p^∞-SELMER GROUPS AND RATIONAL POINTS ON ELLIPTIC CURVES WITH CM

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To Bernadette Perrin-Riou, with admiration

Abstract. Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication by an imaginary quadratic field in which p splits. In this note we prove that if $\mathrm{Sel}_{p^\infty}(E/\mathbb{Q})$ has $\mathbb{Z}_p$-corank one, then $E(\mathbb{Q})$ has a point of infinite order. The non-torsion point arises from a Heegner point construction, and as a result we obtain a converse to a theorem of Gross–Zagier, Kolyvagin, and Rubin in the spirit of Skinner [Ski20]. For $p > 3$, this gives a new proof of a theorem by Burungale–Tian [BT20], which our method extends to small primes.

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Introduction

Following a result by the third author [Ski20], in recent years several works have been devoted to the proof, under varying sets of hypotheses, of a so-called "p-converse" to a celebrated theorem of Gross–Zagier, Kolyvagin, and Rubin; namely, the deduction that $\mathrm{ord}_{s=1}L(E,s) = 1$ for elliptic curves $E/\mathbb{Q}$ provided the p^∞-Selmer group $\mathrm{Sel}_{p^\infty}(E/\mathbb{Q})$ has $\mathbb{Z}_p$-corank 1. An important impetus for the development of these results has arisen from related advances in arithmetic statistics, whereby such p-converse theorems have lead in particular to the proof [BSZ15] (when combined

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with an analogous result in rank 0 deduced from the cyclotomic Iwasawa main conjecture for modular forms [Kat04, SU14, Wan20]) that a sizeable proportion of elliptic curves $E/\mathbb{Q}$ (and conditionally, 100% of them) satisfy the Birch–Swinnerton-Dyer conjecture.

Skinner's original result, whose proof is obtained by showing that a certain Heegner point on E has infinite order assuming $\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) = 1$, was for primes $p > 3$ of good ordinary reduction of E, and under certain technical hypotheses that excluded the CM case.

Our main result in this note is the proof of a p-converse theorem in the spirit of [Ski20] in the CM case. For primes $p > 3$, the result was first obtained by Burungale–Tian [BT20].

Theorem A. *Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication by the ring of integers of an imaginary quadratic field $\mathcal{K}$ of discriminant $-D_{\mathcal{K}} < 0$, and assume that the Hecke character of $\mathcal{K}$ associated to E has conductor exactly divisible by $\mathfrak{d}_{\mathcal{K}} := (\sqrt{-D_{\mathcal{K}}})$. Let p be a prime of good ordinary reduction for E. Then*

$$\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) = 1 \quad \Longrightarrow \quad \mathrm{ord}_{s=1}L(E,s) = 1.$$

In particular[1]*, if* $\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) = 1$ *then* $\mathrm{rank}_{\mathbb{Z}}E(\mathbb{Q}) = 1$ *and* $\#\text{Ш}(E/\mathbb{Q}) < \infty$.

More generally, our main result establishes an analogue of Theorem A for CM abelian varieties $B_\lambda/\mathcal{K}$ associated with Hecke characters of $\mathcal{K}$ of infinity type $(-1, 0)$ (see Theorem 7.0.2). We also note that the assumption on the conductor of the Hecke character associated to E is not an intrinsic limitation of our methods. Rather, it arises from our appeal to results from [CH18] which are only proved under a classical Heegner hypothesis, and it may be removed[2] building on the p-adic Waldspurger formula of Liu–Zhang–Zhang [YZZ13]. (The same hypothesis is not present in [BT20], thanks to the p-adic Gross–Zagier proved by Disegni [Dis17] in the generality of Yuan–Zhang–Zhang [YZZ13].)

Remark B. A recent spectacular result by A. Smith [Smi17] has reduced the proof of Goldfeld's conjecture [Gol79] for elliptic curves $E/\mathbb{Q}$ with $E(\mathbb{Q})[2] \simeq (\mathbb{Z}/2\mathbb{Z})^2$ and admitting no rational cyclic 4-isogeny to the proof of the implication in Theorem A (and its analogue in rank 0, which should follow from a refinement of Rubin's result [Rub91]) for $p = 2$. Unfortunately, Theorem A falls short of providing the desired implication for two reasons. First, E should of be allowed to just have *potentially* good ordinary reduction at p (likely this can be achieved with some more work); the second—and more serious—reason is that $\mathbb{Q}(\sqrt{-7})$ is the only imaginary quadratic field of class number 1 where 2 splits, and unfortunately all elliptic curves $E/\mathbb{Q}$ with complex multiplication by $\mathbb{Q}(\sqrt{-7})$ have $E(\mathbb{Q})[2] \simeq \mathbb{Z}/2\mathbb{Z}$ (see e.g. the table in [Ols74, p. 2]).

Assuming that $\#\text{Ш}(E/\mathbb{Q})[p^\infty] < \infty$ and that p is odd, a p-converse theorem as in Theorem A was first proved by Rubin, [Rub94, Thm. 4], motivated in part by his striking formula [Rub92] expressing the p-adic formal group logarithm of a point $P \in E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ in terms of the value of a Katz p-adic L-function at a point *outside* the range of interpolation. Our proof of Theorem A is inspired by the new proof of Rubin's formula discovered Bertolini–Darmon–Prasanna [BDP12].

More precisely, let $E/\mathbb{Q}$ be an elliptic curve with CM by an imaginary quadratic field $\mathcal{K}$, and assume that p is a prime of good reduction for E which splits in $\mathcal{K}$. Let λ be the Hecke character of $\mathcal{K}$ associated with E, so that

$$L(E,s) = L(\lambda, s).$$

Building on certain nonvanishing results, in [BDP12] it is shown that there exist pairs (ψ, χ) of Hecke characters of $\mathcal{K}$, with χ having finite order, satisfying in particular the conditions

$$\psi\chi = \lambda, \quad L(\psi^*\chi, 1) \neq 0. \tag{0.1}$$

[1] When combined with the work of Gross–Zagier, Kolyvagin, and Rubin.

[2] See [BCK20, §4] for partial progress in this direction.

Letting $f = \theta_\psi$ be the theta series of ψ, the main result of [BDP13] relates the p-adic formal group logarithm of a Heegner point $P_{\psi,\chi} \in B_{\psi,\chi}(\mathcal{K})$ to a value (outside the range of interpolation) of a p-adic Rankin L-series $\mathscr{L}_v(f,\chi)$, where $B_{\psi,\chi}$ is a CM abelian variety over $\mathcal{K}$ admitting a $\mathcal{K}$-rational isogeny $i_\lambda : B_{\psi,\chi} \to E$. By the Gross–Zagier formula (in the generality of [YZZ13]), the point $P_{\psi,\chi}$ is non-torsion if and only if $L'(f,\chi,1) \neq 0$. Setting $P_\mathcal{K} := i_\lambda(P_{\psi,\chi}) \in E(\mathcal{K})$, one thus obtains a point on E which, in light of (0.1) and the factorization

$$L(f,\chi,s) = L(\lambda,s) \cdot L(\psi^*\chi,s), \tag{0.2}$$

is non-torsion if and only if $\mathrm{ord}_{s=1}L(E,s) = 1$, and whose formal group logarithm is related to a value of a Katz p-adic L-function as in Rubin's formula by virtue of a factorization for p-adic L-functions mirroring (0.2).

Our approach to Theorem A consists in showing that $P_\mathcal{K}$ is non-torsion assuming $\mathrm{Sel}_{p^\infty}(E/\mathbb{Q})$ has $\mathbb{Z}_p$-corank 1. From this perspective, in [BT20] this implication is shown by combining:

(1) Rubin's proof of the (anticyclotomic) Iwasawa main conjecture for $\mathcal{K}$ in the root number $+1$ case;
(2) The proof by Agboola–Howard [AH06] and Arnold [Arn07] of the anticyclotomic Iwasawa main conjecture for $\mathcal{K}$ in the root number -1 case;
(3) Burungale's proof [Bur15] that the Λ-adic regulator appearing in the works of Agboola–Howard and Arnold is nonzero;
(4) Disegni's Λ-adic Gross–Zagier formula [Dis17].

Here, instead of (2), (3) and (4) above, we build on the "explicit reciprocity law" established in [CH18], which realizes $\mathscr{L}_v(f,\chi)$ as the image of a Λ-adic Heegner class $\mathbf{z}_{f,\chi}$ under a Perrin-Riou big logarithm map. Similarly as in [BDP12], in §3 we establish a factorization

$$\mathscr{L}_v(\theta_\psi,\chi)^2 \doteq \mathscr{L}_{\overline{v}}(\psi\chi) \cdot \mathscr{L}_{\overline{v}}(\psi^*\chi)$$

relating $\mathscr{L}_v(\theta_\psi,\chi)^2$ to the product of two Katz p-adic L-functions. Combined with an analogous decomposition for Selmer groups shown in §2, we thus deduce from Rubin's work a proof of the Iwasawa–Greenberg main conjecture for $\mathscr{L}_v(\theta_\psi,\chi)^2$. Building on the Λ-adic explicit reciprocity law of [CH18], in §4 we show that this main conjecture is equivalent to another Iwasawa main conjecture formulated in terms of $\mathbf{z}_{f,\chi}$, from where the implication

$$\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) = 1 \quad \Longrightarrow \quad P_\mathcal{K} \neq 0 \in E(\mathcal{K}) \otimes_\mathbb{Z} \mathbb{Q},$$

finally follows from a variant of Mazur's control theorem.

Finally, we conclude this Introduction by recording some standard applications of Theorem A to the Birch–Swinneron-Dyer conjecture. For $p > 3$, these correspond to Corollaries 1.2 and 1.3 in [BT20], respectively, and their proof of the latter (using Theorem A) applies without change.

Corollary C. *Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication by the imaginary quadratic field $\mathcal{K}$, and assume that the Hecke character associated to E has conductor exactly divisible by $\mathfrak{d}_\mathcal{K}$. Let p be a prime of good ordinary reduction for E. If $\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) = 1$, then the p-part of the Birch–Swinnerton-Dyer formula holds for E.*

Proof. By Theorem A, if $\mathrm{corank}_{\mathbb{Z}_p}\mathrm{Sel}_{p^\infty}(E/\mathbb{Q}) = 1$ then

$$\mathrm{ord}_{s=1}L(E,s) = \mathrm{rank}_\mathbb{Z} E(\mathbb{Q}) = 1, \quad \#Ш(E/\mathbb{Q}) < \infty.$$

As in the proof of [Kob13, Cor. 1.4], the result thus follows from the Iwasawa main conjecture for $\mathcal{K}$ [Rub91, JLK11, OV16], the non-triviality of the cyclotomic p-adic height pairing [Ber84], and the p-adic Gross–Zagier formula [PR87b, Dis17]. □

Corollary D. *Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication by imaginary quadratic field $\mathcal{K}$, and assume that the Hecke character associated to E has conductor exactly divisible by $\mathfrak{d}_{\mathcal{K}}$. Let p be a prime of good ordinary reduction for E, and assume that:*

(i) $E(\mathbb{Q})[p]=0$;
(ii) $\mathrm{Sel}_p(E/\mathbb{Q})\simeq\mathbb{Z}/p\mathbb{Z}$, *where* $\mathrm{Sel}_p(E/\mathbb{Q})\subset\mathrm{H}^1(\mathbb{Q},E[p])$ *is the p-Selmer group of E.*

Then $\mathrm{ord}_{s=1}L(E,s)=1$ *and* $Ш(E/\mathbb{Q})[p^\infty]=0$.

Remark E. In forthcoming work, the strategy introduced here will be used to extend the main result of [BT20] to totally real fields (sidestepping the use of elliptic units in *op.cit.*).

As already noted, Theorem A was first proved by Rubin in cases where $\#Ш(E/\mathbb{Q})[p^\infty]<\infty$. That the approach in this note can dispense with this hypothesis can be attributed to the study of Iwasawa theory of Heegner points[3], a study systematically initiated by Perrin-Riou [PR87a]. The theory of p-adic logarithm maps [PR94], another major contribution of Perrin-Riou's work, also plays a key role in our strategy. It is a great pleasure to dedicate this note to Perrin-Riou as a humble gift on the occasion of her 65th birthday.

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1. Preliminaries

Fix throughout a prime p, an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and embeddings $\mathbb{C}\overset{\imath_\infty}{\hookleftarrow}\overline{\mathbb{Q}}\overset{\imath_p}{\hookrightarrow}\mathbb{C}_p$. Fix also an imaginary quadratic field $\mathcal{K}$ of discriminant $-D_{\mathcal{K}}<0$ and ring of integers $\mathcal{O}_{\mathcal{K}}$.

1.1. **CM abelian varieties.** We say that a Hecke character $\psi:\mathcal{K}^\times\backslash\mathbb{A}_{\mathcal{K}}^\times\to\mathbb{C}^\times$ has *infinity type* $(a,b)\in\mathbb{Z}^2$ if, writing $\psi=(\psi_v)_v$ with v running over the places of $\mathcal{K}$, the component ψ_∞ satisfies $\psi_\infty(z)=z^a\bar{z}^b$ for all $z\in(\mathcal{K}\otimes_{\mathbb{Q}}\mathbb{R})^\times\simeq\mathbb{C}^\times$, where the identification is made via $\imath_\infty$. Hence in particular the norm character $\mathbf{N}_{\mathcal{K}}$, given by $\mathfrak{q}\mapsto\#(\mathcal{O}_{\mathcal{K}}/\mathfrak{q})$ on ideals of $\mathcal{O}_{\mathcal{K}}$, has infinity type $(-1,-1)$. The *central character* of such ψ is the character ω_ψ on $\mathbb{A}^\times$ defined by

$$\psi|_{\mathbb{A}^\times}=\omega_\psi\cdot\mathbf{N}^{-(a+b)},$$

where $\mathbf{N}$ is the norm on $\mathbb{A}^\times$.

Our fixed embedding $\imath_p$ defines a natural map $\sigma:\mathcal{K}\otimes_{\mathbb{Q}}\mathbb{Q}_p\to\mathbb{C}_p$, and we let $\overline{\sigma}:\mathcal{K}\otimes_{\mathbb{Q}}\mathbb{Q}_p\to\mathbb{C}_p$ be composition of σ with the non-trivial automorphism of $\mathcal{K}$. The *p-adic avatar* of a Hecke character ψ of infinity type (a,b) is the character $\hat{\psi}:\mathcal{K}^\times\backslash\mathbb{A}_{\mathcal{K},\mathrm{f}}^\times\to\mathbb{C}_p^\times$ given by

$$\hat{\psi}(x)=\imath_p\circ\imath_\infty^{-1}(\psi(x))\sigma(x_p)^a\overline{\sigma}(x_p)^b$$

for all $x\in\mathbb{A}_{\mathcal{K},\mathrm{f}}^\times$, where $x_p\in(\mathcal{K}\otimes\mathbb{Q}_p)^\times$ is the p-component of x.

Thoughout the following, we shall often omit the notational distinction between an algebraic Hecke character and its p-adic avatar, as it will be clear from the context which one is meant.

Let ψ be an algebraic Hecke character of $\mathcal{K}$ infinity type $(-1,0)$ with values in a number field $F_\psi\subset\overline{\mathbb{Q}}$ with ring of integer $\mathcal{O}_\psi$. Let $\mathfrak{P}$ be the prime of F_ψ above p induced by $\imath_p$, and denote by Φ_ψ the completion of F_ψ at $\mathfrak{P}$ and by $\mathscr{O}_\psi$ the ring of integers of Φ_ψ. By a well-known theorem

[3]More precisely, the proof of the "Heegner point main conjecture" formulated in §1.3; one of the divisibilities suffices for the application.

of Casselman's (see [BDP12, Thm. 2.5] and the reference [Shi71, Thm. 6] therein), attached to ψ there is a CM abelian variety $B_{\psi/\mathcal{K}}$, unique up to isogeny over $\mathcal{K}$, with the property that

$$V_{\mathfrak{P}}B_\psi \simeq \hat{\psi}$$

as one-dimensional Φ_ψ-representations of $G_{\mathcal{K}}$, where $V_{\mathfrak{P}}B_\psi = \big(\varprojlim B_\psi[\mathfrak{P}^j]\big)\otimes_{\mathscr{O}_\psi}\Phi_\psi$ is the rational $\mathfrak{P}$-adic Tate module of B_ψ.

1.2. **Heegner points.** Let $f \in S_2(\Gamma_1(N))$ be a normalized eigenform of weight 2, level N prime to p, and nebentypus ε_f. We assume that $\mathcal{K}$ satisfies the *Heegner hypothesis* relative to N:

$$\text{(Heeg)} \qquad \text{there is an ideal } \mathfrak{N} \subset \mathcal{O}_{\mathcal{K}} \text{ with } \mathcal{O}_{\mathcal{K}}/\mathfrak{N} \simeq \mathbb{Z}/N\mathbb{Z},$$

and fix once and for all an ideal $\mathfrak{N}$ as above. We assume also that

$$\text{(spl)} \qquad p\mathcal{O}_{\mathcal{K}} = v\overline{v} \text{ splits in } \mathcal{K},$$

with v the prime of $\mathcal{K}$ above p induced by our fixed embedding $\imath_p$. Let $F \subset \overline{\mathbb{Q}}$ be the number field generated by the Fourier coefficients of f. Denote by $\mathfrak{P}$ the prime of F above p induced by $\imath_p$, and assume that f is $\mathfrak{P}$-*ordinary*, i.e. $v_{\mathfrak{P}}(a_p(f)) = 0$, where $v_{\mathfrak{P}}$ is the $\mathfrak{P}$-adic valuation on F.

Let $A_f/\mathbb{Q}$ be the abelian variety of GL_2-type associated to f, determined up to isogeny over $\mathbb{Q}$ by the equality of L-functions

$$L(A_f, s) = \prod_{\tau : F \hookrightarrow \mathbb{C}} L(f^\tau, s),$$

where f^τ runs over all the conjugates of f. Denote by Φ the completion of F at $\mathfrak{P}$, and let $\mathscr{O}$ be the ring of integers of Φ. Let $T_{\mathfrak{P}}A_f := \varprojlim A_f[\mathfrak{P}^j]$ be the $\mathfrak{P}$-adic Tate module of A_f, which is free of rank two over $\mathscr{O}$.

For every positive integer c, let $\mathcal{K}_c$ be the ring class field of $\mathcal{K}$ of conductor c, so $\mathrm{Gal}(\mathcal{K}_c/\mathcal{K}) \simeq \mathrm{Pic}(\mathcal{O}_c)$ by class field theory, where $\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_{\mathcal{K}}$ is the order of $\mathcal{K}$ of conductor c. For every $c > 0$ prime to N and every ideal $\mathfrak{a}$ of $\mathcal{O}_c$, we consider the CM point $x_{\mathfrak{a}} \in X_1(N)(\tilde{\mathcal{K}}_c)$ constructed in [CH18, §2.3], where $\tilde{\mathcal{K}}_c$ is the compositum of $\mathcal{K}_c$ and the ray class field of $\mathcal{K}$ of conductor $\mathfrak{N}$. Let $\Delta_{\mathfrak{a}}$ be the class of the degree 0 divisor $(x_{\mathfrak{a}}) - (\infty)$ in $J_1(N) = \mathrm{Jac}(X_1(N))$, and denote by $z_{\mathfrak{a}} = \delta(\Delta_{\mathfrak{a}})$ its image under the Kummer map

$$\delta : J_1(N)(\tilde{\mathcal{K}}_c) \to \mathrm{H}^1(\tilde{\mathcal{K}}_c, T_pJ_1(N)).$$

Fix a parametrization $\pi : J_1(N) \to A_f$, and let $y_{f,\mathfrak{a}} \in \mathrm{H}^1(\tilde{\mathcal{K}}_c, T_{\mathfrak{P}}A_f)$ be the image of $y_{\mathfrak{a}}$ under the natural projection

$$\mathrm{H}^1(\tilde{\mathcal{K}}_c, T_pJ_1(N)) \xrightarrow{\pi_*} \mathrm{H}^1(\tilde{\mathcal{K}}_c, T_pA_f) \to \mathrm{H}^1(\tilde{\mathcal{K}}_c, T_{\mathfrak{P}}A_f).$$

For the ease of notation, we set $y_{f,c} = y_{f,\mathfrak{a}}$ for $\mathfrak{a} = \mathcal{O}_c$. A standard calculation shows that if $p \nmid c$, then for every $n > 0$ we have

$$\mathrm{Cor}_{\tilde{\mathcal{K}}_{cp^n}/\tilde{\mathcal{K}}_{cp^{n-1}}}(y_{f,cp^n}) = \begin{cases} a_p(f)\cdot y_{f,cp^{n-1}} - \varepsilon_f(p)\cdot y_{f,cp^{n-2}} & \text{if } n > 1, \\ u_c^{-1}\big(a_p(f) - \sigma_v - \sigma_{\overline{v}}\big)\cdot y_{f,c} & \text{if } n = 1, \end{cases} \tag{1.1}$$

where $u_c := [\mathcal{O}_c^\times : \mathcal{O}_{cp}^\times]$ and $\sigma_v, \sigma_{\overline{v}} \in \mathrm{Gal}(\tilde{\mathcal{K}}_c/\mathcal{K})$ are Frobenius elements at the primes of $\mathcal{K}$ above p (*cf.* [CH18, Prop. 4.4]).

Let α be the $\mathfrak{P}$-adic unit root of $x^2 - a_p(f)x + \varepsilon_f(p)p$, and for any positive integer c prime to N define the α-*stabilized Heegner class* $y_{f,c,\alpha}$ by

$$y_{f,c,\alpha} := \begin{cases} y_{f,c} - \varepsilon_f(p)\alpha^{-1}\cdot y_{f,c/p} & \text{if } p \mid c, \\ u_c^{-1}\big(1 - \sigma_v\alpha^{-1} - \sigma_{\overline{v}}\alpha^{-1}\big)\cdot y_{f,c} & \text{if } p \nmid c. \end{cases}$$

This definition is motivated by the following result.

Lemma 1.2.1. *For all positive integers c prime to N, we have*

$$\mathrm{Cor}_{\mathcal{K}'_{cp}/\mathcal{K}'_c}(y_{f,cp,\alpha}) = \alpha \cdot y_{f,c,\alpha}.$$

Proof. This follows immediately from (1.1). □

1.3. **Heegner point main conjecture.** Let $\mathcal{K}_\infty$ be the anticyclotomic $\mathbb{Z}_p$-extension of $\mathcal{K}$, with Galois group

$$\Gamma = \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K}) \simeq \mathbb{Z}_p,$$

and for every n denote by $\mathcal{K}_n$ the subextension of $\mathcal{K}_\infty$ with $[\mathcal{K}_n : \mathcal{K}] = p^n$. Let χ be a finite order Hecke character of $\mathcal{K}$ with $\chi|_{\mathbb{A}^\times} = \varepsilon_f^{-1}$. Upon enlarging F is necessary, assume that Φ contains the values of χ. For each n, take $m \gg 0$ so that $\tilde{\mathcal{K}}_{cp^m} \supset \mathcal{K}_n$, and set

$$z_{f,\chi,n} := \alpha^{-m} \cdot \mathrm{Cor}_{\tilde{\mathcal{K}}_{cp^m}/\mathcal{K}_n}\Biggl(\sum_{\sigma \in \mathrm{Gal}(\tilde{\mathcal{K}}_{cp^m}/\mathcal{K})} \chi(\sigma) \cdot y_{f,cp^m,\alpha}\Biggr). \tag{1.2}$$

In view of Lemma 1.2.1, the definition of $z_{f,\chi,n}$ does not depend on the choice of m. Moreover, letting $A_{f,\chi}$ be the Serre tensor $A_f \otimes \chi$, we see that $z_{f,\chi,n}$ defines a class $\mathbf{z}_{f,\chi,n} \in \mathrm{H}^1(\mathcal{K}_n, T_{\mathfrak{P}} A_{f,\chi})$.

Let

$$\Lambda_0 = \mathscr{O}[\![\Gamma]\!], \qquad \Lambda = \Lambda_0 \otimes_{\mathscr{O}} \Phi \tag{1.3}$$

be the anticyclotomic Iwasawa algebras. From their construction, the classes $\mathbf{z}_{f,\chi,n}$ are contained in the pro-$\mathfrak{P}$ Selmer group $S_{\mathfrak{P}}(A_{f,\chi}/\mathcal{K}_n) \subset \mathrm{H}^1(\mathcal{K}_n, T_{\mathfrak{P}} A_{f,\chi})$, and by Lemma 1.2.1 they are norm-compatible, hence defining a class $\mathbf{z}_{f,\chi} = \{\mathbf{z}_{f,\chi,n}\}_n$ in the compact Λ_0-adic Selmer group

$$\mathscr{S}(A_{f,\chi}/\mathcal{K}_\infty) := \varprojlim_n S_{\mathfrak{P}}(A_{f,\chi}/\mathcal{K}_n).$$

Denote by $\mathrm{Sel}_{\mathfrak{P}^\infty}(A_{f,\chi}/\mathcal{K}_n) \subset \mathrm{H}^1(\mathcal{K}_n, A_{f,\chi}[\mathfrak{P}^\infty])$ the $\mathfrak{P}^\infty$-Selmer groups of $A_{f,\chi}$, and set

$$\mathscr{X}(A_{f,\chi}/\mathcal{K}_\infty) := \mathrm{Hom}_{\mathbb{Z}_p}\Bigl(\varinjlim_n \mathrm{Sel}_{\mathfrak{P}^\infty}(A_{f,\chi}/\mathcal{K}_n), \Phi/\mathscr{O}\Bigr).$$

Set also

$$\mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty) = \mathscr{S}(A_{f,\chi}/\mathcal{K}_\infty) \otimes_{\mathscr{O}} \Phi, \quad \mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty) = \mathscr{X}(A_{f,\chi}/\mathcal{K}_\infty) \otimes_{\mathscr{O}} \Phi,$$

which are finitely generated Λ-modules.

The following conjecture in a natural extension of Perrin-Riou's Heegner point main conjecture, [PR87a, Conj. B].

Conjecture 1.3.1. *The modules $\mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty)$ and $\mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty)$ have both Λ-rank one, and*

$$\mathrm{char}_\Lambda(\mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty)_{\Lambda\text{-tors}}) = \mathrm{char}_\Lambda\bigl(\mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty)/\Lambda \cdot \mathbf{z}_{f,\chi}\bigr)^2,$$

where the subscript Λ-tors *denotes the maximal Λ-torsion submodule.*

In [BT20, Conj. 2.2] a conjecture similar to Conjecture 1.3.1 is formulated in terms of a Λ-adic Heegner class deduced from work of Disegni [Dis17]. As in [BT20][4], our proof of Theorem A is based on establishing a corresponding result towards Conjecture 1.3.1. The only novelty in our approach is in the proof of cases of this conjecture when f has CM.

[4] Also in other results on the p-converse theorem in rank 1 without a finiteness condition on the Tate–Shafarevich group that appeared after [Ski20]: [Wan14], [CW16], etc.

2. Selmer groups

In this section we introduce the different Selmer groups entering in our arguments. In particular, the decomposition in Proposition 2.3.1 will play a key role.

2.1. Selmer groups of certain Rankin–Selberg convolutions.

As in §1.2, let $f \in S_2(\Gamma_1(N))$ be a $\mathfrak{P}$-ordinary newform with nebentypus ε_f, and let $\mathcal{K}$ be an imaginary quadratic field satisfying (Heeg) and (spl).

Let $c > 0$ be a positive integer prime to N. Similarly as in [BDP12, Def. 3.10], we say that a Hecke character ξ of infinity type $(2+j, -j)$, with $j \in \mathbb{Z}$, has *finite type* $(c, \mathfrak{N}, \varepsilon_f)$ if it satisfies:

(a) $\xi|_{\mathbb{A}^\times} = \varepsilon_f^{-1} \cdot \mathbf{N}^{-2}$, i.e., $\omega_\xi \cdot \varepsilon_f = \mathbb{1}$;
(b) $\mathfrak{f}_\xi = c \cdot \mathfrak{N}'$, where $\mathfrak{N}'$ is the unique divisor of $\mathfrak{N}$ with norm equal to the conductor of ε_f;
(c) the local sign $\epsilon_q(f, \chi) = +1$ for all finite prime q.

These conditions imply that the Rankin–Selberg L-function $L(f, \chi, s)$ is self-dual, with $s = 0$ as the central critical point. The sign in the functional equation is $+1$ (resp. -1) when $j \geqslant 0$ (resp. $j < 0$). Denote by $\Sigma_{\rm cc}(c, \mathfrak{N}, \varepsilon_f)$ the set of such characters ξ, and put

$$\Sigma_{\rm cc}^{(1)}(c, \mathfrak{N}, \varepsilon_f) = \{\xi \in \Sigma_{\rm cc}(c, \mathfrak{N}, \varepsilon_f) \mid j < 0\}, \quad \Sigma_{\rm cc}^{(2)}(c, \mathfrak{N}, \varepsilon_f) = \{\xi \in \Sigma_{\rm cc}(c, \mathfrak{N}, \varepsilon_f) \mid j \geqslant 0\}.$$

Let χ be a finite order character of $\mathcal{K}$ such that $\chi \mathbf{N}_\mathcal{K}^{-1} \in \Sigma_{\rm cc}^{(1)}(c, \mathfrak{N}, \varepsilon_f)$. Denote by $\rho_f : G_\mathbb{Q} \to \mathrm{Aut}_\Phi(V_f)$ the $\mathfrak{P}$-adic Galois representation associated to f, where $V_f = \Phi \otimes_\mathscr{O} T_\mathfrak{P} A_f$, and consider the conjugate self-dual $G_\mathcal{K}$-representation

$$V_{f,\chi} := V_f|_{G_\mathcal{K}} \otimes \chi. \tag{2.1}$$

For any Λ_0-module M, let $M^\vee = \mathrm{Hom}_{\rm cts}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ be the Pontryagin dual. Fix a $G_\mathcal{K}$-stable lattice $T_{f,\chi} \subset V_{f,\chi}$, and define the $G_\mathcal{K}$-modules

$$\mathbf{W}_{f,\chi} := T_{f,\chi} \otimes_\mathscr{O} \Lambda_0^\vee, \qquad \mathbf{T}_{f,\chi} := \mathbf{W}_{f,\chi}^\vee(1) \simeq T_{f,\chi} \otimes_\mathscr{O} \Lambda_0,$$

where the tensor products are endowed with the diagonal Galois actions, with $G_\mathcal{K}$ acting on Λ_0 (resp. $\Lambda_0^\vee$) via the tautological character $\Psi : G_\mathcal{K} \twoheadrightarrow \mathrm{Gal}(\mathcal{K}_\infty/\mathcal{K}) \hookrightarrow \Lambda_0^\times$ (resp. Ψ^{-1}).

Definition 2.1.1. Fix a finite set Σ of places of $\mathcal{K}$ containing ∞ and the primes dividing Np, and denote by $\mathcal{K}^\Sigma \subset \overline{\mathbb{Q}}$ the maximal extension of $\mathcal{K}$ unramified outside Σ. The *BDP Selmer group* of $T_{f,\chi}$ over $\mathcal{K}_\infty$ is defined by

$$\mathscr{X}_v(T_{f,\chi}/\mathcal{K}_\infty) := \ker\left\{\mathrm{H}^1(\mathcal{K}^\Sigma/\mathcal{K}, \mathbf{W}_{f,\chi}) \to \mathrm{H}^1(\mathcal{K}_v, \mathbf{W}_{f,\chi}) \times \prod_{w \in \Sigma, w \nmid p} \mathrm{H}^1(\mathcal{K}_w, \mathbf{W}_{f,\chi})\right\}.$$

We also set

$$\mathcal{X}_v(f, \chi) = \mathscr{X}_v(T_{f,\chi}/\mathcal{K}_\infty) \otimes_\mathscr{O} \Phi,$$

which is independent of the lattice $T_{f,\chi}$. We define the compact version $\mathcal{S}_v(f, \chi) \subset \mathrm{H}^1(\mathcal{K}^\Sigma/\mathcal{K}, \mathbf{T}_{f,\chi})$ in the same manner, replacing $\mathbf{W}_{f,\chi}$ by $\mathbf{T}_{f,\chi}$.

The BDP Selmer groups of Definition 2.1.1 differs from the Selmer group $\mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty)$ and $\mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty)$ in §1.3 in their defining local conditions at the primes above p. More precisely, by $\mathfrak{P}$-ordinarity, for every $w|p$ there is a $G_{\mathcal{K}_w}$-module exact sequence

$$0 \to \mathscr{F}_w^+ T_{f,\chi} \to T_{f,\chi} \to \mathscr{F}_w^- T_{f,\chi} \to 0 \tag{2.2}$$

with $\mathscr{F}_w^\pm T_{f,\chi}$ free of rank one over $\mathscr{O}$, and the $G_{\mathcal{K}_w}$-action on $\mathscr{F}_w^- T_{f,\chi}$ being unramified. Then the Selmer group defined by

$$\mathscr{X}_{\rm ord}(T_{f,\chi}/\mathcal{K}_\infty) := \ker\left\{\mathrm{H}^1(\mathcal{K}^\Sigma/\mathcal{K}, \mathbf{W}_{f,\chi}) \to \prod_{w|p} \frac{\mathrm{H}^1(\mathcal{K}_w, \mathbf{W}_{f,\chi})}{\mathrm{H}^1(\mathcal{K}_w, \mathscr{F}_w^- \mathbf{W}_{f,\chi})} \times \prod_{w \in \Sigma, w \nmid p} \mathrm{H}^1(\mathcal{K}_w, \mathbf{W}_{f,\chi})\right\},$$

where $\mathscr{F}_w^-\mathbf{W}_{f,\chi} = (\mathscr{F}_w^- T_{f,\chi}) \otimes_{\mathscr{O}'} \Lambda_0^\vee$, satisfies

$$\mathcal{X}_{\rm ord}(f,\chi) := \mathscr{X}_{\rm ord}(T_{f,\chi}/\mathcal{K}_\infty) \otimes_{\mathscr{O}'} \Phi' \simeq \mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty) \tag{2.3}$$

Defining $\mathcal{S}_{\rm ord}(f,\chi) \subset \mathrm{H}^1(\mathcal{K}^\Sigma/\mathcal{K}, \mathbf{T}_{f,\chi})$ in the same manner, we similarly have

$$\mathcal{S}_{\rm ord}(f,\chi) \simeq \mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty) \tag{2.4}$$

(see e.g. [CG96, §4]).

2.2. **Selmer groups of characters.** We keep the hypothesis that the imaginary quadratic field $\mathcal{K}$ satisfies (spl), and let ξ be a Hecke character of $\mathcal{K}$ of conductor $\mathfrak{f}_\xi$. Let F be a number field containing the values of ξ. Let Φ be the completion of F at the prime $\mathfrak{P}$ of F above p induced by $\imath_p$, and let $\mathscr{O}$ be the ring of integers of Φ. Denote by T_ξ the free $\mathscr{O}$-module of rank one on which $G_\mathcal{K}$ acts via ξ, and consider the $G_\mathcal{K}$-module

$$\mathbf{W}_\xi := T_\xi \otimes_{\mathscr{O}} \Lambda_0^\vee,$$

where as before the Galois action on $\Lambda_0^\vee$ is given by the character Ψ^{-1}.

Definition 2.2.1. Let Σ be a finite set of places of $\mathcal{K}$ containing ∞ and the primes dividing p or $\mathfrak{f}_\psi$. The *v-Selmer group* of ψ over $\mathcal{K}_\infty$ is defined by

$$\mathscr{X}_v(T_\xi/\mathcal{K}_\infty) := \ker\biggl\{\mathrm{H}^1(\mathcal{K}^\Sigma/\mathcal{K}, \mathbf{W}_\xi) \rightarrow \mathrm{H}^1(\mathcal{K}_v, \mathbf{W}_\xi) \times \prod_{w\in\Sigma, w\nmid p} \mathrm{H}^1(\mathcal{K}_w, \mathbf{W}_\xi)\biggr\}.$$

We also set $\mathcal{X}_v(\xi) = \mathscr{X}_v(T_\xi/\mathcal{K}_\infty) \otimes_{\mathscr{O}} \Phi$.

Remark 2.2.2. Suppose ξ has infinity type $(-1,0)$, and denote by ξ^* the composition of ξ with the non-trivial automorphism of $\mathcal{K}$, so ξ^* has infinity type $(0,-1)$. Then from e.g. [AH06, §1.1] we see that $\mathcal{X}_v(\xi^*)$ corresponds to the Bloch–Kato Selmer group of ξ^* over the tower $\mathcal{K}_\infty/\mathcal{K}$, whereas $\mathcal{X}_v(\xi)$ corresponds to the Bloch–Kato Selmer group for ξ with the *reversed* local conditions at the primes above p.

2.3. **Decomposition.** We now specialize the set-up in §2.1 to the case where $f = \theta_\psi$ is the theta series of a Hecke character ψ of $\mathcal{K}$ of infinity type $(-1,0)$. Then f has level $N = D_\mathcal{K}\cdot\mathbf{N}_\mathcal{K}(\mathfrak{f}_\psi)$ and nebentypus $\varepsilon_f = \eta_\mathcal{K}\cdot\omega_\psi$, where $\eta_\mathcal{K}$ is the quadratic character associated to $\mathcal{K}$.

One easily checks (see [BDP12, Lem. 3.14]) that if $\mathfrak{f}_\psi$ is a cyclic ideal of norm $\mathbf{N}_\mathcal{K}(\mathfrak{f}_\psi)$ prime to $D_\mathcal{K}$, then $\mathcal{K}$ satisfies the Heegner hypothesis (Heeg) relative to N, and one may take

$$\mathfrak{N} = \mathfrak{d}_\mathcal{K}\cdot\mathfrak{f}_\psi, \quad \text{where } \mathfrak{d}_\mathcal{K} := (\sqrt{-D_\mathcal{K}}). \tag{2.5}$$

In the following, we assume that $\mathfrak{f}_\psi$ satisfies the above condition, and take $\mathfrak{N}$ as in (2.5). On the other hand, since we assume (spl), the CM form f is $\mathfrak{P}$-ordinary. FInally, fix a positive integer c prime to Np, and let χ be a finite order character such that $\chi\mathbf{N}_\mathcal{K}^{-1} \in \Sigma_{\rm cc}^{(1)}(c,\mathfrak{N},\varepsilon_f)$.

The following decomposition will play an important role later.

Proposition 2.3.1. *Let ψ and χ be as above. There is a Λ-module isomorphism*

$$\mathcal{X}_v(\theta_\psi,\chi) \simeq \mathcal{X}_v(\psi\chi) \oplus \mathcal{X}_v(\psi^*\chi).$$

Proof. Put $f = \theta_\psi$, and note that there is a $G_\mathcal{K}$-module decomposition

$$V_{f,\chi} \simeq V_{\psi\chi} \oplus V_{\psi^*\chi}. \tag{2.6}$$

Since the module $\mathcal{X}_v(f,\chi) \subset \mathrm{H}^1(\mathcal{K}^\Sigma/\mathcal{K}, \mathbf{W}_{f,\chi}) \otimes_{\mathscr{O}} \Phi$ does not depend on the lattice $T_{f,\chi} \subset V_{f,\chi}$ chosen to define $\mathbf{W}_{f,\chi}$, by (2.6) we may assume that $T_{f,\chi} \simeq T_{\psi\chi} \oplus T_{\psi^*\chi}$ as $G_\mathcal{K}$-modules, and so

$$\mathbf{W}_{f,\chi} \simeq \mathbf{W}_{\psi\chi} \oplus \mathbf{W}_{\psi^*\chi}$$

as $G_{\mathcal{K}}$-modules. The result thus follows immediately by comparing the defining local conditions of the three Selmer groups involved at all places. □

3. p-adic L-functions

In this section we introduce the two p-adic L-functions needed for our arguments, and prove Proposition 3.3.1 relating the two.

3.1. **The BDP p-adic L-function.** As in §2.1, let $f \in S_2(\Gamma_1(N))$ be an eigenform with $p \nmid N$ and nebentypus ε_f, let $\mathcal{K}$ be an imaginary quadratic field satisfying (Heeg) and (spl), and fix an ideal $\mathfrak{N} \subset \mathcal{O}_{\mathcal{K}}$ with cyclic quotient of order N. Let c be a positive integer prime to Np, and let χ be a finite order Hecke character of $\mathcal{K}$ such that $\chi\mathbf{N}_{\mathcal{K}}^{-1} \in \Sigma_{\mathrm{cc}}^{(1)}(c, \mathfrak{N}, \varepsilon_f)$.

Let $F \subset \overline{\mathbb{Q}}$ be a number field containing $\mathcal{K}$, the Fourier coefficients of f, and the values of χ, and let Φ be the completion of F at the prime of F above p induced by $\imath_p$, with ring of integers $\mathscr{O} \subset \Phi$. Let $\Lambda_0 = \mathscr{O}[\![\Gamma]\!]$, and $\Lambda = \Lambda_0 \otimes_{\mathscr{O}} \Phi$ be the anticyclotomic Iwasawa algebras as in (1.3), and set

$$\Lambda_0^{\mathrm{ur}} := \Lambda_0 \hat{\otimes}_{\mathbb{Z}_p} \mathbb{Z}_p^{\mathrm{ur}} \simeq \mathscr{O}^{\mathrm{ur}}[\![\Gamma]\!], \quad \Lambda^{\mathrm{ur}} := \Lambda_0^{\mathrm{ur}} \otimes_{\mathscr{O}} \Phi,$$

where $\mathbb{Z}_p^{\mathrm{ur}}$ is the completion of the ring of integers of the maximal unramified extension of $\mathbb{Q}_p$.

The p-adic L-function in the next theorem was first constructed in [BDP13] as a continuous function on characters of Γ. Its realization as a measure in Λ_0^{ur} was first given in [CH18] following an approach introduced in [Bra11]. As it will suffice for our purposes, we describe below a $\Phi^\times$-multiple of that p-adic L-function.

As in [CH18, §2.3], define $\vartheta \in \mathcal{K}$ by

$$\vartheta := \frac{D' + \sqrt{-D_{\mathcal{K}}}}{2}, \quad \text{where } D' = \begin{cases} D_{\mathcal{K}} & \text{if } 2 \nmid D_{\mathcal{K}}, \\ D_{\mathcal{K}}/2 & \text{else,} \end{cases}$$

and let Ω_p and $\Omega_{\mathcal{K}}$ be CM periods attached to $\mathcal{K}$ as in [*op.cit.*, §2.5].

Theorem 3.1.1. *There exists an element $\mathscr{L}_v(f,\chi) \in \Lambda^{\mathrm{ur}}$ such that for every character ξ of Γ crystalline at both v and $\bar{v}$ and corresponding to a Hecke character of $\mathcal{K}$ of infinity type $(n,-n)$ with*[5] *$n \geqslant 1$, we have*

$$\mathscr{L}_v(f,\chi)^2(\xi) = \frac{\Omega_p^{4n}}{\Omega_{\mathcal{K}}^{4n}} \cdot \frac{\Gamma(n)\Gamma(n+1)\xi(\mathfrak{N}^{-1})}{(4\pi)^{2n+1}(\operatorname{Im}\vartheta)^{2n}} \cdot \big(1 - a_p(f)(\chi\xi)(\overline{v})p^{-1} + \varepsilon_f(p)(\chi\xi)^2(\overline{v})p^{-1}\big)^2 \cdot L(f,\chi\xi,1).$$

Proof. Let η be an anticyclotomic Hecke character of $\mathcal{K}$ of infinity type $(1,-1)$ and conductor dividing $c\mathcal{O}_{\mathcal{K}}$, and define $\mathfrak{L}_{v,\eta}(f,\chi) \in \Lambda_0^{\mathrm{ur}}$ by

$$\mathfrak{L}_{v,\eta}(f,\chi)(\phi) = \sum_{[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O}_c)} (\eta\chi\mathbf{N}_{\mathcal{K}}^{-1})(\mathfrak{a}) \int_{\mathbb{Z}_p^\times} \eta_v(\phi|[\mathfrak{a}])\, \mathrm{d}\mu_{f_{\mathfrak{a}}^\flat},$$

for all continuous characters $\phi : \Gamma \to \overline{\mathbb{Q}}_p^\times$, where:

- $f^\flat = \sum_{p \nmid n} a_n(f) q^n$ is the p-depletion of f,
- $\mu_{f_{\mathfrak{a}}^\flat}$ is the measure on $\mathbb{Z}_p^\times$ corresponding (under the Amice transform) to the power series
$$f^\flat(t_{\mathfrak{a}}^{\mathbf{N}_{\mathcal{K}}(\mathfrak{a})c\sqrt{-D_{\mathcal{K}}}^{-1}}) \in \mathscr{O}'^{\mathrm{ur}}[\![t_{\mathfrak{a}} - 1]\!]$$
with $t_{\mathfrak{a}}$ the Serre–Tate coordinate of the reduction of the point $x_{\mathfrak{a}}$ on the Igusa tower of tame level N constructed in [CH18, (2.5)],
- $\eta_v(x) := \eta(\mathrm{rec}_v(x))$ with $\mathrm{rec}_v : \mathbb{Q}_p^\times = \mathcal{K}_v^\times \to G_{\mathcal{K}}^{\mathrm{ab}} \twoheadrightarrow \Gamma$ the local reciprocity map at v,

[5]Therefore, $\chi\xi\mathbf{N}_{\mathcal{K}}^{-1} \in \Sigma_{\mathrm{cc}}^{(2)}(c, \mathfrak{N}, \varepsilon_f)$.

- $\phi|[\mathfrak{a}] : \mathbb{Z}_p^\times \to \overline{\mathbb{Q}}_p^\times$ is defined by $(\phi|[\mathfrak{a}])(x) = \phi(\mathrm{rec}_v(x)\sigma_{\mathfrak{a}}^{-1})$ with $\sigma_{\mathfrak{a}}$ the Artin symbol of $\mathfrak{a}$.

The same calculation as in [CH18, Prop. 3.8] then shows that the element $\mathscr{L}_v(f,\chi) \in \Lambda^{\mathrm{ur}}$ defined by

$$\mathscr{L}_v(f,\chi)(\xi) := \mathfrak{L}_{v,\eta}(f,\chi)(\eta^{-1}\xi)$$

has, in view of the explicit Waldspurger formula in [Hsi14b, Thm. 3.14], the stated interpolation property up to a fixed element in $\Phi^\times$. The result follows. □

Remark 3.1.2. We our later use, we note that the complex period $\Omega_{\mathcal{K}} \in \mathbb{C}^\times$ in Theorem 6.0.1 (which also agrees with that in [BDP13, (5.1.16)]) is *different* from the complex period $\Omega_\infty \in \mathbb{C}^\times$ defined in [dS87, p. 66] and [HT93, (4.4b)]. In fact, one has

$$\Omega_{\mathcal{K}} = 2\pi i \cdot \Omega_\infty.$$

In terms of Ω_∞, the interpolation formula in Theorem 6.0.1 reads

$$\mathscr{L}_v(f,\chi)^2(\xi) = \frac{\Omega_p^{4n}}{\Omega_\infty^{4n}} \cdot \frac{\Gamma(n)\Gamma(n+1)\xi(\mathfrak{N}^{-1})}{4\pi^{1-2n}(\mathrm{Im}\,\vartheta)^{2n}} \cdot \left(1 - a_p(f)\chi\xi(\overline{v})p^{-1} + \varepsilon_f(p)\chi\xi(\overline{v})^2p^{-1}\right)^2 \cdot L(f,\chi\xi,1).$$

Specialized to the range of critical values for the representation $V_{f,\chi}$, the Iwasawa–Greenberg main conjecture [Gre94] predicts the following.

Conjecture 3.1.3. *The module $\mathcal{X}_v(f,\chi)$ is Λ-torsion, and*

$$\mathrm{char}_\Lambda(\mathcal{X}_v(f,\chi)) = (\mathscr{L}_v(f,\chi)^2).$$

In Theorem 4.0.2, we will explain the relation between Conjectures 1.3.1 and 3.1.3.

3.2. **Katz p-adic L-functions.** Let $\mathcal{K}$ be an imaginary quadratic field satisfying (spl). Let $\mathfrak{c} \subset \mathcal{O}_{\mathcal{K}}$ be an ideal prime to p, and let $\mathcal{K}(\mathfrak{c}p^\infty)$ be the ray class field of $\mathcal{K}$ of conductor $\mathfrak{c}p^\infty$.

The p-adic L-function in the next theorem follows from the work of Katz [Kat78], as extended by Hida–Tilouine [HT93] (see also [dS87]). Here we shall use the construction in [Hsi14a], and similarly as in Theorem 3.1.1, it will suffice for our purposes to describe a fixed $\Phi^\times$-multiple of the integral measure constructed in *op.cit.*.

For any Hecke character ξ of $\mathcal{K}$, we denote by $L_{\mathfrak{c}}(\xi,s)$ the Hecke L-function $L(\xi,s)$ with the Euler factors at the primes $\mathfrak{l}|\mathfrak{c}$ removed.

Theorem 3.2.1. *Let ϕ be a character of* $\mathrm{Gal}(\mathcal{K}(\mathfrak{c}p^\infty)/\mathcal{K})$ *corresponding to a Hecke character of $\mathcal{K}$ of infinity type $(-1-j,j)$, with $j \in \mathbb{Z}_{\geqslant 0}$ and conductor prime to p. There exists an element $\mathscr{L}_v(\phi) \in \Lambda^{\mathrm{ur}}$ such that for every character ξ of Γ crystalline at both v and $\bar{v}$ and corresponding to a Hecke character of $\mathcal{K}$ of infinity type $(-n,n)$ with $n+j \in \mathbb{Z}_{\geqslant 0}$, we have*

$$\mathscr{L}_v(\phi)(\xi) = \frac{\Omega_p^{2n+2j+1}}{\Omega_\infty^{2n+2j+1}} \cdot \Gamma(n+j+1) \cdot \frac{(2\pi)^{n+j}}{(\mathrm{Im}\,\vartheta)^{n+j}} \cdot \left(1 - \phi^{-1}\xi^{-1}(\overline{v})\right)^2 \cdot L_{\mathfrak{c}}(\phi^{-1}\xi^{-1},0),$$

where Ω_p, $\Omega_\infty = (2\pi i)^{-1}\Omega_{\mathcal{K}}$, and ϑ are as in Theorem 6.0.1. Similarly, if ϕ as above has infinity type $(j,-1-j)$, with $j \in \mathbb{Z}_{\geqslant 0}$, there exists an element $\mathscr{L}_{\overline{v}}(\phi) \in \Lambda^{\mathrm{ur}}$ such that for every character ξ of Γ crystalline at both v and $\bar{v}$ and corresponding to a Hecke character of $\mathcal{K}$ of infinity type $(n,-n)$ with $n+j \in \mathbb{Z}_{\geqslant 0}$, we have

$$\mathscr{L}_{\overline{v}}(\phi)(\xi) = \frac{\Omega_p^{2n+2j+1}}{\Omega_\infty^{2n+2j+1}} \cdot \Gamma(n+j+1) \cdot \frac{(2\pi)^{n+j}}{(\mathrm{Im}\,\vartheta)^{n+j}} \cdot \left(1 - \phi^{-1}\xi^{-1}(v)\right)^2 \cdot L_{\mathfrak{c}}(\phi^{-1}\xi^{-1},0).$$

Proof. Let $\mathfrak{L}_v$ be the integral p-adic measure on $\mathrm{Gal}(\mathcal{K}(\mathfrak{c}p^\infty)/\mathcal{K})$ constructed in [Hsi14a, §4.8], associated to the p-adic CM type corresponding to our fixed embedding $\imath_\infty, \imath_p$. Setting

$$\mathscr{L}_v(\phi)(\xi) := \mathfrak{L}_v(\phi^{-1} \cdot \pi^*\xi^{-1})$$

for all characters ξ of Γ, where $\pi^*\xi$ is the pullback under the projection $\pi : \mathrm{Gal}(\mathcal{K}(\mathfrak{c}p^\infty)/\mathcal{K}) \to \Gamma$, that $\mathscr{L}_v(\phi)$ satisfies the claimed interpolation property follows from [Hsi14a, Prop. 4.9]. Taking the conjugate CM type, the result for $\mathscr{L}_{\overline{v}}(\phi)$ also follows. ☐

3.3. **Factorization.** As in §2.3, we now specialize to the case where $f = \theta_\psi$ for a Hecke character ψ of $\mathcal{K}$ of infinity type $(-1, 0)$ and conductor $\mathfrak{f}_\psi$ with cyclic quotient of norm prime to $D_\mathcal{K}$, so that $\mathcal{K}$ satisfies hypothesis (Heeg) relative to $N = D_\mathcal{K} \cdot \mathbf{N}_\mathcal{K}(\mathfrak{f}_\psi)$.

Fix an integer $c > 0$ prime to Np, and let χ be a finite order Hecke character of $\mathcal{K}$ such that $\chi\mathbf{N}_\mathcal{K}^{-1} \in \Sigma_{\mathrm{cc}}^{(1)}(c, \mathfrak{N}, \varepsilon_f)$. Then we have a $G_\mathcal{K}$-module decomposition

$$V_{f,\chi} \simeq V_{\psi\chi} \oplus V_{\psi^*\chi}, \tag{3.1}$$

where $V_{f,\chi}$ is as in (2.1). Note that each of the characters appearing in the right-hand side are *self-dual*, in the sense that

$$\psi\chi\psi^*\chi^* = \mathbf{N}_\mathcal{K}$$

(see e.g. [BDP12, Rem. 3.7]).

The following result is a manifestation of the Artin formalism arising from the decomposition (3.1). A similar result in shown in [BDP12, Thm. 3.17]. As we shall see in §6, this is a counterpart on the analytic side of the Selmer group decomposition in Proposition 2.3.1.

Proposition 3.3.1. *Suppose that $f = \theta_\psi$ and χ are as above. Then*

$$\mathscr{L}_v(f, \chi)^2 = u \cdot \mathscr{L}_{\overline{v}}(\psi\chi) \cdot \mathscr{L}_{\overline{v}}(\psi^*\chi),$$

where u is a unit in $(\Lambda^{\mathrm{ur}})^\times$.

Proof. This will follow by comparing the values interpolated by each side of the desired equality, using that an element in Λ^{ur} is uniquely determined by its values at infinitely many characters. Denote by $\iota : \Lambda^{\mathrm{ur}} \to \Lambda^{\mathrm{ur}}$ the involution given by $\gamma \mapsto \gamma^{-1}$ for $\gamma \in \Gamma$. We first claim that

$$\mathscr{L}_v(f, \chi)^2 = u \cdot \mathscr{L}_v(\psi^{-1}\chi^{-1}\mathbf{N}_\mathcal{K})^\iota \cdot \mathscr{L}_v((\psi^*)^{-1}\chi^{-1}\mathbf{N}_\mathcal{K})^\iota \tag{3.2}$$

for some unit $u \in (\Lambda^{\mathrm{ur}})^\times$. Indeed, let ξ be a character of Γ as in the statement of Theorem 3.1.1, of infinity type $(n, -n)$ with $n \geqslant 1$. The decomposition (2.6) yields

$$L(f, \chi\xi, 1) = L(f, \chi\xi\mathbf{N}_\mathcal{K}^{-1}, 0) = L(\psi\chi\xi\mathbf{N}_\mathcal{K}^{-1}, 0) \cdot L(\psi^*\chi\xi\mathbf{N}_\mathcal{K}^{-1}, 0). \tag{3.3}$$

The factors in the right-hand side of (3.3) are interpolated by the values at ξ^{-1} of $\mathscr{L}_v(\psi^{-1}\chi^{-1}\mathbf{N}_\mathcal{K})$ and $\mathscr{L}_v((\psi^*)^{-1}\chi^{-1}\mathbf{N}_\mathcal{K})$, respectively. Noting that

$$\big(1 - \psi\chi\xi(\overline{v})p^{-1}\big) \cdot \big(1 - \psi^*\chi\xi(\overline{v})p^{-1}\big) = \big(1 - a_p(f)\chi\xi(\overline{v})p^{-1} + \varepsilon_f(p)\chi\xi(\overline{v})^2p^{-1}\big),$$

from Theorem 3.2.1 with $j = -1$ and $j = 0$ we find

$$\begin{aligned}\mathscr{L}_v(\psi^{-1}\chi^{-1}\mathbf{N}_\mathcal{K})(\xi^{-1}) \cdot \mathscr{L}_v((\psi^*)^{-1}\chi^{-1}\mathbf{N}_\mathcal{K})(\xi^{-1}) &= \frac{\Omega_p^{2n-1}}{\Omega_\infty^{2n-1}} \cdot \frac{\Omega_p^{2n}}{\Omega_\infty^{2n}} \cdot \Gamma(n)\Gamma(n+1) \cdot \frac{(2\pi)^{n-1}}{(\mathrm{Im}\,\vartheta)^{n-1}} \frac{(2\pi)^n}{(\mathrm{Im}\,\vartheta)^n} \\ &\times \big(1 - a_p(f)\chi\xi(\overline{v})p^{-1} + \varepsilon_f(p)\chi\xi(\overline{v})^2p^{-1}\big) \cdot L(f, \chi\xi, 1).\end{aligned}$$

The proof of (3.2) thus follows from Theorem 3.1.1 and Remark 3.1.2.

Now, noting that the characters $\psi\chi$ and $\psi^*\chi$ are both self-dual and ξ is anticyclotomic, we find

$$\mathscr{L}_v(\psi^{-1}\chi^{-1}\mathbf{N}_\mathcal{K})(\xi^{-1}) = \mathscr{L}_v(\psi^*\chi^*)(\xi^*) = \mathscr{L}_{\overline{v}}(\psi\chi)(\xi),$$

where the last equality follows from another direct comparison of interpolation properties (see e.g. [BCG+19, Lem. 3.3.2(a)]). Similarly, we find $\mathscr{L}_v((\psi^*)^{-1}\chi^{-1}\mathbf{N}_\mathcal{K})(\xi^{-1}) = \mathscr{L}_{\overline{v}}(\psi^*\chi)(\xi)$, so the result follows from (3.2). ☐

4. Explicit reciprocity law

In this section we explain a variant of the explicit reciprocity law [CH18] relating the Λ-adic Heegner class $\mathbf{z}_{f,\chi}$ to the p-adic L-function $\mathscr{L}_v(f,\chi)$ via a Perrin-Riou big logarithm map, and record a key consequence.

Let (f,χ) be as in §2.1. For every $w|p$, the natural map $\mathrm{H}^1(\mathcal{K}_w, \mathscr{F}_w^+\mathbf{T}_{f,\chi}) \to \mathrm{H}^1(\mathcal{K}_w, \mathbf{T}_{f,\chi})$ induced by (2.2) is injective, since its kernel is $\mathrm{H}^0(\mathcal{K}_w, \mathscr{F}_w^-\mathbf{T}_{f,\chi}) = 0$. Therefore, in view of (2.4) the image of $\mathbf{z}_{f,\chi}$ under the restriction map

$$\mathrm{loc}_w : \mathrm{H}^1(\mathcal{K}, \mathbf{T}_{f,\chi}) \to \mathrm{H}^1(\mathcal{K}_w, \mathbf{T}_{f,\chi})$$

is naturally contained in $\mathrm{H}^1(\mathcal{K}_w, \mathscr{F}_w^+\mathbf{T}_{f,\chi})$. Let Φ^{ur} the compositum of Φ and $\mathbb{Q}_p^{\mathrm{ur}}$.

Recall that $\mathcal{K}$ is assumed to satisfy (spl), and v denotes the prime of $\mathcal{K}$ above p induced by our fixed embedding $\imath_p$.

Theorem 4.0.1. *There is a Λ^{ur}-linear isomorphism $\mathrm{Log}_v : \mathrm{H}^1(\mathcal{K}_v, \mathscr{F}_v^+\mathbf{T}_{f,\chi}) \otimes \Lambda^{\mathrm{ur}} \to \Lambda^{\mathrm{ur}}$ such that*

$$\mathrm{Log}_v\big(\mathrm{loc}_v(\mathbf{z}_{f,\chi})\big) = c \cdot \mathscr{L}_v(f,\chi)$$

for some $c \in (\Phi^{\mathrm{ur}})^\times$.

Proof. The existence of the map Log_v (with coefficients in Λ_0^{ur}, rather than Λ^{ur}) follows from the two-variable extension by Loeffler–Zerbes [LZ14] of Perrin-Riou's big logarithm map [PR94], and the proof of the explicit reciprocity law (integrally) is given in [CH18, §5.3]. That the Λ^{ur}-linear map Log_v is injective follows from [LZ14, Prop. 4.11], and so it becomes an isomorphism after extending scalars to $\Lambda^{\mathrm{ur}} = \Lambda_0^{\mathrm{ur}} \otimes_{\mathscr{O}} \Phi$. □

Similarly as observed in [Cas13] and [Wan14], the equivalence between Conjectures 1.3.1 and 3.1.3 can be deduced from Theorem 4.0.1 using Poitou–Tate global duality.

Theorem 4.0.2. *Assume that the class $\mathbf{z}_{f,\chi}$ is not Λ-torsion. Then Conjectures 1.3.1 and 3.1.3 are equivalent.*

Proof. This can be shown in the same way as [CW16, Thm. 5.16], but since here we are working in a different setting, we provide the details. We explain the implication from Conjecture 3.1.3 to Conjecture 1.3.1 (the only implication we will need later), and note that the converse follows from the same ideas.

Following [Cas17, §2.1], below we denote by $\mathcal{S}_{\mathrm{str,rel}}(f,\chi)$ (resp. $\mathcal{S}_{\mathrm{ord,rel}}(f,\chi)$, etc.) the Selmer group defined as in §2.1 but with the strict at v and relaxed at $\overline{v}$ (resp. ordinary at v and relaxed at $\overline{v}$, etc.) local conditions. We also use implicitly use the isomorphisms (2.3) and (2.4).

Now assume Conjecture 3.1.3, so in particular $\mathcal{X}_v(f,\chi)$ is Λ-torsion. Then $\mathcal{S}_{\mathrm{str,rel}}(f,\chi)$ is also Λ-torsion, and global duality yields the following exact sequence

$$0 \to \mathcal{S}_{\mathrm{str,rel}}(f,\chi) \to \mathcal{S}_{\mathrm{ord,rel}}(f,\chi) \xrightarrow{\mathrm{loc}_v} \mathrm{H}^1(\mathcal{K}_v, \mathscr{F}_v^+\mathbf{T}_{f,\chi}) \to \mathcal{X}_v(f,\chi) \to \mathcal{X}_{\mathrm{str,ord}}(f,\chi) \to 0. \tag{4.1}$$

Since $\mathrm{H}^1(\mathcal{K}_v, \mathscr{F}_v^+\mathbf{T}_{f,\chi})$ has Λ-rank one, the assumption that $\mathbf{z}_{f,\chi}$ is non-torsion together with Theorem 4.0.1 implies that $\mathcal{S}_{\mathrm{ord,rel}}(f,\chi)$ has Λ-rank one. Since $\mathbf{z}_{f,\chi} \in \mathcal{S}_{\mathrm{ord}}(f,\chi) \subset \mathcal{S}_{\mathrm{ord,rel}}(f,\chi)$, it follows that $\mathcal{S}_{\mathrm{ord}}(f,\chi)$ also has Λ-rank one, and by [Cas17, Lem. 2.3(1)] so does $\mathcal{X}_{\mathrm{ord}}(f,\chi)$.

Hence the quotient $\mathcal{S}_{\mathrm{ord,rel}}(f,\chi)/\mathcal{S}_{\mathrm{ord}}(f,\chi)$ is Λ-torsion, and since it injects in $\mathrm{H}^1(\mathcal{K}_v, \mathscr{F}_v^+\mathbf{T}_{f,\chi})$ which is Λ-torsion-free, this shows the equality $\mathcal{S}_{\mathrm{ord}}(f,\chi) = \mathcal{S}_{\mathrm{ord,rel}}(f,\chi)$. Thus we see that (4.1) reduces to the exact sequence

$$0 \to \mathcal{S}_{\mathrm{str,ord}}(f,\chi) \to \mathcal{S}_{\mathrm{ord}}(f,\chi) \xrightarrow{\mathrm{loc}_v} \mathrm{H}^1(\mathcal{K}_v, \mathscr{F}_v^+\mathbf{T}_{f,\chi}) \to \mathcal{X}_{\mathrm{ord,rel}}(f,\chi) \to \mathcal{X}_{\mathrm{ord}}(f,\chi) \to 0. \tag{4.2}$$

Since $\mathcal{S}_{\mathrm{str,ord}}(f,\chi)$ is Λ-torsion and $\mathrm{H}^1(\mathcal{K},\mathbf{T}_{f,\chi})$ trivial Λ-torsion-free, $\mathcal{S}_{\mathrm{str,ord}}(f,\chi)$ vanishes, and therefore (4.2) yields

$$0\to\frac{\mathcal{S}_{\mathrm{ord}}(f,\chi)}{\Lambda\cdot\mathbf{z}_{f,\chi}}\xrightarrow{\mathrm{loc}_v}\frac{\mathrm{H}^1(\mathcal{K}_v,\mathbf{T}_{f,\chi})}{\Lambda\cdot\mathrm{loc}_v(\mathbf{z}_{f,\chi})}\to\mathrm{coker}(\mathrm{loc}_v)\to 0.$$

In view of Theorem 4.0.1, it follows that

$$\mathrm{char}_\Lambda\Big(\frac{\mathcal{S}_{\mathrm{ord}}(f,\chi)}{\Lambda\cdot\mathbf{z}_{f,\chi}}\Big)\cdot\mathrm{char}_\Lambda\big(\mathrm{coker}(\mathrm{loc}_v)\big)\Lambda^{\mathrm{ur}}=(\mathscr{L}_v(f,\chi)).\tag{4.3}$$

Next, from (4.1) and (4.2) we can extract the short exact sequences

$$0\to\mathrm{coker}(\mathrm{loc}_v)\to\mathcal{X}_v(f,\chi)\to\mathcal{X}_{\mathrm{str,ord}}(f,\chi)\to 0,$$
$$0\to\mathrm{coker}(\mathrm{loc}_v)\to\mathcal{X}_{\mathrm{ord,rel}}(f,\chi)\to\mathcal{X}_{\mathrm{ord}}(f,\chi)\to 0,$$

from which we readily obtain (taking Λ-torsion in the first exact sequence and using a straightforward variant of [Cas17, Lem. 2.3] or [BL18, Prop. 3.14]) the relations

$$\begin{aligned}\mathrm{char}_\Lambda\big(\mathcal{X}_v(f,\chi)\big)&=\mathrm{char}_\Lambda\big(\mathcal{X}_{\mathrm{str,ord}}(f,\chi)\big)\cdot\mathrm{char}_\Lambda\big(\mathrm{coker}(\mathrm{loc}_v)\big)\\&=\mathrm{char}_\Lambda\big(\mathcal{X}_{\mathrm{ord,rel}}(f,\chi)_{\Lambda\text{-tors}}\big)\cdot\mathrm{char}_\Lambda\big(\mathrm{coker}(\mathrm{loc}_v)\big)\\&=\mathrm{char}_\Lambda\big(\mathcal{X}_{\mathrm{ord}}(f,\chi)_{\Lambda\text{-tors}}\big)\cdot\mathrm{char}_\Lambda(\mathrm{coker}(\mathrm{loc}_v))^2.\end{aligned}$$

Combined with (4.3), we thus obtain

$$\mathrm{char}_\Lambda\big(\mathcal{X}_v(f,\chi)\big)\cdot\mathrm{char}_\Lambda\Big(\frac{\mathcal{S}_{\mathrm{ord}}(f,\chi)}{\Lambda\cdot\mathbf{z}_{f,\chi}}\Big)^2\Lambda^{\mathrm{ur}}=\mathrm{char}_\Lambda\big(\mathcal{X}_{\mathrm{ord}}(f,\chi)_{\Lambda\text{-tors}}\big)\cdot\big(\mathscr{L}_v(f,\chi)^2\big).$$

The result follows. □

5. Twisted anticyclotomic main conjectures for $\mathcal{K}$

Let $\mathcal{K}$ be an imaginary quadratic field satisfying (spl). The Iwasawa main conjecture for $\mathcal{K}$ was proved by Rubin [Rub91] under some restrictions on p (including $p\nmid\mathcal{O}_{\mathcal{K}}^\times$) that were removed in subsequent work by Johnson-Leung–Kings [JLK11] and Oukhaba–Viguié [OV16]. In this section we record a consequence of these results for the anticyclotomic $\mathbb{Z}_p$-extension.

Note that if ξ is a self-dual Hecke character, i.e.,

$$\xi\xi^*=\mathbf{N}_{\mathcal{K}}$$

(so ξ is necessarily of infinity type $(-1-j,j)$, for some $j\in\mathbb{Z}$), then the Hecke L-function $L(\xi^{-1},s)$ is self-dual, with a functional equation relating its values at s and $-s$. In the following, by the *sign* of ξ we refer to the sign appearing in this functional equation.

Theorem 5.0.1. *Let ψ be a Hecke character of $\mathcal{K}$ of infinity type $(-1,0)$, and let χ be a finite order of character of such that the product $\psi\chi$ is self-dual. Assume that $\psi^*\chi$ has sign $+1$. Then the modules $\mathcal{X}_v(\psi\chi)$ and $\mathcal{X}_v(\psi^*\chi)$ are both Λ-torsion, and we have*

$$\mathrm{char}_\Lambda\big(\mathcal{X}_v(\psi\chi)\big)=\big(\mathscr{L}_{\overline{v}}(\psi\chi)\big),\quad\mathrm{char}_\Lambda\big(\mathcal{X}_v(\psi^*\chi)\big)=\big(\mathscr{L}_{\overline{v}}(\psi^*\chi)\big).$$

Proof. Consider first the result for $\psi^*\chi$. The Iwasawa module $\mathcal{X}_v(\psi^*\chi)$ recovers the Bloch–Kato Selmer group for $\psi^*\chi$ over the anticyclotomic $\mathbb{Z}_p$-extension (see Remark 2.2.2), and so the result follows from [AH06, Thm. 2.4.17], as extended in [Arn07, Thm. 3.9]. (In these references, the hypothesis $p>3$ arises from their appearance in [Rub91], but as already noted this restriction can be removed thanks to [JLK11, OV16].)

On the other hand, similarly as in [Agb07, Cor. 3.3] (an adaptation of the argument in [Coa83, Thm. 12]), we see that $\mathcal{X}_v(\psi\chi)$ is isomorphic as a Λ-module to the twist of $\mathcal{X}_v(\psi^*\chi)$ by $\psi(\psi^*)^{-1}$, and therefore is also Λ-torsion if $\psi^*\chi$ has sign $+1$. Since by definition $\mathscr{L}_{\overline{v}}(\psi\chi)$ is similarly the twist of $\mathscr{L}_{\overline{v}}(\psi^*\chi)$ by the character $\psi(\psi^*)^{-1}$, the first equality of characteristic ideals follows from the second. □

6. The main results

Recall that $\mathcal{K}$ is an imaginary quadratic field of discriminant $-D_{\mathcal{K}} < 0$ satisfying (spl), with v the prime of $\mathcal{K}$ above p induced by our fixed embedding $\imath_p$.

Theorem 6.0.1. *Let ψ be a Hecke character of $\mathcal{K}$ of infinity type $(-1,0)$ and conductor $\mathfrak{f}_\psi$ with cyclic quotient of norm prime to $D_{\mathcal{K}}$, and set*

$$f = \theta_\psi, \quad N = D_{\mathcal{K}} \cdot \mathbf{N}_{\mathcal{K}}(\mathfrak{f}_\psi), \quad \mathfrak{N} = \mathfrak{d}_{\mathcal{K}} \cdot \mathfrak{f}_\psi.$$

Let c be a positive integer prime to Np, and let χ be a finite order character such that $\chi\mathbf{N}_{\mathcal{K}}^{-1} \in \Sigma_{\rm cc}^{(1)}(c, \mathfrak{N}, \varepsilon_f)$. Assume that $\psi\chi$ has sign -1. Then:

(i) *The class $\mathbf{z}_{f,\chi}$ is not Λ-torsion.*
(ii) *The the module $\mathcal{X}_v(f,\chi)$ is Λ-torsion, and*

$$\mathrm{char}_\Lambda\big(\mathcal{X}_v(f,\chi)\big)\Lambda^{\rm ur} = \big(\mathscr{L}_v(f,\chi)^2\big).$$

In other words, Conjecture 3.1.3 holds.

Proof. We begin by showing that the class $\mathbf{z}_{f,\chi}$ is not Λ-torsion. As noted in §3.3, our assumption on $\mathfrak{f}_\psi$ implies that $\mathcal{K}$ satisfies hypothesis (Heeg) relative to the level of f. In particular, the Rankin–Selberg L-function $L(f,\chi,s)$ has a functional equation relating its values at s and $2-s$ with sign -1. The $G_{\mathcal{K}}$-module decomposition (2.6) yields

$$L(f,\chi,s) = L(\psi\chi,s) \cdot L(\psi^*\chi,s),$$

and each of the factors in the right-hand side has a functional equation relating its values at s and $2-s$. Since we assume that $\psi\chi$ has sign -1, the self-dual character $\psi^*\chi$ has sign $+1$. By Rohrlich's theorem [Roh84], for all but finitely characters ξ of Γ, we have

$$L(\psi^*\chi\xi, 1) \neq 0.$$

Hence in view of the interpolation property in Theorem 3.2.1, the p-adic L-function $\mathscr{L}_{\overline{v}}(\psi^*\chi)$ is nonzero, and therefore so is its twist $\mathscr{L}_{\overline{v}}(\psi\chi)$. By the factorization in Proposition 3.3.1, we conclude that $\mathscr{L}_v(f,\chi)$ is also nonzero, and the claim that $\mathbf{z}_{f,\chi}$ is not Λ-torsion now follows from the explicit reciprocity law of Theorem 4.0.1.

For part (ii), by Theorem 5.0.1 the modules $\mathcal{X}_v(\psi\chi)$ and $\mathcal{X}_v(\psi^*\chi)$ are both Λ-torsion, with characteristic ideals generated by $\mathscr{L}_{\overline{v}}(\psi\chi)$ and $\mathscr{L}_{\overline{v}}(\psi^*\chi)$ over $\Lambda^{\rm ur}$, respectively. Thus from Propositions 2.3.1 and 3.3.1 we obtain that $\mathcal{X}_v(f,\chi)$ is Λ-torsion, with characteristic ideal generated by $\mathscr{L}_v(f,\chi)^2$ over $\Lambda^{\rm ur}$. □

Corollary 6.0.2. *Let $f = \theta_\psi$ and χ be an in Theorem 6.0.1, and assume that $\psi\chi$ has sign -1. Then the modules $\mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty)$ and $\mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty)$ have both Λ-rank one, and*

$$\mathrm{char}_\Lambda(\mathcal{X}(A_{f,\chi}/\mathcal{K}_\infty)_{\Lambda\text{-tors}}) = \mathrm{char}_\Lambda\big(\mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty)/\Lambda\cdot\mathbf{z}_{f,\chi}\big)^2.$$

In other words, Conjecture 1.3.1 holds.

Proof. This is the combination of Theorem 4.0.2 and Theorem 6.0.1. □

7. Application to p-converse

In this section we deduce from our main results the proof of Theorem A in the Introduction. Let λ be a self-dual Hecke character of infinity type $(-1,0)$ and conductor $\mathfrak{f}_\psi$, and suppose that:

(a) λ has sign -1;
(b) λ has central character $\omega_\lambda = \eta_\mathcal{K}$;
(c) $\mathfrak{d}_\mathcal{K} \| \mathfrak{f}_\lambda$.

Note that $\mathfrak{f}_\lambda$ is divisible by $\mathfrak{d}_\mathcal{K} = (\sqrt{-D_\mathcal{K}})$ by condition (b). Since λ is self-dual, $\mathfrak{f}_\lambda$ is invariant under complex conjucation, so by condition (c) we can write $\mathfrak{f}_\lambda = (c)\mathfrak{d}_\mathcal{K}$ for a unique $c > 0$.

We shall apply Corollary 6.0.2 for a pair (ψ,χ) which is *good for* λ in the sense of [BDP12, Def. 3.19]:

(G1) ψ has infinity type $(-1,0)$ and conductor $\mathfrak{f}_\psi$ with cyclic quotient of norm prime to $pD_\mathcal{K}$;
(G2) χ is a finite order character such that $\chi\mathbf{N}_\mathcal{K}^{-1} \in \Sigma_{\rm cc}^{(1)}(c,\mathfrak{N},\varepsilon_f)$, where $f = \theta_\psi$ and $\mathfrak{N} = \mathfrak{f}_\psi\mathfrak{d}_\mathcal{K}$;
(G3) $\psi\chi = \lambda$;
(G4) $L(\psi^*\chi, 1) \neq 0$.

The existence of good pairs for λ is shown in [BDP12, Lem. 3.29] building on the non-vanishing results of Greenberg [Gre85] and Rohrlich [Roh84].

Fix a good pair (ψ,χ) for λ, and let $F \subset \overline{\mathbb{Q}}$ be a number field of containing the values of ψ and χ. Let $\mathfrak{P}$ be the prime of F above p induced by our fixed embedding $\imath_p$, let Φ be the completion of F at $\mathfrak{P}$, and let $\mathscr{O}$ be the ring of integers of Φ. Let $W_{\psi\chi}$ be the module $\Phi/\mathscr{O}$ equipped with the $G_\mathcal{K}$-action via $\psi\chi$, and define $W_{\psi^*\chi}$ similarly.

Let Σ a finite set of places of $\mathcal{K}$ containing ∞, p, and the primes of $\mathcal{K}$ dividing the conductor of λ. Denote by $\mathrm{H}^1_{\rm f}(\mathcal{K}, W_{\psi\chi})$ the *Bloch–Kato Selmer group* for $\psi\chi$:

$$\mathrm{H}^1_{\rm f}(\mathcal{K}, W_{\psi\chi}) = \ker\left\{\mathrm{H}^1(\mathcal{K}^\Sigma/\mathcal{K}, W_{\psi\chi}) \to \frac{\mathrm{H}^1(\mathcal{K}_v, W_{\psi\chi})}{\mathrm{H}^1(\mathcal{K}_v, W_{\psi\chi})_{\rm div}} \times \mathrm{H}^1(\mathcal{K}_{\overline{v}}, W_{\psi\chi}) \times \prod_{w\in\Sigma, w\nmid p} \mathrm{H}^1(\mathcal{K}_w^{\rm ur}, W_{\psi\chi})\right\},$$

where $\mathrm{H}^1(\mathcal{K}_v, W_{\psi\chi})_{\rm div} \subset \mathrm{H}^1(\mathcal{K}_v, W_{\psi\chi})$ is the maximal divisible submodule and $\mathcal{K}_w^{\rm ur}$ denote the maximal unramified extension of $\mathcal{K}_w$. Similarly, let

$$\mathrm{H}^1_{\rm f}(\mathcal{K}, W_{\psi^*\chi}) = \ker\left\{\mathrm{H}^1(\mathcal{K}^\Sigma/\mathcal{K}, W_{\psi^*\chi}) \to \mathrm{H}^1(\mathcal{K}_v, W_{\psi^*\chi}) \times \frac{\mathrm{H}^1(\mathcal{K}_v, W_{\psi^*\chi})}{\mathrm{H}^1(\mathcal{K}_v, W_{\psi^*\chi})_{\rm div}} \times \prod_{w\in\Sigma, w\nmid p} \mathrm{H}^1(\mathcal{K}_w^{\rm ur}, W_{\psi^*\chi})\right\}$$

be the Bloch–Kato Selmer group for $\psi^*\chi$ (see also [AH06, §1.1]). Finally, let $V_{f,\chi}$ be as in (2.1).

Lemma 7.0.1. *In the above setting, we have*

$$\mathrm{corank}_\mathscr{O}\mathrm{Sel}_{\mathfrak{P}^\infty}(A_{f,\chi}/\mathcal{K}) = \mathrm{corank}_\mathscr{O}\mathrm{H}^1_{\rm f}(\mathcal{K}, W_{\psi\chi}) + \mathrm{corank}_\mathscr{O}\mathrm{H}^1_{\rm f}(\mathcal{K}, W_{\psi^*\chi}).$$

Proof. As is well-known (see e.g. [BK90]), $\mathrm{Sel}_{\mathfrak{P}^\infty}(A_{f,\chi}/\mathcal{K})$ agrees with the Bloch–Kato Selmer group

$$\mathrm{H}^1_{\rm f}(\mathcal{K}, W_{f,\chi}) \subset \mathrm{H}^1(\mathcal{K}, W_{f,\chi}),$$

where $W_{f,\chi} := V_{f,\chi}/T_{f,\chi}$ for any $G_\mathcal{K}$-stable $\mathscr{O}$-lattice $T_{f,\chi} \subset V_{f,\chi}$. In turn (see [Gre99, Prop. 2.2]), the local conditions defining $\mathrm{H}^1_{\rm f}(\mathcal{K}, W_{f,\chi})$ at the primes $w|p$ can be described in terms of the filtration (2.2). Since different lattices $T_{f,\chi}$ give rise to Selmer groups $\mathrm{H}^1_{\rm f}(\mathcal{K}, W_{f,\chi})$ with the same $\mathscr{O}$-corank, taking $T_{f,\chi}$ so that $W_{f,\chi} \simeq W_{\psi\chi} \oplus W_{\psi^*\chi}$, similarly as in the proof of Proposition 2.3.1 the result follows by comparing the local conditions defining the Bloch–Kato Selmer groups for $W_{f,\chi}$, $W_{\psi\chi}$ and $W_{\psi^*\chi}$. □

The following recovers Theorem A in the introduction as a special case.

Theorem 7.0.2. *Let λ be a self-dual Hecke character of $\mathcal{K}$ of infinity type $(-1,0)$ with central character $\omega_\lambda = \eta_{\mathcal{K}}$ and whose conductor $\mathfrak{f}_\lambda$ satisfies $\mathfrak{d}_{\mathcal{K}} \| \mathfrak{f}_\lambda$. Then*

$$\mathrm{corank}_{\mathscr{O}} \mathrm{Sel}_{\mathfrak{P}^\infty}(B_\lambda/\mathcal{K}) = 1 \quad \Longrightarrow \quad \mathrm{ord}_{s=1} L(\lambda, s) = 1.$$

Proof. Note that by the p-parity conjecture [Nek01], if $\mathrm{Sel}_{\mathfrak{P}^\infty}(B_\lambda/\mathcal{K})$ has $\mathscr{O}$-corank 1, then λ has sign -1. Let (ψ, χ) be a good pair for λ, i.e., satisfying conditions (G1)–(G4) above, so in particular

$$L(\psi^* \chi \mathbf{N}_{\mathcal{K}}^{-1}, 0) = L(\psi^* \chi, 1) \neq 0. \tag{7.1}$$

Since by Theorem 3.2.1 the value $L(\psi\chi^* \mathbf{N}_{\mathcal{K}}^{-1}, 0)$ is a nonzero multiple of the value of $\mathscr{L}_{\overline{v}}(\psi^*\chi)$ at the trivial character $\mathbb{1}$ of Γ, by Theorem 3.2.1 it follows that

$$\#\big(\mathscr{X}_v(\psi^*\chi)/(\gamma - 1)\mathscr{X}_v(\psi^*\chi)\big) < \infty,$$

where $\gamma \in \Gamma$ is any topological generator. Since $\mathscr{X}_v(\psi^*\chi)$ corresponds to the Bloch–Kato Selmer group for $\psi^*\chi$ over $\mathcal{K}_\infty/\mathcal{K}$ (see Remark 2.2.2), it follows that $\mathrm{corank}_{\mathscr{O}} \mathrm{H}^1_{\mathrm{f}}(\mathcal{K}, W_{\psi^*\chi}) = 0$. Since $\mathrm{Sel}_{\mathfrak{P}^\infty}(B_\lambda/\mathcal{K}) \simeq \mathrm{H}^1_{\mathrm{f}}(\mathcal{K}, W_{\psi\chi})$, from Lemma 7.0.1 we thus obtain

$$\mathrm{corank}_{\mathscr{O}} \mathrm{Sel}_{\mathfrak{P}^\infty}(B_\lambda/\mathcal{K}) = 1 \quad \Longrightarrow \quad \mathrm{corank}_{\mathscr{O}} \mathrm{Sel}_{\mathfrak{P}^\infty}(A_{f,\chi}/\mathcal{K}) = 1. \tag{7.2}$$

Now, Corollary 6.0.2 together with a variant of Mazur's control theorem immediately yields the implication

$$\mathrm{corank}_{\mathscr{O}} \mathrm{Sel}_{\mathfrak{P}^\infty}(A_{f,\chi}/\mathcal{K}) = 1 \quad \Longrightarrow \quad \mathbf{z}_{f,\chi,0} \neq 0 \in S_{\mathfrak{P}}(A_{f,\chi}/\mathcal{K}) \otimes_{\mathscr{O}} \Phi. \tag{7.3}$$

Here $\mathbf{z}_{f,\chi,0}$ is the image of $\mathbf{z}_{f,\chi}$ under the specialization map $\mathcal{S}(A_{f,\chi}/\mathcal{K}_\infty) \to \mathrm{H}^1(\mathcal{K}, V_{f,\chi})$ at the trivial character, which agrees with the Heegner class $z_{f,\chi,0}$ (see e.g. [CH18, Lem. 5.4]).

Finally, we note once more that (2.6) yields the factorization

$$L(f, \chi, s) = L(\psi\chi, s) \cdot L(\psi^*\chi, s).$$

Since $\mathbf{z}_{f,\chi,0} \neq 0 \Longleftrightarrow \mathrm{ord}_{s=1} L(f,\chi,s) = 1$ by the general Gross–Zagier formula [YZZ13, CST14], together with (7.2) and (7.3), we conclude that

$$\begin{aligned} \mathrm{corank}_{\mathscr{O}} \mathrm{Sel}_{\mathfrak{P}^\infty}(B_\lambda/\mathcal{K}) = 1 \quad &\Longrightarrow \quad \mathrm{ord}_{s=1} L(f,\chi,s) = 1 \\ &\Longrightarrow \quad \mathrm{ord}_{s=1} L(\psi\chi, s) = 1, \end{aligned}$$

where the second equality follows from (7.1). Since $\psi\chi = \lambda$, this concludes the proof. □

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