$ and $\mathcal{L}_\mathcal{V}(\psi^*\chi^{-1}N_\mathcal{K})^\iota$, respectively. Noting that

$$(1 - \psi\chi(\pi)p^{-1})(1 - \psi^*\chi(\pi)p^{-1}) = (1 - a_p(f)\chi(\pi)p^{-1} + \varepsilon_f(p)\chi(\pi)^2p^{-1}),$$

from Theorem 3.2.1 with $j = -1$ and $j = 0$ we find

$$\mathcal{L}_\mathcal{V}(\psi^*\chi^{-1}N_\mathcal{K})(\xi^{-1}) \cdot \mathcal{L}_\mathcal{V}(\psi^*\chi^{-1}N_\mathcal{K})(\xi^{-1}) = \frac{\Omega_{\mathfrak{p}}^{2n-1}}{\Omega_{\mathfrak{p}}^{2n}} \cdot \frac{\Omega_f^{2n}}{\Omega_{\mathfrak{p}}^{2n}} \cdot \Gamma(n) \Gamma(n + 1) \cdot (2\pi)^{n-1} \cdot (2\pi)^n

\times (1 - a_p(f)\chi(\pi)p^{-1} + \varepsilon_f(p)\chi(\pi)^2p^{-1}) \cdot L(f, \chi, 1).$$

The proof of (3.2) thus follows from Theorem 3.1.1 and Remark 3.1.2.

Now, noting that the characters $\psi\chi$ and $\psi^*\chi$ are both self-dual and $\xi$ is anticyclotomic, we find

$$\mathcal{L}_\mathcal{V}(\psi^*\chi^{-1}N_\mathcal{K})(\xi^{-1}) = \mathcal{L}_\mathcal{V}(\psi^*\chi)(\xi^*) = \mathcal{L}_\mathcal{V}(\psi\chi)(\xi),$$

where the last equality follows from another direct comparison of interpolation properties (see e.g. [BCG ∗ , Lem. 3.3.2(a)]). Similarly, we find $\mathcal{L}_\mathcal{V}(\psi^*\chi^{-1}N_\mathcal{K})(\xi^{-1}) = \mathcal{L}_\mathcal{V}(\psi\chi)(\xi)$, so the result follows from (3.2).
4. Explicit reciprocity law

In this section we explain a variant of the explicit reciprocity law [CH18] relating the $\Lambda$-adic Heegner class $\mathbf{z}_{f,\chi}$ to the $p$-adic $L$-function $L_v(f, \chi)$ via a Perrin-Riou big logarithm map, and record a key consequence.

Let $(f, \chi)$ be as in §2.1. For every $w | p$, the natural map $H^1(K_w, \mathcal{F}_w^+ T_{f,\chi}) \to H^1(K_w, T_{f,\chi})$ induced by (2.2) is injective, since its kernel is $H^0(K_w, \mathcal{F}_w T_{f,\chi}) = 0$. Therefore, in view of (2.4) the image of $\mathbf{z}_{f,\chi}$ under the restriction map

$$\text{loc}_w : H^1(K, T_{f,\chi}) \to H^1(K_w, T_{f,\chi})$$

is naturally contained in $H^1(K_w, \mathcal{F}_w^+ T_{f,\chi})$. Let $\Phi^{ur}$ the compositum of $\Phi$ and $\mathbb{Q}_p^{ur}$.

Recall that $K$ is assumed to satisfy (spl), and $v$ denotes the prime of $K$ above $p$ induced by our fixed embedding $\iota_p$.

**Theorem 4.0.1.** There is a $\Lambda^{ur}$-linear isomorphism $\Log_v : H^1(K_v, \mathcal{F}_v^+ T_{f,\chi}) \otimes \Lambda^{ur} \to \Lambda^{ur}$ such that

$$\Log_v(\text{loc}_v(\mathbf{z}_{f,\chi})) = c \cdot L_v(f, \chi)$$

for some $c \in (\Phi^{ur})^\times$.

**Proof.** The existence of the map $\Log_v$ (with coefficients in $\Lambda_0^{ur}$, rather than $\Lambda^{ur}$) follows from the two-variable extension by Loeffler–Zerbes [LZ14] of Perrin-Riou’s big logarithm map [PR94], and the proof of the explicit reciprocity law (integrally) is given in [CH18, §5.3]. That the $\Lambda^{ur}$-linear map $\Log_v$ is injective follows from [LZ14, Prop. 4.11], and so it becomes an isomorphism after extending scalars to $\Lambda^{ur} = \Lambda_0^{ur} \otimes_{\mathcal{O}} \Phi$.

Similarly as observed in [Cas13] and [Wan14], the equivalence between Conjectures 1.3.1 and 3.1.3 can be deduced from Theorem 4.0.1 using Poitou–Tate global duality.

**Theorem 4.0.2.** Assume that the class $\mathbf{z}_{f,\chi}$ is not $\Lambda$-torsion. Then Conjectures 1.3.1 and 3.1.3 are equivalent.

**Proof.** This can be shown in the same way as [CW16, Thm. 5.16], but since here we are working in a different setting, we provide the details. We explain the implication from Conjecture 3.1.3 to Conjecture 1.3.1 (the only implication we will need later), and note that the converse follows from the same ideas.

Following [Cas17, §2.1], below we denote by $\mathcal{S}_{str,\text{rel}}(f, \chi)$ (resp. $\mathcal{S}_{\text{ord},\text{rel}}(f, \chi)$, etc.) the Selmer group defined as in §2.1 but with the strict at $v$ and relaxed at $\varpi$ (resp. ordinary at $v$ and relaxed at $\varpi$, etc.) local conditions. We also use implicitly use the isomorphisms (2.3) and (2.4).

Now assume Conjecture 3.1.3, so in particular $\mathcal{X}_v(f, \chi)$ is $\Lambda$-torsion. Then $\mathcal{S}_{str,\text{rel}}(f, \chi)$ is also $\Lambda$-torsion, and global duality yields the following exact sequence

$$0 \to \mathcal{S}_{str,\text{rel}}(f, \chi) \to \mathcal{S}_{\text{ord},\text{rel}}(f, \chi) \xrightarrow{\text{loc}_v} H^1(K_v, \mathcal{F}_v^+ T_{f,\chi}) \to \mathcal{X}_v(f, \chi) \to \mathcal{X}_{str,\text{ord}}(f, \chi) \to 0. \quad (4.1)$$

Since $H^1(K_v, \mathcal{F}_v^+ T_{f,\chi})$ has $\Lambda$-rank one, the assumption that $\mathbf{z}_{f,\chi}$ is non-torsion together with Theorem 4.0.1 implies that $\mathcal{S}_{\text{ord},\text{rel}}(f, \chi)$ has $\Lambda$-rank one. Since $\mathbf{z}_{f,\chi} \in \mathcal{S}_{\text{ord}}(f, \chi) \subset \mathcal{S}_{\text{ord},\text{rel}}(f, \chi)$, it follows that $\mathcal{S}_{\text{ord}}(f, \chi)$ also has $\Lambda$-rank one, and by [Cas17, Lem. 2.3(1)] so does $\mathcal{X}_{\text{ord}}(f, \chi)$.

Hence the quotient $\mathcal{S}_{\text{ord},\text{rel}}(f, \chi)/\mathcal{S}_{\text{ord}}(f, \chi)$ is $\Lambda$-torsion, and since it injects in $H^1(K_v, \mathcal{F}_v^+ T_{f,\chi})$ which is $\Lambda$-torsion-free, this shows the equality $\mathcal{S}_{\text{ord}}(f, \chi) = \mathcal{S}_{\text{ord},\text{rel}}(f, \chi)$. Thus we see that (4.1) reduces to the exact sequence

$$0 \to \mathcal{S}_{str,\text{ord}}(f, \chi) \to \mathcal{S}_{\text{ord}}(f, \chi) \xrightarrow{\text{loc}_v} H^1(K_v, \mathcal{F}_v^+ T_{f,\chi}) \to \mathcal{X}_{\text{ord},\text{rel}}(f, \chi) \to \mathcal{X}_{\text{ord}}(f, \chi) \to 0. \quad (4.2)$$
Since $S_{\text{str,ord}}(f, \chi)$ is $\Lambda$-torsion and $H^1(K, T_{f, \chi})$ trivial $\Lambda$-torsion-free, $S_{\text{str,ord}}(f, \chi)$ vanishes, and therefore (4.2) yields

$$0 \to \frac{S_{\text{ord}}(f, \chi)}{\Lambda \cdot z_{f, \chi}} \xrightarrow{\text{loc}_v} \frac{H^1(K_v, T_{f, \chi})}{\Lambda \cdot \text{loc}_v(z_{f, \chi})} \to \text{coker}(\text{loc}_v) \to 0.$$ 

In view of Theorem 4.0.1, it follows that

$$\text{char}_\Lambda \left( \frac{S_{\text{ord}}(f, \chi)}{\Lambda \cdot z_{f, \chi}} \right) \cdot \text{char}_\Lambda \left( \text{coker}(\text{loc}_v) \right) \Lambda_{\text{ur}} \equiv (\mathcal{L}_v(f, \chi)). \quad (4.3)$$

Next, from (4.1) and (4.2) we can extract the short exact sequences

$$0 \to \text{coker}(\text{loc}_v) \to X_v(f, \chi) \to X_{\text{str,ord}}(f, \chi) \to 0,$$

$$0 \to \text{coker}(\text{loc}_v) \to X_{\text{ord,rel}}(f, \chi) \to X_{\text{ord}}(f, \chi) \to 0,$$

from which we readily obtain (taking $\Lambda$-torsion in the first exact sequence and using a straightforward variant of [Cas17, Lem. 2.3] or [BL18, Prop. 3.14]) the relations

$$\text{char}_\Lambda(X_v(f, \chi)) = \text{char}_\Lambda(X_{\text{str,ord}}(f, \chi)) \cdot \text{char}_\Lambda(\text{coker}(\text{loc}_v))$$

$$= \text{char}_\Lambda(X_{\text{ord,rel}}(f, \chi)_{\Lambda\text{-tors}}) \cdot \text{char}_\Lambda(\text{coker}(\text{loc}_v))$$

$$= \text{char}_\Lambda(X_{\text{ord}}(f, \chi)_{\Lambda\text{-tors}}) \cdot \text{char}_\Lambda(\text{coker}(\text{loc}_v))^2.$$

Combined with (4.3), we thus obtain

$$\text{char}_\Lambda(X_v(f, \chi)) \cdot \text{char}_\Lambda \left( \frac{S_{\text{ord}}(f, \chi)}{\Lambda \cdot z_{f, \chi}} \right)^2 \Lambda_{\text{ur}} \equiv \text{char}_\Lambda(X_{\text{ord}}(f, \chi)_{\Lambda\text{-tors}}) \cdot (\mathcal{L}_v(f, \chi))^2.$$

The result follows. \hfill \square

5. Twisted anticyclotomic main conjectures for $K$

Let $K$ be an imaginary quadratic field satisfying (spl). The Iwasawa main conjecture for $K$ was proved by Rubin [Rub91] under some restrictions on $p$ (including $p \nmid \mathcal{O}_K^\times$) that were removed in subsequent work by Johnson-Leung–Kings [JKL11] and Oukhaba–Vigué [OV16]. In this section we record a consequence of these results for the anticyclotomic $Z_p$-extension.

Note that if $\xi$ is a self-dual Hecke character, i.e.,

$$\xi^{*} = N_K$$

(so $\xi$ is necessarily of infinity type $(-1-j, j)$, for some $j \in \mathbb{Z}$), then the Hecke $L$-function $L(\xi^{-1}, s)$ is self-dual, with a functional equation relating its values at $s$ and $-s$. In the following, by the sign of $\xi$ we refer to the sign appearing in this functional equation.

**Theorem 5.0.1.** Let $\psi$ be a Hecke character of $K$ of infinity type $(-1, 0)$, and let $\chi$ be a finite order character of such that the product $\psi \chi$ is self-dual. Assume that $\psi^{*} \chi$ has sign $+1$. Then the modules $X_v(\psi \chi)$ and $X_v(\psi^{*} \chi)$ are both $\Lambda$-torsion, and we have

$$\text{char}_\Lambda(X_v(\psi \chi)) = (\mathcal{L}_T(\psi \chi)), \quad \text{char}_\Lambda(X_v(\psi^{*} \chi)) = (\mathcal{L}_T(\psi^{*} \chi)).$$

**Proof.** Consider first the result for $\psi \chi$. The Iwasawa module $X_v(\psi \chi)$ recovers the Bloch–Kato Selmer group for $\psi \chi$ over the anticyclotomic $Z_p$-extension (see Remark 2.2.2), and so the result follows from [AH06, Thm. 2.4.17], as extended in [Arn07, Thm. 3.9]. (In these references, the hypothesis $p > 3$ arises from their appearance in [Rub91], but as already mentioned this restriction can be removed thanks to [JKL11, OV16]. Recall that $p$ is inverted in our $\Lambda$.)

On the other hand, similarly as in [Agb07, Cor. 3.3] (an adaptation of the argument in [Coa83, Thm. 12]), we see that $X_v(\psi \chi)$ is isomorphic as a $\Lambda$-module to the twist of $X_v(\psi^{*} \chi)$ by $\psi(\psi^{*})^{-1}$, and therefore is also $\Lambda$-torsion if $\psi^{*} \chi$ has sign $+1$. Since by definition $\mathcal{L}_T(\psi \chi)$ is similarly the
twist of $\mathcal{L}_\tau(\psi^*\chi)$ by the character $\psi(\psi^*)^{-1}$, the first equality of characteristic ideals follows from the second.

6. The main results

Recall that $K$ is an imaginary quadratic field of discriminant $-D_K < 0$ satisfying (spl), with $v$ the prime of $K$ above $p$ induced by our fixed embedding $\iota_p$.

Theorem 6.0.1. Let $\psi$ be a Hecke character of $K$ of infinity type $(-1,0)$ and conductor $f_\psi$ with cyclic quotient of norm prime to $D_K$, and set

$$f = \theta_\psi, \quad N = D_K \cdot N_K(f_\psi), \quad \mathfrak{N} = \mathfrak{d}_K \cdot f_\psi.$$ 

Let $c$ be a positive integer prime to $N_p$, and let $\chi$ be a finite order character such that $\chi N^{-1}_K \in \Sigma^{(1)}(c, \mathfrak{N}, \varepsilon_f)$. Assume that $\psi\chi$ has sign $-1$. Then:

(i) The class $z_{f,\chi}$ is not $\Lambda$-torsion.

(ii) The the module $X_v(f,\chi)$ is $\Lambda$-torsion, and

$$\text{char}_\Lambda(X_v(f,\chi))\Lambda^\text{ur} = (\mathcal{L}_v(f,\chi)^2).$$

In other words, Conjecture 3.1.3 holds.

Proof. We begin by showing that the class $z_{f,\chi}$ is not $\Lambda$-torsion. As noted in §3.3, our assumption on $f_\psi$ implies that $K$ satisfies hypothesis (Heeg) relative to the level of $f$. In particular, the Rankin–Selberg $L$-function $L(f,\chi,s)$ has a functional equation relating its values at $s$ and $2-s$ with sign $-1$. The $G_K$-module decomposition (2.6) yields

$$L(f,\chi,s) = L(\psi,\chi,s) \cdot L(\psi^*\chi,s),$$

and each of the factors in the right-hand side has a functional equation relating its values at $s$ and $2-s$. Since we assume that $\psi\chi$ has sign $-1$, the self-dual character $\psi^*\chi$ has sign $+1$. By Rohrlich’s theorem [Roh84], for all but finitely characters $\xi$ of $\Gamma$, we have

$$L(\psi^*\chi\xi,1) \neq 0.$$ 

Hence in view of the interpolation property in Theorem 3.2.1, the $p$-adic $L$-function $\mathcal{L}_\tau(\psi^*\chi)$ is nonzero, and therefore so is its twist $\mathcal{L}_\tau(\psi_\chi)$. By the factorization in Proposition 3.3.1, we conclude that $\mathcal{L}_v(f,\chi)$ is also nonzero, and the claim that $z_{f,\chi}$ is not $\Lambda$-torsion now follows from the explicit reciprocity law of Theorem 4.0.1.

For part (ii), by Theorem 5.0.1 the modules $X_v(\psi\chi)$ and $X_v(\psi^*\chi)$ are both $\Lambda$-torsion, with characteristic ideals generated by $\mathcal{L}_\tau(\psi\chi)$ and $\mathcal{L}_\tau(\psi^*\chi)$ over $\Lambda^\text{ur}$, respectively. Thus from Propositions 2.3.1 and 3.3.1 we obtain that $X_v(f,\chi)$ is $\Lambda$-torsion, with characteristic ideal generated by $\mathcal{L}_v(f,\chi)^2$ over $\Lambda^\text{ur}$. 

Corollary 6.0.2. Let $f = \theta_\psi$ and $\chi$ be an in Theorem 6.0.1, and assume that $\psi\chi$ has sign $-1$. Then the modules $S(A_{f,\chi}/K_\infty)$ and $X(A_{f,\chi}/K_\infty)$ have both $\Lambda$-rank one, and

$$\text{char}_\Lambda(X(A_{f,\chi}/K_\infty))_{\text{\Lambda-tors}} = \text{char}_\Lambda(S(A_{f,\chi}/K_\infty)/\Lambda \cdot z_{f,\chi})^2.$$ 

In other words, Conjecture 1.3.1 holds.

Proof. This is the combination of Theorem 4.0.2 and Theorem 6.0.1.
7. The $p$-converse

In this section we deduce from our main results the proof of Theorem A in the Introduction. Let $\lambda$ be a self-dual Hecke character of infinity type $(-1, 0)$ and conductor $f_\lambda$, and suppose that:

(a) $\lambda$ has sign $-1$;
(b) $\lambda$ has central character $\omega_\lambda = \eta_K$;
(c) $\delta_K || f_\lambda$.

Note that $f_\lambda$ is divisible by $\delta_K = (\sqrt{-D_K})$ by condition (b). Since $\lambda$ is self-dual, $f_\lambda$ is invariant under complex conjugation, so by condition (c) we can write $f_\lambda = (c)\delta_K$ for a unique $c > 0$.

We shall apply Corollary 6.0.2 for a pair $(\psi, \chi)$ which is good for $\lambda$ in the sense of [BDP12, Def. 3.19]:

(G1) $\psi$ has infinity type $(-1, 0)$ and conductor $f_\psi$ with cyclic quotient of norm prime to $pD_K$;

(G2) $\chi$ is a finite order character such that $\chi N_K^{-1} \in \Sigma_{cc}(1, c, \epsilon_f)$, where $f = \theta_\psi$ and $\mathfrak{N} = f_\psi \mathfrak{d}_K$;

(G3) $\psi \chi = \lambda$;

(G4) $L(\psi, 1) \neq 0$.

The existence of good pairs for $\lambda$ is shown in [BDP12, Lem. 3.29] building on the non-vanishing results of Greenberg [Gre85] and Rohrlich [Roh84].

Fix a good pair $(\psi, \chi)$ for $\lambda$, and let $F, \mathfrak{Q}$ be a number field of containing the values of $\psi$ and $\chi$. Let $\mathfrak{p}$ be the prime of $F$ above $p$ induced by our fixed embedding $\iota_p$, let $\mathfrak{K}$ be the completion of $F$ at $\mathfrak{p}$, and let $\mathfrak{O}$ be the ring of integers of $\mathfrak{K}$. Let $W_{\psi, \chi}$ be the ring of integers of $\Phi$ equipped with the $G_\mathfrak{K}$-action via $\psi$, and define $W_{\psi^* \chi}$ similarly.

Let $\Sigma$ a finite set of places of $\mathfrak{K}$ containing $\infty, p$, and the primes of $\mathfrak{K}$ dividing the conductor of $\lambda$. Denote by $H^1_2(\mathfrak{K}, W_{\psi, \chi})$ the Bloch–Kato Selmer group for $\psi$:

$$H^1_2(\mathfrak{K}, W_{\psi, \chi}) = \ker \left( H^1(\mathfrak{K}^{ur}/\mathfrak{K}, W_{\psi, \chi}) \right) \rightarrow H^1(\mathfrak{K}_{ur}, W_{\psi, \chi}) \times H^1(\mathfrak{K}_\mathfrak{p}, W_{\psi, \chi}) \times \prod_{w \in \Sigma, \mathfrak{w} \mid \mathfrak{p}} H^1(\mathfrak{K}_{ur}, W_{\psi, \chi}),$$

where $H^1(\mathfrak{K}_\mathfrak{p}, W_{\psi, \chi})$ is the maximal divisible submodule and $\mathfrak{K}_{ur}$ denote the maximal unramified extension of $\mathfrak{K}_\mathfrak{w}$. Similarly, let

$$H^1_2(\mathfrak{K}, W_{\psi^* \chi}) = \ker \left( H^1(\mathfrak{K}^{ur}/\mathfrak{K}, W_{\psi^* \chi}) \rightarrow H^1(\mathfrak{K}_{ur}, W_{\psi^* \chi}) \times H^1(\mathfrak{K}_\mathfrak{p}, W_{\psi^* \chi}) \times \prod_{w \in \Sigma, \mathfrak{w} \mid \mathfrak{p}} H^1(\mathfrak{K}_{ur}, W_{\psi^* \chi}) \right)$$

be the Bloch–Kato Selmer group for $\psi^* \chi$ (see also [AH06, §1.1]). Finally, let $V_{f, \chi}$ be as in (2.1).

**Lemma 7.0.1.** In the above setting, we have

$$\text{corank}_{\mathfrak{O}} \text{Sel}_{p^\infty}(A_{f, \chi}/\mathfrak{K}) = \text{corank}_{\mathfrak{O}} H^1_2(\mathfrak{K}, W_{\psi, \chi}) + \text{corank}_{\mathfrak{O}} H^1_2(\mathfrak{K}, W_{\psi^* \chi}).$$

**Proof.** As is well-known (see e.g. [BK90]), $\text{Sel}_{p^\infty}(A_{f, \chi}/\mathfrak{K})$ agrees with the Bloch–Kato Selmer group

$$H^1_2(\mathfrak{K}, W_{f, \chi}) \subset H^1(\mathfrak{K}, W_{f, \chi}),$$

where $W_{f, \chi} := V_{f, \chi}/T_{f, \chi}$ for any $G_{\mathfrak{K}}$-stable $\mathfrak{O}$-lattice $T_{f, \chi} \subset V_{f, \chi}$. In turn (see [Gre99, Prop. 2.2]), the local conditions defining $H^1_2(\mathfrak{K}, W_{f, \chi})$ at the primes $w|p$ can be described in terms of the filtration (2.2). Since different lattices $T_{f, \chi}$ give rise to Selmer groups $H^1_2(\mathfrak{K}, W_{f, \chi})$ with the same $\mathfrak{O}$-corank, taking $T_{f, \chi}$ so that $W_{f, \chi} \simeq W_{\psi, \chi} \oplus W_{\psi^* \chi}$, similarly as in the proof of Proposition 2.3.1 the result follows by comparing the local conditions defining the Bloch–Kato Selmer groups for $W_{f, \chi}$, $W_{\psi, \chi}$ and $W_{\psi^* \chi}$. \qed

The following recovers Theorem A in the introduction as a special case.
Theorem 7.0.2. Let \( \lambda \) be a self-dual Hecke character of \( \mathcal{K} \) of infinity type \((-1, 0)\) with central character \( \omega_{\lambda} = \eta_{\mathcal{K}} \) and whose conductor \( f_{\lambda} \) satisfies \( \mathfrak{d}_{\mathcal{K}} | f_{\lambda} \). Then
\[
\text{corank}_\mathcal{O} \text{Sel}_{\mathfrak{p}}(B_{\lambda}/\mathcal{K}) = 1 \implies \text{ord}_{s=1} L(\lambda, s) = 1.
\]

Proof. Note that by the \( p \)-parity conjecture [Nek01], if \( \text{Sel}_{\mathfrak{p}}(B_{\lambda}/\mathcal{K}) \) has \( \mathcal{O} \)-corank 1, then \( \lambda \) has sign \(-1\). Let \((\psi, \chi)\) be a good pair for \( \lambda \), i.e., satisfying conditions (G1)–(G4) above, so in particular
\[
L(\psi^* \chi \mathcal{N}_{\mathcal{K}}^{-1}, 0) = L(\psi^* \chi, 1) \neq 0. \quad (7.1)
\]
Since by Theorem 3.2.1 the value \( L(\psi^* \chi \mathcal{N}_{\mathcal{K}}^{-1}, 0) \) is a nonzero multiple of the value \( \mathcal{L}_{\pi}(\psi^* \chi) \) at the trivial character \( 1 \) of \( \Gamma \), by Theorem 3.2.1 it follows that
\[
\#(\mathcal{Z}^\circ(\psi^* \chi)/(\gamma - 1)(\mathcal{Z}^\circ(\psi^* \chi))) < \infty,
\]
where \( \gamma \in \Gamma \) is any topological generator. Since \( \mathcal{Z}^\circ(\psi^* \chi) \) corresponds to the Bloch–Kato Selmer group for \( \psi^* \chi \) over \( \mathcal{K}_{\infty}/\mathcal{K} \) (see Remark 2.2.2), it follows that \( \text{corank}_\mathcal{O} H^1(\mathcal{K}, W_{\psi^* \chi}) = 0 \). Since \( \text{Sel}_{\mathfrak{p}}(B_{\lambda}/\mathcal{K}) \simeq H^1(\mathcal{K}, W_{\psi^* \chi}) \), from Lemma 7.0.1 we thus obtain
\[
\text{corank}_\mathcal{O} \text{Sel}_{\mathfrak{p}}(B_{\lambda}/\mathcal{K}) = 1 \implies \text{corank}_\mathcal{O} \text{Sel}_{\mathfrak{p}}(A_{f, \lambda}/\mathcal{K}) = 1. \quad (7.2)
\]

Now, Corollary 6.0.2 together with a variant of Mazur’s control theorem immediately yields the implication
\[
\text{corank}_\mathcal{O} \text{Sel}_{\mathfrak{p}}(A_{f, \lambda}/\mathcal{K}) = 1 \implies z_{f, \chi, 0} \neq 0 \in S^\circ_{\mathfrak{p}}(A_{f, \lambda}/\mathcal{K}) \otimes \mathcal{O} \Phi. \quad (7.3)
\]
Here \( z_{f, \chi, 0} \) is the image of \( z_{f, \lambda} \) under the specialization map \( \mathcal{S}(A_{f, \chi}/\mathcal{K}_{\infty}) \to H^1(\mathcal{K}, V_{f, \chi}) \) at the trivial character, which agrees with the Heegner class \( z_{f, \chi, 0} \) (see e.g. [CH18, Lem. 5.4]).

Finally, we note once more that (2.6) yields the factorization
\[
L(f, \chi, s) = L(\psi \chi, s) \cdot L(\psi^* \chi, s).
\]
Since \( z_{f, \chi, 0} \neq 0 \iff \text{ord}_{s=1} L(f, \chi, s) = 1 \) by the general Gross–Zagier formula [YZZ13, CST14], together with (7.2) and (7.3), we conclude that
\[
\text{corank}_\mathcal{O} \text{Sel}_{\mathfrak{p}}(B_{\lambda}/\mathcal{K}) = 1 \implies \text{ord}_{s=1} L(f, \chi, s) = 1
\]
\[
\implies \text{ord}_{s=1} L(\psi \chi, s) = 1,
\]
where the second equality follows from (7.1). Since \( \psi \chi = \lambda \), this concludes the proof. \( \square \)

References


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