ON THE EXCEPTIONAL SPECIALIZATIONS OF BIG HEEGNER POINTS

FRANCESC CASTELLA

Abstract. We extend the $p$-adic Gross–Zagier formula of Bertolini, Darmon, and Prasanna [BDP13] to the semistable non-crystalline setting, and combine it with our previous work [Cas14] to obtain a derivative formula for the specializations of Howard’s big Heegner points [How07b] at exceptional primes in the Hida family.

Contents

Introduction 1
1. Preliminaries 4
  1.1. Modular curves 4
  1.2. Modular forms and cohomology 4
  1.3. $p$-new forms 6
  1.4. $p$-adic modular forms 6
  1.5. Ordinary CM points 7
  1.6. Generalized Heegner cycles 7
2. A semistable non-crystalline setting 8
  2.1. $p$-adic Abel–Jacobi maps 8
  2.2. Rigid cohomology 9
  2.3. Coleman’s $p$-adic integration 11
  2.4. Frobenius and monodromy 12
  2.5. $p$-adic Gross–Zagier formula 13
3. Main result 16
  3.1. Heegner points in Hida families 16
  3.2. Two-variable $p$-adic $L$-functions 18
  3.3. Ochiai’s big logarithm maps 18
  3.4. Exceptional zero formula 20
References 23

Introduction

Fix a prime $p \geq 5$, an integer $N > 0$ prime to $p$, and let $f \in S_2(\Gamma_0(Np))$ be a newform. Throughout this paper, we shall assume that $f$ is split multiplicative at $p$, meaning that

$$f(q) = q + \sum_{n=2}^{\infty} a_n(f)q^n \quad \text{with} \quad a_p(f) = 1.$$ 

Fix embeddings $\mathbb{C} \leftarrow \overline{\mathbb{Q}} \leftarrow \mathbb{C}_p$, let $L$ be a finite extension of $\mathbb{Q}_p$ containing $\mu_{p^\infty}^{-1}(a_n(f))$ for all $n$, and let $\mathcal{O}_L$ be the ring of integers of $L$. Since the $U_p$-eigenvalue of $f$ is $a_p(f) = 1$ by hypothesis, the form $f$ is ordinary at $p$, and hence there is a Hida family

$$f = \sum_{n=1}^{\infty} a_n q^n \in \mathbb{Z}[[q]].$$

Date: July 23, 2015.

Research supported in part by Grant MTM2012-34611 and by Prof. Hida’s NSF Grant DMS-0753991.
passing through \( f \). Here \( \mathcal{I} \) is a finite flat extension of the power series ring \( \mathcal{O}_L[[T]] \), which for simplicity in this introduction it will be assumed to be \( \mathcal{O}_L[[T]] \) itself. Embed \( \mathbb{Z} \) in the space \( \mathcal{X}_{\mathcal{O}_L}([1]) \) of continuous \( \mathcal{O}_L \)-algebra homomorphisms \( \nu : \mathbb{I} \to \overline{\mathbb{Q}}_p \), by identifying \( k \in \mathbb{Z} \) with the homomorphism \( \nu_k : \mathbb{I} \to \overline{\mathbb{Q}}_p \) defined by \( 1 + T \mapsto (1 + p)^{k-2} \). The Hida family \( f \) is then uniquely characterized by the property that for every \( k \in \mathbb{Z}_{\geq 2} \) its weight \( k \) specialization

\[
f_k := \sum_{n=1}^{\infty} \nu_k(a_n)q^n
\]
gives the \( q \)-expansion of a \( p \)-ordinary \( p \)-stabilized newform \( f_k \in S_k(\Gamma_0(Np)) \) with \( f_2 = f \).

Let \( K \) be an imaginary quadratic field equipped with an integral ideal \( \mathfrak{m} \subseteq \mathcal{O}_K \) with \( \mathcal{O}_K/\mathfrak{m} \cong \mathbb{Z}/N\mathbb{Z} \), assume that \( p \) splits in \( K \), and write \( p\mathcal{O}_K = \mathfrak{p}\mathfrak{f} \) with \( \mathfrak{p} \) the prime above \( p \) induced by \( v_p \). If \( A \) is an elliptic curve with CM by \( \mathcal{O}_K \), then the pair \((A, A[9\mathfrak{p}])\) defines a Heegner point \( \mathfrak{P}_A \) on \( X_0(Np) \) defined over the Hilbert class field \( H \) of \( K \). Taking the image of the degree zero divisor \((\mathfrak{P}_A - (\infty)) \) under the composite map

\[
J_0(Np) \xrightarrow{\text{Kum}} H^1(H, \mathfrak{T}_\mathfrak{p}(J_0(Np))) \xrightarrow{\text{Cor}_{H/K}} H^1(H, V_f) \xrightarrow{\text{Aut}_{H/K}} H^1(K, V_f)
\]
yields a class \( \kappa_f \in \text{Sel}(K,V_f) \) in the Selmer group for the \( p \)-adic Galois representation

\[
\rho_f : G_\mathbb{Q} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}_L(V_f) \cong \text{GL}_2(L)
\]
associated to \( f \). On the other hand, by working over a \( p \)-tower of modular curves, Howard [How07b] constructed a so-called big Heegner point \( \mathfrak{z}_0 \in \text{Sel}_{\mathfrak{c}}(K,T^1) \) in the Selmer group for a self-dual twist of the big Galois representation

\[
\rho_{\mathfrak{z}} : G_\mathbb{Q} \to \text{Aut}_\mathbb{Q}(T) \cong \text{GL}_2(\mathbb{Q})
\]
associated to \( f \). The image of \( \mathfrak{z}_0 \) under the specialization map \( \nu_2 : \text{Sel}_{\mathfrak{c}}(K,T^1) \to \text{Sel}(K,V_f) \) induced by \( \nu_2 : 1 \to \overline{\mathbb{Q}}_p \) yields a second class of “Heegner type” in \( \text{Sel}(K,V_f) \); the question of comparing \( \kappa_f \) with \( \nu_2(\mathfrak{z}_0) \) thus naturally arises.

For \( k > 2 \), the question of relating the specializations \( \nu_k(\mathfrak{z}_0) \) to higher dimensional Heegner cycles was considered in [Cas13]. In that case, one could show (see [loc.cit., (5.31)]) that

\[
\log_p(\nu_k(\mathfrak{z}_0)) = u^{-1}\left(1 - \frac{p^{k/2-1}}{\nu_k(\mathfrak{a}_p)}\right)^2 \cdot \log_p(\kappa_{\mathfrak{c}}),
\]

where \( u := |\mathcal{O}_K^*|/2, \) \( \log_p : H^1(K,\mathfrak{V}_f) \to H^1(K_p,\mathfrak{V}_f) \) is the localization map, and \( \kappa_{\mathfrak{c}} \) is a class given by the \( p \)-adic étale Abel–Jacobi images of certain Heegner cycles on a Kuga–Sato variety of dimension \( k - 1 \). However, for the above newform \( f \), the main result of [Cas13] does not immediately yield a similar relation between \( \nu_2(\mathfrak{z}_0) \) and \( \kappa_{\mathfrak{c}} \), since in loc.cit. a crucial use is made of the fact that the \( p \)-adic Galois representations associated with the eigenforms under consideration are (potentially) crystalline at \( p \), whereas \( V_f \) is well-known to be semistable but non-crystalline at \( p \). Moreover, it is easy to see that the expected relation between these two classes may not be given by the naïve extension of (0.2) with \( k = 2 \); indeed, granted the injectivity of \( \log_p \), by the Gross–Zagier formula the class \( \log_p(\kappa_{\mathfrak{c}}) \) is nonzero as long as \( L'(f/K,1) \neq 0 \), whilst (0.2) for \( k = 2 \) would imply the vanishing of \( \log_p(\nu_2(\mathfrak{z}_0)) \) in all cases, since

\[
\left(1 - \frac{p^{k/2-1}}{\nu_k(\mathfrak{a}_p)}\right)|_{k=2} = \left(1 - \frac{1}{\mathfrak{a}_p(f)}\right) = 0.
\]

As shown in [How07b], the class \( \mathfrak{z}_0 \) fits in compatible system of similar classes \( \mathfrak{z}_\infty = \{\mathfrak{z}_n\}_{n \geq 0} \) over the anticyclotomic \( \mathbb{Z}_p \)-extension of \( K \); thus \( \mathfrak{z}_0 \) might be seen as the value of \( \mathfrak{z}_\infty \) at the trivial character. As suggested by the above discussion, in this paper we will show that the class \( \log_p(\nu_2(\mathfrak{z}_0)) \) vanishes, and prove an “exceptional zero formula” relating its derivative at the trivial character (in a precise sense to be defined) to the geometric class \( \kappa_f \). To state the main result, let \( h \) be the class number of \( K \), write \( \mathfrak{p}^h = \pi_p \mathcal{O}_K \), and define

\[
\mathcal{L}_p(f,K) := L_p(f) - \frac{\log_p(\varpi_p)}{\ord_p(\varpi_p)}.
\]
where \( \mathcal{L}_p(f) \) is the \( \mathcal{L} \)-invariant of \( f \) (see [MTT86, §II.14] for example), \( \omega_p := \pi_p / \tau_p \in K_p \simeq Q_p \), and \( \log_p : Q_p^\times \to Z_p \) is Iwasawa’s branch of the \( p \)-adic logarithm.

**Theorem.** Let \( f \in \text{S}_2(\Gamma_0(Np)) \) be a newform split multiplicative at \( p \), and define \( Z_{p, f, \infty} = \{ Z_{p, f, n} \}_{n \geq 0} \) by \( Z_{p, f, n} := \text{loc}_p(\nu_2(3_n)) \). Then \( Z_{p, f, 0} = 0 \) and

\[
Z'_{p, f, 0} = \mathcal{L}_p(f, K) \cdot \text{loc}_p(\kappa_f).
\]

In Lemma 3.10 below, we define the “derivative” \( Z'_{\infty} \) for any compatible system of classes \( Z_{\infty} = \{ Z_n \}_{n \geq 0} \) with \( Z_0 = 0 \). The above result, which corresponds to Theorem 3.11 in the body of the paper, may thus be seen as an exceptional zero formula relating the derivative of \( \text{loc}_p(\nu_2(3_{\infty})) \) at the trivial character to classical Heegner points.

**Remark 1.** As suggested in [LLZ14, §8], one might view \( p \)-adic \( \mathcal{L} \)-functions (as described in [PR00] and [Rub00, Ch. 8]) as “rank 0” Euler–Iwasawa systems. In this view, it is natural to expect higher rank Euler–Iwasawa systems to exhibit exceptional zero phenomena similar to their rank 0 counterparts. We would like to see the main result of this paper as an instance of this phenomenon in “rank 1”.

**Remark 2.** It would be interesting to study the formulation of our main result in the framework afforded by Nekovář’s theory of Selmer complexes [Nek06], similarly as the exceptional zero conjecture of Mazur–Tate–Teitelbaum [MTT86] has recently been proved by Venerucci [Ven15] in the rank 1 case.

**Remark 3.** The second term in the definition (0.4) is precisely the \( \mathcal{L} \)-invariant \( \mathcal{L}_p(\chi_K) \) appearing in the exceptional zero formula of Ferrero–Greenberg [FG79] and Gross–Koblitz [GK79] for the Kubota–Leopoldt \( p \)-adic \( \mathcal{L} \)-function associated to the quadratic Dirichlet character \( \chi_K \) corresponding to \( K \). It would be interesting to find a conceptual explanation for the rather surprising appearance of \( \mathcal{L}_p(\chi_K) \) in our derivative formula; we expect this to be related to a comparison of \( p \)-adic periods (cf. [Cas15]).

The proof of the above theorem is obtained by computing in two different ways the value of a certain anticyclotomic \( p \)-adic \( \mathcal{L} \)-function \( L_p(f) \) at the norm character \( N_{K} \). The \( p \)-adic \( \mathcal{L} \)-function \( L_p(f) \) is defined by the interpolation of the central critical values for the Rankin–Selberg convolution of \( f \) with the theta series attached to Hecke characters of \( \text{S}_2(\Gamma_0(Np)) \), and via a suitable extension of the methods of Bertolini–Darmon–Prasanna [BDP13] to our setting, in Theorem 2.11 we show that

\[
L_p(f) = (1 - a_p(f)p^{-1}) \cdot (\log_{V'_{\kappa}}(\text{loc}_p(\kappa_f), \omega_f)).
\]

On the other hand, in [Cas14] we constructed a two-variable \( p \)-adic \( \mathcal{L} \)-function \( L_p(\xi, f) \) (of the variables \( (\nu, \phi) \)) interpolating (a shift of) the \( p \)-adic \( \mathcal{L} \)-functions \( L_p(\xi, f_k) \) for all \( k \geq 2 \), and established the equality

\[
L_p(\xi, f) = \mathcal{L}_{X + \tau}^\omega(\text{loc}_p(\kappa_{f}^{-1})),
\]

where \( \mathcal{L}_{X + \tau} \) is a two-variable Coleman power series map whose restriction to a certain “line” interpolates

\[
\left( 1 - \frac{p^{k/2-1}}{\nu_k(a_p)} \right)^{-1} \left( 1 + \frac{\nu_k(a_p)}{p^{k/2}} \right) \cdot \log_{V'_{\kappa}}
\]

for all \( k > 2 \). A second evaluation of \( L_p(f)(N_K) \) should thus follow by specializing (0.6) at \( (\nu_2, 1) \). However, because of the vanishing (0.3), we may not directly specialize \( \mathcal{L}_{X + \tau}^\omega \) at \( (\nu_2, 1) \), and we are led to utilize a different argument reminiscent of Greenberg–Stevens’ [GS93]. In fact, from the form of the \( p \)-adic multipliers appearing in the interpolation property defining \( \mathcal{L}_{X + \tau} \), we deduce a factorization

\[
E_p(f) \cdot L_p(f) = \mathcal{L}_{X + \tau}^\omega(\text{loc}_p(\kappa_{f}^{-1})),
\]

upon restricting (0.6) to an appropriate “line” (different from the above) passing through \( (\nu_2, 1) \), where \( \mathcal{L}_{X + \tau}^\omega \) is a modification of \( \mathcal{L}_{X + \tau} \) and \( E_p(f) \) is a \( p \)-adic analytic function vanishing at that point. The vanishing of \( Z_{p, f, 0} \) thus follows, and exploiting the “functional equation” satisfied by \( \mathcal{L}_{X, \infty} \), we arrive at the equality

\[
Z_p(f, K) \cdot L_p(f)(N_K) = (1 - a_p(f)p^{-1}) \cdot (\log_{V'_{\kappa}}(Z_{p, f, 0}) \cdot \omega_f)
\]

using a well-known formula for the \( \mathcal{L} \)-invariant as a logarithmic derivative of \( \nu_k(a_p) \) at \( k = 2 \). The proof of our exceptional zero formula then follows by combining (0.5) and (0.7).
Acknowledgements. We would like to thank Daniel Disegni for conversations related to this work, and the referee for pointing out a number of inaccuracies in an earlier version of this paper, as well as for helpful suggestions which led to significant improvements in the article.

1. Preliminaries

For a more complete and detailed discussion of the topics that we touch upon in this section, we refer the reader to [Col94] and [BDP13].

1.1. Modular curves. Keep \( N \) and \( p \nmid N \) as in the Introduction, and let

\[ \Gamma := \Gamma_1(N) \cap \Gamma_0(p) \subset \text{SL}_2(\mathbb{Z}). \]

An elliptic curve with \( \Gamma \)-level structure over a \( \mathbb{Z}[1/N] \)-scheme \( S \) is a triple \((E,t,\alpha)\) consisting of

\begin{itemize}
  \item an elliptic curve \( E \) over \( S \);
  \item a section \( t : S \to E \) of the structure morphism of \( E/S \) of exact order \( N \); and
  \item a \( p \)-isogeny \( \alpha : E \to E' \).
\end{itemize}

The functor on \( \mathbb{Z}[1/N] \)-schemes assigning to \( S \) the set of isomorphism classes of elliptic curves with \( \Gamma \)-level structure over \( S \) is representable, and we let \( Y/\mathbb{Z}[1/N] \) be the corresponding fine moduli scheme. The same moduli problem for generalized elliptic curves with \( \Gamma \)-level structure defines a smooth geometrically connected curve \( X/\mathbb{Z}[1/N] \) containing \( Y \) as an open subscheme, and we refer to \( Z_X := X \setminus Y \) as the cuspidal subscheme of \( X \). Removing the data of \( \alpha \) from the above moduli problem, we obtain the modular curve \( X_1(N) \) of level \( \Gamma_1(N) \).

For our later use (see esp. Theorem 2.4), recall that if \( a \) is any integer coprime to \( N \), the rule

\[ (a)(E,t,\alpha) = (E,a \cdot t,\alpha) \]

defines an action of \( (\mathbb{Z}/N\mathbb{Z})^\times \) on \( X \) defined over \( \mathbb{Z}[1/N] \), and we let \( X_0(Np) = X/(\mathbb{Z}/N\mathbb{Z})^\times \) be the quotient of \( X \) by this action.

The special fiber \( X_{\mathbb{F}_p} := X \times_{\mathbb{Z}[1/N]} \mathbb{F}_p \) is non-smooth. In fact, it consists of two irreducible components, denoted \( C_0 \) and \( C_\infty \), meeting transversally at the singular points \( SS \). Let Frob be the absolute Frobenius of an elliptic curve over \( \mathbb{F}_p \), and Ver = Frob\(^\vee \) be the Verschiebung. The maps

\[ \gamma_V : X_1(N)_{\mathbb{F}_p} := X_1(N) \times_{\mathbb{Z}[1/N]} \mathbb{F}_p \to X_{\mathbb{F}_p}, \quad \gamma_F : X_1(N)_{\mathbb{F}_p} \to X_{\mathbb{F}_p} \]

defined by sending a pair \((E,t)_{\mathbb{F}_p}\) to \((E,t,\ker(\text{Ver}))\) and \((E,t,\ker(\text{Frob}))\) respectively, are closed immersions sending \( X_1(N)_{\mathbb{F}_p} \) isomorphically onto \( C_0 \) and \( C_\infty \), and mapping the supersingular points in \( X_1(N)_{\mathbb{F}_p} \) bijectively onto \( SS \). The non-singular geometric points of \( C_0 \) (resp. \( C_\infty \)) thus correspond to the moduli of triples \((E,t,\alpha)\) in characteristic \( p \) with \( \ker(\alpha) \) étale (resp. connected).

Corresponding to the preceding description of \( X_{\mathbb{F}_p} \), there is a covering of \( X \) as rigid analytic space over \( \mathbb{Q}_p \). Consider the reduction map

\[ (1.1) \quad \text{red}_p : X(\mathbb{C}_p) \to X_{\mathbb{F}_p}(\bar{\mathbb{F}}_p), \]

let \( W_0 \) and \( W_\infty \) be the inverse image of \( C_0 \) and \( C_\infty \), respectively, and let \( Z_0 \subset W_0 \) and \( Z_\infty \subset W_\infty \) be the inverse image of their non-singular points. In the terminology of [Col89], \( W_0 \) (resp. \( W_\infty \)) is a basic wide open with underlying affinoid \( Z_0 \) (resp. \( Z_\infty \)). If \( x \in SS \), then \( A_x := \text{red}_p^{-1}(x) \) is conformal to an open annulus in \( \mathbb{C}_p \), and by definition we have

\[ X(\mathbb{C}_p) = W_0 \cup W_\infty = Z_0 \cup Z_\infty \cup W, \]

where \( W = W_0 \cap W_\infty = \bigcup_{x \in SS} A_x \) is the union of the supersingular annuli.

1.2. Modular forms and cohomology. In this section, we regard the modular curve \( X \) as a scheme over a fixed base field \( F \). Let \( \mathcal{E} \to X \) be the universal generalized elliptic curve with \( \Gamma \)-level structure, set \( \tilde{Z}_X = \pi^{-1}(Z_X) \), and consider the invertible sheaf on \( X \) given by

\[ \omega := \pi_! \Omega^1_{\mathcal{E}/X}(\log \tilde{Z}_X). \]

The space of algebraic modular forms (resp. cusp forms) of weight \( k \) and level \( \Gamma \) defined over \( F \) is

\[ M_k(X; F) := H^0(X, \omega^\otimes k) \quad \text{(resp. } S_k(X; F) := H^0(X, \omega^\otimes k \otimes I), \text{)} \]
where $\omega_F$ is the pullback of $\omega$ to $X \times \mathbb{Q} F$, and $\mathcal{I}$ is the ideal sheaf of $Z_X \subset X$. If there is no risk of confusion, $F$ will be often suppressed from the notation. Alternatively, on the open modular curve $Y$ a form $f \in \mathcal{S}_k(X; F) \subset M_k(X; F)$ is a rule on quadruples $(E, t, \alpha, \omega) / A$, consisting of an $A$-valued point $(E, t, \alpha) \in Y(A)$ and a differential $\omega \in \Omega^1_{E/A}$ over arbitrary $F$-algebras $A$, assigning to any such quadruple a value $f(E, t, \alpha, \omega) \in A$ subject to the weight $k$ condition

$$f(E, t, \alpha, \omega) = \lambda^{-k} \cdot f(E, t, \alpha, \omega)$$

for all $\lambda \in A^\times$,

depending only on the isomorphism class of the quadruple, and compatible with base change of $F$-algebras. The two descriptions are related by

$$f(E, t, \alpha) = f(E, t, \alpha, \omega)^k,$$

for any chosen generator $\omega \in \Omega^1_{E/A}$.

There is a third way of thinking about modular forms that will be useful in the following. Consider the relative de Rham cohomology of $\mathcal{E}/X$:

$$\mathcal{L} := \mathbb{R}^1 \pi_*(0 \to \mathcal{O}_E \to \Omega^1_{E/X}(\log \mathbb{Z}_X) \to 0),$$

which fits in a short exact sequence

$$(1.2) \quad 0 \to \omega \to \mathcal{L} \to \omega^{-1} \to 0$$

of sheaves on $X$ and is equipped with a non-degenerate pairing

$$(1.3) \quad \langle , \rangle : \mathcal{L} \times \mathcal{L} \to \mathcal{O}_X$$

coming from the Hodge filtration and the Poincaré pairing on the de Rham cohomology of the fibers. By the Kodaira–Spencer isomorphism

$$\sigma : \omega^{\otimes 2} \cong \Omega^1_X(\log Z_X)$$

given by $\sigma(\omega \otimes \eta) = \langle \omega, \nabla \eta \rangle$, where

$$\nabla : \mathcal{L} \to \mathcal{L} \otimes \Omega^1_X(\log Z_X)$$

is the Gauss–Manin connection, a modular form $f$ of weight $r + 2$ and level $\Gamma$ defines a section $\omega_f$ of the sheaf $\omega^{\otimes r} \otimes \Omega^1_X(\log Z_X)$ by the rule

$$\omega_f(E, t, \alpha) := f(E, t, \alpha, \omega^r) \otimes \sigma(\omega^2).$$

If $f$ is a cusp form, then the above rule defines a section $\omega_f$ of $\omega^{\otimes r} \otimes \Omega^1_X$, thus yielding an identification

$$S_{r+2}(X) \simeq H^0(X, \omega^{\otimes r} \otimes \Omega^1_X).$$

For each $r \geq 0$, let $\mathcal{L}_r := \text{Sym}^r \mathcal{L}$ (with $\mathcal{L}_0 := \mathcal{O}_X$), and define the de Rham cohomology of $X$ (attached to $\mathcal{L}_r$) as the hypercohomology group

$$(1.4) \quad H^1_{\text{DR}}(X, \mathcal{L}_r, \nabla) := H^1(\mathcal{L}^*_r : \mathcal{L}_r \to \nabla \otimes \Omega^1_X(\log Z_X)).$$

Twisting by the ideal sheaf $\mathcal{I}$ gives rise to the subcomplex $\mathcal{L}^*_r \otimes \mathcal{I} \to \mathcal{L}^*_r$, and the weight $r + 2$ parabolic cohomology of $X$ is defined by

$$(1.5) \quad H^1_{\text{par}}(X, \mathcal{L}_r, \nabla) := \text{image}(H^1(\mathcal{L}^*_r \otimes \mathcal{I}) \to H^1_{\text{DR}}(X, \mathcal{L}_r, \nabla)).$$

The exact sequence $(1.2)$ induces the short exact sequence

$$(1.6) \quad 0 \to H^0(X, \omega^{\otimes r} \otimes \Omega^1_X) \to H^1_{\text{par}}(X, \mathcal{L}_r, \nabla) \to H^1(X, \omega^{\otimes r} - r) \to 0,$$

and hence the above assignment $f \mapsto \omega_f$ identifies $S_{r+2}(X)$ with a subspace of $H^1_{\text{par}}(X, \mathcal{L}_r, \nabla)$. In addition, the pairing $(1.3)$ induces a non-degenerate pairing

$$(1.7) \quad \langle , \rangle : H^1_{\text{par}}(X, \mathcal{L}_r, \nabla) \times H^1_{\text{par}}(X, \mathcal{L}_r, \nabla) \to F$$

with respect to which $(1.6)$ is self-dual.
1.3. \textit{p-new forms.} Consider the two degeneracy maps

\[ \pi_1, \pi_2 : X \rightarrow X_1(N) \]

defined by sending, under the moduli interpretation, a triple \((E, t, \alpha)\) to the pairs \((E, t)\) and \((\alpha(E), \alpha(t))\), respectively. These morphisms induce maps

\[ \pi_1^*, \pi_2^* : H^1_{\text{par}}(X_1(N), L, \nabla) \rightarrow H^1_{\text{par}}(X, L, \nabla), \]

where \(H^1_{\text{par}}(X_1(N), L, \nabla)\) is defined as in (1.5) using the analogous objects over \(X_1(N)\).

**Lemma 1.1.** The map \(\pi_1^* \oplus \pi_2^*\) is injective.

**Proof.** This is [Col94, Prop. 4.1]. \qed

Define the \(p\)-old subspace \(H^1_{\text{par}}(X, L, \nabla)^{p\text{-old}}\) of \(H^1_{\text{par}}(X, L, \nabla)\) to be the image of \(\pi_1^* \oplus \pi_2^*\), and the \(p\)-new subspace \(H^1_{\text{par}}(X, L, \nabla)^{p\text{-new}}\) to be the orthogonal complement of the \(p\)-old subspace under the Poincaré pairing (1.7). The space of \(p\)-new cusp forms of weight \(k\) and level \(\Gamma\) is defined by

\[ S_{r+2}(X)^{p\text{-new}} := S_{r+2}(X) \cap H^1_{\text{par}}(X, L, \nabla)^{p\text{-new}}, \]

viewing \(S_{r+2}(X)\) as subspace of \(H^1_{\text{par}}(X, L, \nabla)\) in the form described above.

1.4. \textit{p-adic modular forms.} Recall that the Hasse invariant is a modular form \(H\) over \(\mathbb{F}_p\) of level 1 and weight \(p-1\) with the property that an elliptic curve \(E\) over an \(\mathbb{F}_p\)-algebra \(B\) is ordinary if and only if \(H(E, \omega)\) is a unit in \(B\) for some (or equivalently, any) generator \(\omega \in \Omega^1_{E/B}\).

Let \(R\) be a \(p\)-adic ring, i.e., a ring which is isomorphic to its \(p\)-pro completion. A \(p\)-adic modular form of tame level \(N\) and weight \(k\) defined over \(R\) is a rule assigning to every triple \((E, t, \omega)_{/A}\), over an arbitrary \(p\)-adic \(R\)-algebra \(A\), consisting of:

- an elliptic curve \(E/A\) such that the reduction \(E \times_A A/pA\) is ordinary;
- a section \(t : \text{Spec}(A) \rightarrow E\) of the structure morphism of \(E/A\) of exact order \(N\); and
- a differential \(\omega \in \Omega^1_{E/A}\).

an element \(f(E, t, \omega) \in A\) depending only on the isomorphism class of \((E, t, \omega)_{/A}\), homogeneous of degree \(-k\), viewed over \(A\), and compatible with base change of \(p\)-adic \(R\)-algebras. Let \(\mathcal{M}_k(N; R)\) be the \(R\)-module of \(p\)-adic modular forms of weight \(k\) and level \(N\) defined over \(R\); as before, if there is no risk of confusion \(R\) will be often suppressed from the notation.

Similarly as for classical forms, it will be convenient to think of \(p\)-adic modular forms of weight \(k\) as sections of the sheaf \(\omega^{\otimes k}\) over a certain subset of the rigid analytic space \(X(\mathbb{C}_p)\). Let \(E_{p-1}\) be the normalized Eisenstein series of weight \(p-1\) (recall that \(p \geq 5\), and define the ordinary locus of \(X_1(N)\) by

\[ X_1(N)_{\text{ord}} := \{ x \in X_1(N)(\mathbb{C}_p) : |E_{p-1}(E_x, \omega_x)|_{p} \geq 1 \}, \]

where \(E_x / \mathbb{C}_p\) is a generalized elliptic curve corresponding to \(x\) under the moduli interpretation, \(\omega_x \in \Omega^1_{E_x/\mathbb{C}_p}\) is a regular differential on \(E_x\), chosen so that it extends to a regular differential over \(\mathcal{O}_{C_p}\) if \(E_x\) has good reduction at \(p\), or corresponds to the canonical differential on the Tate curve if \(x\) lies in the residue disc of a cusp, and \(|\cdot|_p\) is the absolute value on \(\mathbb{C}_p\) normalized so that \(|p|_p = p^{-1}\). Since \(E_{p-1}\) reduces to the Hasse invariant \(H\) modulo \(p\), it follows that the points \(x \in X_1(N)_{\text{ord}}\) correspond to pairs \((E_x, t_x)\) with \(E_x\) having ordinary reduction modulo \(p\). Thus the assignment \(f \mapsto (x \mapsto f(E_x, t_x, \omega_x)\omega_x^{k})\), for any chosen generator \(\omega_x \in \Omega^1_{E_x/\mathbb{C}_p}\), defines an identification

\[ \mathcal{M}_k(N) \cong H^0(X_1(N)_{\text{ord}}; \omega^{\otimes k}). \]

Let \(I := \{ v \in Q : 0 < v \leq \frac{1}{p^{r-1}}\} \), and for any \(v \in I\) define

\[ X_1(N)(v) := \{ x \in X_1(N)(\mathbb{C}_p) : |E_{p-1}(E_x, \omega_x)|_p > p^{-v}\}. \]

The space of \textit{overconvergent \(p\)-adic modular forms} of weight \(k\) and tame level \(N\) is given by

\[ \mathcal{M}_k^I(N) = \lim_{\text{proj}} H^0(X_1(N)(v); \omega^{\otimes k}), \]

where the transition maps \(H^0(X_1(N)(v); \omega^{\otimes k}) \rightarrow H^0(X_1(N)(v'); \omega^{\otimes k})\), for \(v' < v\) in \(I\), are given by restriction; since these maps are injective, \(\mathcal{M}_k^I(N)\) is naturally a subspace of \(\mathcal{M}_k(N)\).
By the theory of the canonical subgroup (see [Kat73, Thm. 3.1]), if \((E_x, t_x)\) corresponds to a point \(x\) in \(X_1(N)(\mathbb{F}_p)\), the elliptic curve \(E_x\) admits a distinguished subgroup \(\text{can}(E_x) \subset E_x[p]\) of order \(p\) reducing to the kernel of Frobenius in characteristic \(p\). The rule
\[
(E_x, t_x) \mapsto (E_x, t_x, \alpha_{\text{can}}),
\]
where \(\alpha_{\text{can}} : E_x \mapsto E_x/\text{can}(E_x)\) is the projection, defines rigid morphism \(X_1(N)(\mathbb{F}_p) \rightarrow W_{\infty}\), and hence if \(f\) is a modular form of weight \(k\) and level \(\Gamma\), then the restriction \(f|_{W_{\infty}}\) gives an overconvergent \(p\)-adic modular form of weight \(k\) and tame level \(N\).

1.5. Ordinary CM points. Let \(K\) be an imaginary quadratic field with ring of integers \(\mathcal{O}_K\) equipped with a cyclic ideal \(\mathfrak{N} \subset \mathcal{O}_K\) such that
\[
\mathcal{O}_K/\mathfrak{N} \simeq \mathbb{Z}/N\mathbb{Z}.
\]
Fix an elliptic curve \(A\) defined over the Hilbert class field \(H\) of \(K\) with \(\text{End}_H(A) \simeq \mathcal{O}_K\) having good reduction at the primes above \(p\), and choose a \(\Gamma_1(N)\)-level structure \(t_A \in A[9\mathfrak{N}]\) and a regular differential \(\omega_A \in \Omega^1_{A/H}\). The identification \(\text{End}_H(A) = \mathcal{O}_K\) is normalized so that \(\lambda \in \mathcal{O}_K\) acts as
\[
\lambda^* \omega = \omega
\]
for all \(\omega \in \Omega^1_{A/H}\).

For every integer \(c \geq 1\) prime to \(N_p\), let \(\mathcal{O}_c = \mathbb{Z} + c\mathcal{O}_K\) be the order of \(K\) of conductor \(c\), and denote by \(\text{Isog}^{\mathfrak{N}}_c(A)\) the set of elliptic curves \(A'\) with CM by \(\mathcal{O}_c\) equipped with an isogeny \(\varphi : A \rightarrow A'\) satisfying \(\ker(\varphi) \cap A[9\mathfrak{N}] = \{0\}\).

The semigroup of projective rank one \(\mathcal{O}_c\)-modules \(a \subset \mathcal{O}_c\) prime to \(\mathfrak{N} \cap \mathcal{O}_c\) acts on \(\text{Isog}^{\mathfrak{N}}_c(A)\) by the rule
\[
a \ast (\varphi : A \rightarrow A') = \varphi_a \varphi : A \rightarrow A' \rightarrow A'_a,
\]
where \(A'_a := A'/A'[a]\) and \(\varphi_a : A' \rightarrow A'_a\) is the natural projection. It is easily seen that this induces an action of \(\text{Pic}(\mathcal{O}_c)\) on \(\text{Isog}^{\mathfrak{N}}_c(A)\).

Throughout this paper, we shall assume that \(p = \overline{\mathfrak{p}}\) splits in \(K\), and let \(\mathfrak{p}\) be the prime of \(K\) above \(p\) induced by our fixed embedding \(\overline{\mathcal{O}}_p \overset{\iota_p}{\rightarrow} \mathbb{C}_p\). Thus if \(A'\) is an elliptic curve with CM by \(\mathcal{O}_c\) defined over the ring class field \(H_c\) of \(K\) of conductor \(c\), then \(A'\) has ordinary reduction at \(p\), and \(A'[\mathfrak{p}] \subset A'[\mathfrak{p}]\) is the canonical subgroup. In the following, we will let \(\alpha'_p = \alpha_{\text{can}} : A' \rightarrow A'/A'[\mathfrak{p}]\) denote the projection.

1.6. Generalized Heegner cycles. For any \(r > 0\), let \(W_r\) be the Kuga–Sato variety over
\[
X_0 := X_1(Np)
\]
obtained as the canonical desingularization of the \(r\)-fold self-product of the universal generalized elliptic curve over \(X_0\), and define
\[
X_r := W_r \times A',
\]
where \(A/H\) is the elliptic curve with CM by \(\mathcal{O}_K\) fixed in the preceding section.

The variety \(X_r\) is fibered over \(X_0\), and the fiber over a non-cuspidal point \(x\) associated with a pair \((E_x, t_x)\) is identified with \(E_x^r \times A'\). Thus for every isogeny \(\varphi : A \rightarrow A'\) in \(\text{Isog}^{\mathfrak{N}}_c(A)\), we may consider the cycle
\[
\Upsilon_\varphi := (\Gamma^r_x) \subset (A' \times A)^r \subset X_r,
\]
where \(\Gamma^r_x\) is the transpose of the graph of \(\varphi\), and following [BDP13, §2.3] define the generalized Heegner cycle associated with \(\varphi\) by
\[
\Delta_\varphi := \epsilon_X \Upsilon_\varphi,
\]
where \(\epsilon_X\) is the ideomorphism defined in [loc.cit., (2.1.1)] (with \(X_0\) in place of their curve \(C = X_1(N)\)). By [BDP13, Prop. 2.7], the cycles \(\Delta_\varphi\) are homologically trivial; by abuse of notation, we shall still denote by \(\Delta_\varphi\) the classes they define in the Chow group \(\text{CH}^{r+1}(X_r)_0\) with rational coefficients. For \(r = 0\), set
\[
\Delta_\varphi := (A', t_{A'}) - (\infty),
\]
where \(t_{A'} \in A'[Np]\) is a \(\Gamma_1(Np)\)-level structure contained in \(A'[9\mathfrak{p}]\), and \(\infty\) is the cusp (Tate\((q), \zeta_{Np}\)).
2. A SEMISTABLE NON-CRISTALLINE SETTING

This section is aimed at proving Theorem 2.11 below, which extends the $p$-adic Gross–Zagier formula due Bertolini–Darmon–Prasanna [BDP13] in the good reduction case to the semistable non-crystalline setting.

2.1. $p$-adic Abel–Jacobi maps. Let $F$ be a finite unramified extension of $\mathbb{Q}_p$, denote by $\mathcal{O}_F$ the ring of integers of $F$, and let $\kappa$ be the residue field. The generalized Kuga–Sato variety $X_r$, which was defined in (1.8) as a scheme over $\mathbb{Z}[1/Np]$, has semistable reduction at $p$. In other words, there exists a proper scheme $\mathcal{X}_r$ over $\mathcal{O}_F$ with generic fiber $X_r \times_{\mathbb{Z}[1/Np]} F$ and with special fiber $X_r \times_{\mathcal{O}_F} \kappa$ whose only singularities are divisors with normal crossings.

By the work of Hyodo–Kato [HK94], attached to $X_r$ there are log-crystalline cohomology groups $H^j_{\log,\text{cris}}(X_r \times_{\mathcal{O}_F} \kappa)$, which are $\mathcal{O}_F$-modules of finite rank equipped with a semilinear Frobenius automorphism $\Phi$ and a linear nilpotent monodromy operator $N$ satisfying

$$N\Phi = p\Phi N.$$ 

Moreover, for each choice of a uniformizer of $\mathcal{O}_F$ there is a comparison isomorphism

$$H^j_{\log,\text{cris}}(X_r \times_{\mathcal{O}_F} \kappa) \otimes \mathcal{O}_F \cong H^j_{\text{dR}}(X_r/F)$$

endowing the algebraic de Rham cohomology groups $H^j_{\text{dR}}(X_r/F)$ with the structure of filtered $(\Phi, N)$-modules. In the following, we shall restrict our attention to the middle degree cohomology, i.e., we set $j = 2r + 1$.

Let $G_F := \text{Gal}(\overline{F}/F)$ be the absolute Galois group of $F$, and consider the $p$-adic $G_F$-representation given by

$$V_r := H^1_{\text{dR}}(X_r \times_F F, \mathbb{Q}_p).$$

Applying Fontaine’s functor $\mathbf{D}_{\text{st}}$ to $V_r$ yields another filtered $(\Phi, N)$-module associated to $X_r$.

**Theorem 2.1** (Tsuji). The $p$-adic $G_F$-representation $V_r$ is semistable, and there is a natural isomorphism

$$\mathbf{D}_{\text{st}}(V_r) \cong H^1_{\text{dR}}(X_r/F)$$

compatible with all structures. In particular, the assignment $V \mapsto \mathbf{D}_{\text{st}}(V)$ induces an isomorphism

$$\text{Ext}_{\text{st}}(\mathbb{Q}_p, V_r) \cong \text{Ext}_{\text{Mod}_{r}(\Phi, N)}(F, H^1_{\text{dR}}(X_r/F)).$$

Here, $\text{Ext}_{\text{st}}(\mathbb{Q}_p, V_r) \cong H^1_{\text{st}}(F, V_r) := \ker(H^1(F, V_r) \rightarrow H^1(F, V_r \otimes_{\mathbb{Q}_p} \mathbf{B}_{\text{st}}))$ is the group of extensions of the trivial representation $\mathbb{Q}_p$ by $\mathcal{O}_F$. It is computed in the following proposition.

The idempotent $\epsilon_X$ used in the definition (1.9) of the generalized Heegner cycles $\Delta_z$ acts as a projector on the various cohomology groups associated to the variety $X_r$. Let $V_r(r+1)$ be the $(r+1)$-st Tate twist of $V_r$, and consider the étale Abel–Jacobi map

$$\text{AJ}_{F}^{\text{et}} : CH^{r+1}(X_r)_0(F) \rightarrow \text{Ext}_{\text{Rep}_{F}}(\mathbb{Q}_p, \epsilon_X V_r(r+1)) = H^1(F, \epsilon_X V_r(r+1))$$

constructed in [Nek00]. By [loc.cit., Thm. 3.1(ii)], the image of $\text{AJ}_{F}^{\text{et}}$ lands in $H^1_{\text{dR}}(F, \epsilon_X V_r(r+1))$, and hence via the comparison isomorphism (2.1) it can be seen as taking values in the group

$$\text{Ext}_{\text{Mod}_{r}(\Phi, N)}(F, \epsilon_X H^1_{\text{dR}}(X_r/F)(r+1)).$$

of extensions of $F$ by the twist $\epsilon_X H^1_{\text{dR}}(X_r/F)(r+1)$ in the category of filtered $(\Phi, N)$-modules over $F$. This group admits the following explicit description.

**Lemma 2.2.** Set $H_r := H^1_{\text{dR}}(X_r/F)$ and let $n = [F: \mathbb{Q}_p]$. The assignment

$$\{0 \rightarrow \epsilon_X H_r(r+1) \rightarrow E^\wedge_n F \rightarrow 0\} \mapsto \eta = \eta^{\text{hol}}_E(1) - \eta^{\text{frob}}_E(1),$$

where $\eta^{\text{hol}}_E : F \rightarrow \text{Fil}^0 E$ (resp. $\eta^{\text{frob}}_E : F \rightarrow E^{0,n=1,N=0}$) is a section of $p$ compatible with filtrations (resp. with Frobenius and monodromy) yields an isomorphism

$$\text{Ext}_{\text{Mod}_{r}(\Phi, N)}(F, \epsilon_X H_r(r+1)) \cong \epsilon_X H_r(r+1)/\text{Fil}^0 \epsilon_X H_r(r+1).$$

**Proof.** See [IS03, Lemma 2.1], for example. \qed
Define the \( p \)-adic Abel–Jacobi map

\[
\text{AJ}_F : \text{CH}^{r+1}(X_r)_0(F) \longrightarrow \epsilon_X H^{2r+1}_\text{dR}(X_r/F)(r+1)/\text{Fil}^0\epsilon_X H^{2r+1}_\text{dR}(X_r/F)(r+1)
\]

to be the composite of \( \text{AJ}_F^0 \) with the isomorphisms of Theorem 2.1 and Lemma 2.1. Since the filtered pieces \( \text{Fil}^1\epsilon_X H^{2r+1}_\text{dR}(X_r/F)(r) \) and \( \text{Fil}^0\epsilon_X H^{2r+1}_\text{dR}(X_r/F)(r+1) \) are exact annihilators under the Poincaré duality

\[
\epsilon_X H^{2r+1}_\text{dR}(X_r/F)(r) \times \epsilon_X H^{2r+1}_\text{dR}(X_r/F)(r+1) \longrightarrow F,
\]

the target of \( \text{AJ}_F \) may be identified with the linear dual (\( \text{Fil}^r\epsilon_X H^{2r+1}_\text{dR}(X_r/F) \))\(^r\).

Recall the coherent sheaf of \( \mathcal{O}_X \)-modules \( \mathcal{L}_r = \text{Sym}^r \mathcal{L} \) on \( X \) introduced in Section 1.2, and set

\[
\mathcal{L}_{r,r} := \mathcal{L}_r \otimes \text{Sym}^r\bigl(H^1_\text{dR}(A/F)\bigr).
\]

With the trivial extension of the Gauss–Manin connection \( \nabla \) on \( \mathcal{L}_r \) to \( \mathcal{L}_{r,r} \), consider the complex

\[
\mathcal{L}_{r,r}^* : \mathcal{L}_{r,r} \xrightarrow{\nabla} \mathcal{L}_{r,r} \otimes \Omega^1_X(\log Z_X),
\]

and define \( H^1_{\text{par}}(X, \mathcal{L}_{r,r}, \nabla) \) as in (1.5). By [BDP13, Prop. 2.4], we then have

\[
\epsilon_X H^{2r+1}_\text{dR}(X_r/F) \simeq H^1_{\text{par}}(X, \mathcal{L}_{r,r}, \nabla) = H^1_{\text{par}}(X, \mathcal{L}_{r,r}, \nabla) \otimes \text{Sym}^r H^1_\text{dR}(A/F)
\]

and

\[
\text{Fil}^r\epsilon_X H^{2r+1}_\text{dR}(X_r/F) \simeq H^0(X, \omega^{\text{par}}_X \otimes \Omega^1_X) \otimes \text{Sym}^r H^1_\text{dR}(A/F)\bigl)\).}

As a result of these identifications, we shall view the \( p \)-adic Abel–Jacobi map (2.1) as a map

\[
\text{AJ}_F : \text{CH}^{r+1}(X_r)_0(F) \longrightarrow \bigl(H^0(X, \omega^{\text{par}}_X \otimes \Omega^1_X) \otimes \text{Sym}^r H^1_\text{dR}(A/F)\bigr)\)^r.
\]

Moreover, if \( \Delta = \epsilon_X \Delta \in \text{CH}^{r+1}(X_r)_0(F) \) is the class of a cycle in the image of the idempotent \( \epsilon_X \) supported on the fiber of \( X_r \longrightarrow X \) over a point \( P \in X(F) \), we see that \( \text{AJ}_F(\Delta) \) may be computed using the following recipe. Consider the commutative diagram with Cartesian squares:

\[
\begin{array}{ccc}
0 & \longrightarrow & H^1_{\text{par}}(X, \mathcal{L}_{r,r}, \nabla)(r+1) \quad \longrightarrow \quad D_\Delta \quad \longrightarrow \quad F \quad \longrightarrow \quad 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & H^1_{\text{par}}(X, \mathcal{L}_{r,r}, \nabla)(r+1) \quad \longrightarrow \quad H^1_{\text{par}}(X \smallsetminus P, \mathcal{L}_{r,r}, \nabla)(r+1) \quad \longrightarrow \quad \mathcal{L}_{r,r}(P)(r) \quad \longrightarrow \quad 0,
\end{array}
\]

where the rightmost vertical map is defined by sending \( 1 \in F \) to the cycle class \( \mathcal{L}_r(P)(\Delta) \). Then \( \text{AJ}_F(\Delta) \) is given by the linear functional

\[
\text{AJ}_F(\Delta) = (-, \eta_\Delta),
\]

where \( \eta_\Delta = \eta_\Delta^\text{hol} - \eta_\Delta^\text{frob} := H^0_{\text{dR}}(1) - \eta_\Delta^\text{frob}(1) \) is the “tangent vector” associated as in Lemma 2.2 to the extension \( D_\Delta \) as filtered \((\Phi, N)\)-modules, and

\[
\langle , \rangle : H^1_{\text{par}}(X, \mathcal{L}_{r,r}, \nabla)(r) \times H^1_{\text{par}}(X, \mathcal{L}_{r,r}, \nabla)(r+1) \longrightarrow F
\]

is the Poincaré duality.

2.2. Rigid cohomology. Recall the rigid spaces \( \mathbb{Z}_\infty \subset \mathbb{W}_\infty \), \( \mathbb{Z}_0 \subset \mathbb{W}_0 \) introduced in Section 1.1. Fix a collection of points \( \{P_1, \ldots, P_t\} \) of \( X(F) \) contained in \( \mathbb{Z}_\infty \), containing all the cusps of \( \mathbb{Z}_\infty \), and such that \( \text{red}_p(P_i) \neq \text{red}_p(P_j) \) for \( i \neq j \). Let \( w_p \) be the automorphism of \( X \) defined in terms of moduli by

\[
w_p(E, t, \alpha) = (\alpha(E), \alpha(t), \alpha^\vee),
\]

where \( \alpha^\vee \) is the isogeny dual to \( \alpha \), and set \( P^*_j := w_p P_j \). Then the points \( P^*_j \) factor through \( \mathbb{Z}_0 \), and the set

\[
S := \{P_1, \ldots, P_t, P^*_1, \ldots, P^*_t\}
\]

contains all the cusps of \( X \). Since the points \( Q \in S \) reduce to smooth points \( \bar{Q} \) in the special fiber, the spaces \( D(Q) := \text{red}^{-1}_p(\bar{Q}) \) are conformal to the open disc in \( D(0; 1) \) in \( \mathbb{C}_p \). Fix isomorphisms \( h_Q : D(Q) \longrightarrow D(0; 1) \) mapping the point \( Q \) to 0, and for a collection of real numbers \( r_Q < 1 \) consider the annuli

\[
\mathcal{V}_Q := \{x \in D(Q) : r_Q < |h_Q(x)|_p < 1\}.
\]
Denote by \( \mathcal{L}^{rig}_{\tau,r} \) the sheaf for the rigid analytic topology on \( X(\mathbb{C}_p) \) defined by the algebraic vector bundle \( \mathcal{L}_{\tau,r} \). If \( V \subset X(\mathbb{C}_p) \) is a connected wide open contained in \( Y(\mathbb{C}_p) \), the Gauss–Manin connection yields a connection

\[
\nabla : \mathcal{L}^{rig}_{\tau,r}|_V \to \mathcal{L}^{rig}_{\tau,r}|_V \otimes \Omega^1_V,
\]

and similarly as in (1.4) we define the \( i \)-th de Rham cohomology of \( V \) attached to \( \mathcal{L}^{rig}_{\tau,r} \) by

\[
H^i(\mathcal{L}^{*,r}_r|_V) = H^i_{dR}(V, \mathcal{L}^{rig}_{\tau,r}, \nabla) := H^i(\mathcal{L}^{rig}_{\tau,r}|_V \to \mathcal{L}^{rig}_{\tau,r}|_V \otimes \Omega^1_V).
\]

In particular, if \( V \) is a basic wide open, then

\[
H^1(\mathcal{L}^{*,r}_r|_V) \simeq \frac{\mathcal{L}^{rig}_r(V) \otimes \Omega^1_V}{\nabla \mathcal{L}^{rig}_r(V)},
\]

and \( H^0(\mathcal{L}^{*,r}_r|_V) \cong \mathcal{L}^{rig}_r(V)^{\nabla = 0} \) is the space of horizontal sections of \( \mathcal{L}^{rig}_{\tau,r} \) over \( V \). For \( r = 0 \), we set

\[
H^1(\mathcal{L}^{*,r}_r|_V) = H^1(V) := \Omega^1_V/d\Omega_V, \quad H^0(\mathcal{L}^{*,r}_r|_V) = H^0(V) := \mathcal{O}_V^{d=0},
\]

where \( d : \mathcal{O}_V \to \Omega^1_V \) is the differentiation map.

In terms of the admissible cover of \( X(\mathbb{C}_p) \) by basic wide opens described in Section 1.1, the classes in \( H^0_{dR}(X(\mathbb{C}_p), \mathcal{L}^{rig}_{\tau,r}, \nabla) \) may be represented by hypercocycles \((\omega_0, \omega_\infty; f_0)\), where \( \omega_0 \) and \( \omega_\infty \) are \( \mathcal{L}^{rig}_{\tau,r} \)-valued differentials on \( \mathcal{W}_0 \) and \( \mathcal{W}_\infty \), respectively, and \( f_0 \in \mathcal{L}^{rig}_{\tau,r}(\mathcal{W}) \) is such that \((\omega_\infty - \omega_0)|_{\mathcal{W}} = \nabla f_0|_{\mathcal{W}}\); and two hypercocycles represent the same class if their difference is of the form \((\nabla f_0, \nabla f_\infty; (f_\infty - f_0)|_{\mathcal{W}})\) for some \( f_\infty \in \mathcal{L}^{rig}_{\tau,r}(\mathcal{W}_\infty) \) and \( f_0 \in \mathcal{L}^{rig}_{\tau,r}(\mathcal{W}_0) \).

If \( V \) is a wide open annulus, associated with an orientation of \( V \) there is a \( p \)-adic annular residue

\[
res_V : \Omega^1_V \to \mathbb{C}_p
\]

defined by expanding \( \omega = \sum_n a_n T^n \frac{dT}{T} \in \Omega^1_V \) with respect to a fixed uniformizing parameter \( T \) of compatible with the orientation, and setting \( res_V(\omega) := a_0 \) (see [Col89, Lemma 2.1]). Combined with the natural pairing

\[
\langle \cdot, \cdot \rangle : \mathcal{L}^{rig}_r(V) \times \mathcal{L}^{rig}_r(V) \otimes \Omega^1_V \to \Omega^1_V
\]

induced by the Poincaré duality (1.3) on \( \mathcal{L}_r \) (extended to \( \mathcal{L}_{\tau,r} \) in the obvious manner), we obtain a higher \( p \)-adic annular residue map

\[
Res_V : \mathcal{L}^{rig}_r(V) \otimes \Omega^1_V \to \mathcal{L}^{rig}_r(V)\!,
\]

by setting

\[
Res_V(\omega)(\alpha) = res_V(\alpha, \omega)
\]

for every \( \mathcal{L}^{rig}_r \)-valued differential \( \omega \) on \( V \) and every section \( \alpha \in \mathcal{L}^{rig}_r(V) \). Since \( res_V \) clearly descends to a map \( H^1(V) = \Omega^1_V/d\Omega_V \to \mathbb{C}_p \), by composing \( Res_V \) with the projection \( \mathcal{L}^{rig}_r(V) \to H^0(V) \), it is easily seen from the Leibniz rule that we obtain a well-defined map

\[
Res_V : H^1(\mathcal{L}^{*,r}_r|_V) \to H^0(\mathcal{L}^{*,r}_r|_V).
\]

If \( V_Q \subset D(Q) \) is the annulus attached to a non-cuspidal point \( Q \in S \), it will be convenient, following the discussion after [BDP13, Cor. 3.7], to view \( Res_{V_Q} \) as taking values on the fiber \( \mathcal{L}_{\tau,r}(Q) \), using the sequence of identifications

\[
H^0(\mathcal{L}^{*,r}_r|_{V_Q})\!\! = (H^0(D(Q), \mathcal{L}_{\tau,r})\!\!^{\nabla = 0})\!\! = \mathcal{L}_{\tau,r}(Q)\!\! = \mathcal{L}_{\tau,r}(Q)
\]

arising from “analytic continuation”, the choice of an “initial condition”, and the self-duality of \( \mathcal{L}_{\tau,r}(Q) \), respectively. (See loc.cit. for the case of a cusp \( Q \in S \).)

For a supersingular annulus \( \mathcal{A}_x \), the vector space \( H^0(\mathcal{L}^{*,r}_r|_{\mathcal{A}_x}) \) is equipped with a pairing \( \langle \cdot, \cdot \rangle_{\mathcal{A}_x} \), arising from an identification (similar to (2.9)) with the de Rham cohomology of a supersingular elliptic curve in characteristic \( p \) corresponding to \( x \in SS \). Moreover, since \( H^0(\mathcal{A}_x) \simeq \mathbb{C}_p \), the residue map (2.6) yields an isomorphism \( \text{res}_{\mathcal{A}_x} : H^1(\mathcal{A}_x) \to H^0(\mathcal{A}_x) \), and using a trivialization of \( \mathcal{L}^{rig}_{\tau,r}|_{\mathcal{A}_x} \) it may be extended to an isomorphism

\[
\text{res}_{\mathcal{A}_x} : H^1(\mathcal{L}^{*,r}_r|_{\mathcal{A}_x}) \simeq H^0(\mathcal{L}^{*,r}_r|_{\mathcal{A}_x})
\]
(see [Col94, Prop. 7.1]). It is then easily checked that (2.8) and (2.10) correspond to each other under the identification $H^0(L^*_{\varphi}(A_s))^\vee = H^0(L^*_{\varphi}(A_s))$ defined by $(,)_A$.

Let $S$ be a set of points as introduced above, and define

$$W^S_\infty := \mathbb{Z}_\infty \setminus \bigcup_{Q \in S \setminus \mathbb{Z}_\infty} D(Q) \setminus V_Q, \quad U := W^S_\infty \cup W^0_\infty,$$

where $W^0_\infty := w_p W^S_\infty$, and $U := X \setminus S$. The restriction of an $L_{\varphi}$-valued differential on $X$ which is regular on $U$ defines a section of $\mathcal{L}^{\rig}_{\varphi} \otimes \Omega^1_X$ over $U$. As argued in the proof of [Col94, Prop. 7.2], this yields an isomorphism

$$H^1_{\text{dR}}(U, L_{\varphi}, \nabla) \simeq H^1(L^*_{\varphi}|U)$$

between algebraic and rigid de Rham cohomology.

**Proposition 2.3.** Let the notations be as before.

1. A class $\kappa \in H^1(L^*_{\varphi}|U)$ belongs to the image of $H^1_{\text{par}}(X, L_{\varphi}, \nabla)$ under restriction

$$H^1_{\text{par}}(X, L_{\varphi}, \nabla) \to H^1_{\text{dR}}(U, L_{\varphi}, \nabla) \simeq H^1(L^*_{\varphi}|U)$$

if and only if $\text{Res}_{V/Q}(\kappa) = 0$ for all $Q \in S$.

2. Let $V$ be such that $\{U, V\}$ is an admissible covering of $X$. If $\kappa_\omega, \kappa_\eta \in H^1_{\text{par}}(X, L_{\varphi}, \nabla)$ are represented by the hypercocycles $(\omega_U, \omega_V; \omega_W|_V)$, $(\eta_U, \eta_V; \eta_W|_V)$ respectively, with respect to this covering, then the value $(\kappa_\omega, \kappa_\eta)$ under the Poincaré duality (2.3) is given by

$$(\kappa_\omega, \kappa_\eta) = \sum_{Q \in S} \text{Res}_{V/Q}(F_{\omega,Q}, \eta_U),$$

where $F_{\omega,Q}$ is any local primitive of $\omega_U$ on $V_Q$, i.e., such that $\nabla F_{\omega,Q} = \omega_U|_{V_Q}$.

**Proof.** The first assertion follows from the same argument as in [BDP13, Prop. 3.8], and the second is [Col94, Lemma 7.1]. \qed

### 2.3. Coleman’s $p$-adic integration

In this section, we give an explicit description of the filtered $(\Phi, N)$-module structure on $H^1_{\text{par}}(X, L_{\varphi}, \nabla)$, following the work of Coleman–Iovita [CI10]. We state the results for $L_{\varphi}$, leaving their trivial extension to $L_{\varphi} = \mathcal{L} \otimes \text{Sym}^i H^1_{\text{dR}}(A)$ to the reader.

As recalled in Section 1.4, for every pair $(E_x, t_x)$ corresponding to a point $x \in X_1(N)(\mathbb{F}_p)$ there is a canonical $p$-isogeny $\alpha_{\text{can}} : E_x \to E_x/\text{can}(E_x)$, where $\text{can}(E_x) \subset E_x[p]$ is the canonical subgroup. The map $V : X_1(N)(\mathbb{F}_p) \to X_1(N)(\mathbb{F}_p)$ defined in terms of moduli by

$$V(E_x, t_x) = (\alpha_{\text{can}}(E_x), \alpha_{\text{can}}(t_x))$$

is then a lift of the absolute Frobenius on $X_1(N)_{F_x}$. Letting $s_1 : X_1(N)(\mathbb{F}_p) \to W_\infty$ be defined by $(E_x, t_x) \mapsto (E_x, t_x, \text{can}(E_x))$, and letting $W^{s_1}_\infty \subset W_\infty$ be the image of $X_1(N)(\mathbb{F}_p)$ under $s_1$, the map $\phi_{s_1}$ defined by the commutativity of the diagram

$$
\begin{array}{ccc}
W^{s_1}_\infty & \xrightarrow{\phi_{s_1}} & W_\infty \\
\downarrow{\pi_1} & & \downarrow{s_1} \\
X_1(N)(\mathbb{F}_p) & \xrightarrow{V} & X_1(N)(\mathbb{F}_p),
\end{array}
$$

is therefore a lift of the absolute Frobenius on $X_{F_x}$.

As explained in [Col94, p.41] (see also the more detailed discussion in [Col96, p.218]), the canonical subgroup yields a horizontal morphism $\text{Fr}_\infty : \phi_{\infty}^* L_{\varphi} \to L_{\varphi}|W^{s_1}_\infty$. Define the *Frobenius endomorphism* $\Phi_{s_1}$ on $H^1(L^*_{\varphi}|W^{s_1}_\infty)$ by the composite map

$$H^1(L^*_{\varphi}|W^{s_1}_\infty) \simeq \frac{\mathcal{L}^{\rig}_{\varphi} \otimes \Omega^1_{W^{s_1}_\infty}(W^{s_1}_\infty)}{\nabla \mathcal{L}^{\rig}_{\varphi}(W^{s_1}_\infty)} \xrightarrow{(\text{Fr}_\infty \otimes \text{id}) \phi_{s_1}^*} \frac{\mathcal{L}^{\rig}_{\varphi} \otimes \Omega^1_{W^{s_1}_\infty}(W^{s_1}_\infty)}{\nabla \mathcal{L}^{\rig}_{\varphi}(W^{s_1}_\infty)} \simeq H^1(L^*_{\varphi}|W^{s_1}_\infty),$$
where the last isomorphism is given by restriction (see [Col94, Prop. 10.3]). Setting $W^p := w_p W_{\infty} \subset W_0 = w_p W_{\infty}$ and $\delta_0 := w_p^{-1} \phi_{\infty} w_p$, where $w_p$ is the automorphism of $X$ given by (2.4), we similarly define a Frobenius endomorphism $\Phi_0$ of $H^1(L^\bullet \vert W_0)$.

**Theorem 2.4** (Coleman). Let $f = q + \sum_{n=2}^{\infty} a_n(f)q^n \in S_{r+2}(\Gamma_0(Np))$ be a $p$-new eigenform of weight $r + 2 \geq 2$, and let $\omega_f \in H^0(X, \omega^r \otimes \Omega_X) \subset H^1_{\text{par}}(X, \mathcal{L}_r, \nabla)$ be the associated differential. Then for each $r \in \{\infty, 0\}$ there exists a locally analytic section $F_{f, r}$ of $\mathcal{L}_r$ on $W_r$ such that

(i) $\nabla F_{f, r} = \omega_f \vert_{W_r}$; and

(ii) $F_{f, r} - \frac{a_p(f)}{p^{r+1}} \phi_* F_{f, r}$ is rigid analytic on $W_r$.

Moreover, $F_{f, r}$ is unique modulo $H^0(L^\bullet_r \vert W_r)$.

**Proof.** This follows from the discussion in [Col94, §11]. By [loc.cit, Lemma 11.1] we have $\Phi_\infty = pU_p$ on the image of $S_{r+2}(X)p_{\text{new}}$ in $H^1(L^\bullet_r \vert W_0)$. Since $U_p^2 = p^{r}(p)$ on the former space and we have the relations $U_p \omega_f = a_p(f) \omega_f$ and $\langle p \rangle \omega_f = \omega_f$ by hypothesis, it follows that the polynomial

$$P(T) = 1 - \frac{a_p(f)}{p^{r+1}} T$$

is such that $P(\Phi_\infty)(\omega_f \vert_{W_0}) = 0$, and hence also $P(\Phi_0)(\omega_f \vert_{W_0}) = 0$. The result thus follows from [Col94, Thm. 10.1]. □

A locally analytic section $F_{f, r}$ as in Theorem 2.4 is called a Coleman primitive of $f$ on $W_r$.

**Remark 2.5.** For $r > 0$, the spaces $H^0(L^\bullet_r \vert W_r)$ are trivial, and so the Coleman primitives $F_{f, r}$ are unique. On the other hand, for $r = 0$ we have $H^0(L^\bullet_r \vert W_0) \simeq C_p$, and so the $F_{f, r}$ are unique modulo a global constant on $W_0$.

### 2.4. Frobenius and monodromy.

Denote by $\iota$ the inclusion of any rigid subspace of $X$ into $X$. Associated with the exact sequence of complexes of sheaves on $X$

$$0 \longrightarrow L^\bullet_r \longrightarrow \iota^*(L^\bullet_r \vert W_0) \oplus \iota^*(L^\bullet_r \vert W_{\infty}) \overset{\rho_{\infty} - \rho_0}{\longrightarrow} \iota^*(L^\bullet_r \vert W) \longrightarrow 0,$$

there is a Mayer–Vietoris long exact sequence

$$\cdots \longrightarrow H^0_{\text{par}}(L^\bullet_r \vert W_0 \cup W_{\infty}) \overset{\delta^0}{\longrightarrow} H^0_{\text{par}}(L^\bullet_r \vert W) \overset{\delta}{\longrightarrow} H^1_{\text{par}}(X, \mathcal{L}_r^{\text{rig}}, \nabla) \longrightarrow$$

$$\longrightarrow H^1_{\text{par}}(L^\bullet_r \vert W) \overset{\delta^1}{\longrightarrow} H^1_{\text{par}}(L^\bullet_r \vert W) \longrightarrow \cdots$$

in hypercohomology. By [Col94, §10] and the discussion in the preceding section, each of the non-central spaces in the resulting short exact sequence

$$0 \longrightarrow \frac{H^0_{\text{par}}(L^\bullet_r \vert W)}{\delta^0(H^0_{\text{par}}(L^\bullet_r \vert W_0 \cup W_{\infty}))} \overset{\delta}{\longrightarrow} H^1_{\text{par}}(X, \mathcal{L}_r, \nabla) \longrightarrow H^1_{\text{par}}(L^\bullet_r \vert W) \overset{\delta^1=0}{\longrightarrow} 0$$

is equipped with a Frobenius endomorphism. Therefore, to define a Frobenius action on $H^1_{\text{par}}(X, \mathcal{L}_r, \nabla)$ it suffices to construct a splitting of (2.12).

As shown in [Col94, §A.5], this may be obtained as follows. Assume that $\kappa \in H^1_{\text{par}}(L^\bullet_r, \nabla)$ is represented by the hypercyclo (ω0, ω0; f) with respect to the covering {W0, W∞} of X. Since $W = \bigcup_{x \in SS} A_x$ is the union of the supersingular annuli, we may write $f_W = \{f_x\}_{x \in SS}$ with $f_x \in L^\bullet_{rig}(A_x)$. The assignment

$$A_x \mapsto F_{\infty} \mid_{A_x} - F_{\omega_0} \mid_{A_x} - f_x,$$

where $F_{\omega_0}$ is a Coleman primitive of ω0 on $W_r$, defines a horizontal section of $L^{rig}$ on $W_r$, and its image modulo $\delta^0(H^0_{\text{par}}(L^\bullet_r \vert W_0 \cup W_{\infty}))$ is independent of the chosen $F_{\omega_0}$ (see Remark 2.5). It is easily checked that $s \delta = \text{id}$, and hence we may define a Frobenius operator $\Phi$ on $H^1_{\text{par}}(X, \mathcal{L}_r, \nabla)$ by requiring that its action be compatible with the resulting splitting of (2.12).

On the other hand, define the monodromy operator $N$ on $H^1_{\text{par}}(X, \mathcal{L}_r, \nabla)$ by the composite map

$$H^1_{\text{par}}(X, \mathcal{L}_r, \nabla) \longrightarrow H^1(L^\bullet_r \vert W) \oplus_{x \in SS} \text{Res}_{A_x} H^0(L^\bullet_r \vert W) \overset{\delta}{\longrightarrow} H^1_{\text{par}}(X, \mathcal{L}_r, \nabla),$$

where Res$_{A_x}$ are the p-adic residue maps (2.10).
Lemma 2.6. Let $\kappa \in H^1_{\text{par}}(X, \mathcal{L}_r, \nabla)$. Then:

(i) For $r > 0$, $N(\kappa) = 0$ if and only if $\text{Res}_{A_{\kappa}}(\kappa) = 0$ for all $x \in SS$.
(ii) For $r = 0$, $N(\kappa) = 0$ if and only if there is $C \in C_p$ such that $\text{res}_{A_{\kappa}}(\kappa) = C$ for all $x \in SS$.

Proof. This follows immediately from the exact sequence (2.12) and the determination of the spaces $H^0(\mathcal{L}^*_r|_{W_r})$ recalled in Remark 2.5.

By the main result of [CI10], the operators $\Phi$ and $N$ on $H^1_{\text{par}}(X, \mathcal{L}_r, \nabla)$ defined above agree with the corresponding structures deduced from the comparison isomorphism of Theorem 2.1.

2.5. p-adic Gross–Zagier formula. Fix a finite extension $F/Q_p$ containing the image of the Hilbert class field $H$ of $K$ under our fixed embedding $\mathbb{Q} \hookrightarrow C_p$, and let $c \geq 1$ be an integer prime to $Np$.

Proposition 2.7. Let $f = q + \sum_{n=2}^{\infty} a_n(f)q^n \in S_{r+2}(\Gamma_0(Np))$ be a $p$-new eigenform of weight $r+2 \geq 2$. Let $\varphi : A \to A'$ be an isogeny in $\text{Isog}^{\text{rig}}_{\kappa}(A)$, let $P_{A'} \in X(F)$ be the point defined by $(A', t_{A'})$, and let $\Delta_x$ be general ized Heegner cycle associated to $\varphi$. Then for all $\alpha \in \text{Sym}^1 H^1_{\text{dr}}(A/F)$, we have

$$AJ_F(\Delta_\varphi)(\omega_f \wedge \alpha) = \langle F_{f, \infty}(P_{A'}) \wedge \alpha, \text{cl}_{P_{A'}}(\Delta_x) \rangle,$$

where $F_{f, \infty}$ is the Coleman primitive of $\omega_f \in H^0(\mathcal{X}_r, \Omega_{\mathcal{X}r}^1 \otimes \Omega^1_{\kappa})$ on $W_\infty$ (vanishing at infinity if $r = 0$), and the pairing on the right-hand side is the natural one on $L_{r}(P_{A'})$.

Proof. Following the recipe described at the end of Section 2.1, we have

$$AJ_F(\Delta_\varphi)(\omega_f \wedge \alpha) = \langle \omega_f \wedge \alpha, \eta_{\Delta}^{\text{hol}} - \eta_{\Delta}^{\text{frob}} \rangle,$$

where:

- $\eta_{\Delta}^{\text{hol}}$ is a cohomology class represented by a section (still denoted $\eta_{\Delta}^{\text{hol}}$) of $\mathcal{L}_{r} \otimes \Omega^1_{\mathcal{X}}(\log Z_{\mathcal{X}})$ over $U$ having residue 0 at the cusps, and with a simple pole at $P_{A'}$ with residue $\text{cl}_{P_{A'}}(\Delta_x)$;
- $\eta_{\Delta}^{\text{frob}}$ is section of $\mathcal{L}_{r}^{\text{rig}} \otimes \Omega^1_{\kappa}$ over $U$ having the same residues as $\eta_{\Delta}^{\text{hol}}$, and satisfying $N(\eta_{\Delta}^{\text{frob}}) = 0$.

$$\Phi \eta_{\Delta}^{\text{frob}} = \eta_{\Delta}^{\text{frob}} + \nabla G,$$

for some rigid section $G$ of $\mathcal{L}_{r}^{\text{rig}}$ on a strict neighborhood of $(\mathbb{Z}_0 \cap W^0_\kappa) \cup (\mathbb{Z}_\infty \cap W^\infty_\kappa)$ in $U$.

By the formula for the Poincaré pairing in Proposition 2.3, equation (2.14) may be rewritten as

$$AJ_F(\Delta_\varphi)(\omega_f \wedge \alpha) = \sum_{Q \in S} \text{Res}_{V_Q} \langle F_{f, Q} \wedge \alpha, \eta_{\Delta}^{\text{hol}} - \eta_{\Delta}^{\text{frob}} \rangle,$$

where $F_{f, Q} \in \mathcal{L}_r(V_Q)$ is an arbitrary local primitive of $\omega_f$ on the annulus $V_Q$. (Note that here we are using the fact that the connection $\nabla$ on $\mathcal{L}_{r} \otimes \text{Sym}^1 H^1_{\text{dr}}(A/F)$ is defined from the Gauss–Manin connection on $L_r$ by extending it trivially on the second factor.)

If $F_{f, \infty}$ is a Coleman primitive of $\omega_f$ on $W_\infty$, then $F_{f, \infty}^{[p]} := F_{f, \infty} - \frac{a_2(f)}{p-1} \phi_{\infty} F_{f, \infty}$ is rigid analytic on $W^\infty_\kappa < W_\kappa$ by Theorem 2.4, and hence

$$\sum_{Q \in S \cap W_\infty} \text{Res}_{V_Q} \langle F_{f, Q}^{[p]} \wedge \alpha, \eta_{\Delta}^{\text{hol}} \rangle + \sum_{\kappa \in SS} \text{Res}_{A_{\kappa}} \langle F_{f, \infty}^{[p]} \wedge \alpha, \eta_{\Delta}^{\text{frob}} \rangle = 0,$$

by the Residue Theorem (see [BDP13, Thm. 3.8]). Since $N(\eta_{\Delta}^{\text{frob}}) = 0$, Lemma 2.6 implies that we can write $\eta_{\Delta}^{\text{frob}} = \nabla G_x$ for some rigid section $G_x \in \mathcal{L}_{r}^{\text{rig}}(A_x)$ on each supersingular annulus $A_x$, and hence

$$d(F_{f, \infty}^{[p]} \wedge \alpha, G_x) = \langle \nabla F_{f, \infty}^{[p]} \wedge \alpha, G_x \rangle = \langle F_{f, \infty}^{[p]} \wedge \alpha, \eta_{\Delta}^{\text{frob}} \rangle.$$

In particular, the right-hand side in the last equality has residue 0, and hence

$$\text{Res}_{A_{x}} \langle F_{f, \infty}^{[p]} \wedge \alpha, \eta_{\Delta}^{\text{frob}} \rangle = -\text{Res}_{A_{x}} \langle \nabla F_{f, \infty}^{[p]} \wedge \alpha, G_x \rangle,$$

Plugging (2.18) into (2.17), we arrive at

$$\sum_{Q \in S \cap W_\infty} \text{Res}_{V_Q} \langle F_{f, Q}^{[p]} \wedge \alpha, \eta_{\Delta}^{\text{frob}} \rangle - \sum_{\kappa \in SS} \text{Res}_{A_{\kappa}} \langle \nabla F_{f, \infty}^{[p]} \wedge \alpha, \eta_{\Delta}^{\text{frob}} \rangle = 0.$$
Lemma 2.8. Let the notations be as in Proposition 2.7. Then, for each \( \lambda \in \mathcal{O}_K \), \( \langle \omega_f, \eta \rangle = 1 \), where \( \lambda \mapsto \lambda^\rho \) denotes the action of the non-trivial automorphism of \( K \), and \( \langle \cdot, \cdot \rangle \) is the cup product pairing on \( H^1_{\dR}(A/F) \). If \( (A', t_{A'}, \omega_{A'}) \) is the CM triple induced from \( (A, t_A, \omega_A) \) by an isogeny \( \varphi \in \text{Isog}^\text{et}_r(\Lambda) \), we define \( \eta_{A'} \in H^1_{\dR}(A'/F) \) by the analogous recipe. For the integers \( j \) with \( 0 \leq j \leq r \), the classes \( \omega_{A'}^j \eta_{A'}^{r-j} \) defined in [BDP13, (1.4.6)] then form a basis of \( \text{Sym}^r H^1_{\dR}(A'/F) \).

Lemma 2.8. Let the notations be as in Proposition 2.7. Then, for each \( 0 \leq j \leq r \), we have

\[
\lambda^* \eta_{A} = \lambda^\rho \eta_{A} \quad \text{for all} \ \lambda \in \mathcal{O}_K, \quad \langle \omega_f, \eta \rangle = 1,
\]

where \( \lambda \mapsto \lambda^\rho \) denotes the action of the non-trivial automorphism of \( K \), and \( \langle \cdot, \cdot \rangle \) is the cup product pairing on \( H^1_{\dR}(A/F) \). If \( (A', t_{A'}, \omega_{A'}) \) is the CM triple induced from \( (A, t_A, \omega_A) \) by an isogeny \( \varphi \in \text{Isog}^\text{et}_r(\Lambda) \), we define \( \eta_{A'} \in H^1_{\dR}(A'/F) \) by the analogous recipe. For the integers \( j \) with \( 0 \leq j \leq r \), the classes \( \omega_{A'}^j \eta_{A'}^{r-j} \) defined in [BDP13, (1.4.6)] then form a basis of \( \text{Sym}^r H^1_{\dR}(A'/F) \).

Proof. This follows from Proposition 2.7 as in [BDP13, Lemma 3.22]. \( \square \)

Recall that if \( f \in S_k(X) \) is a cuspidal form of weight \( k \) and level \( \Gamma \), then \( f|_{\nu_\infty} \) defines a \( p \)-adic modular form \( f_p \in \mathcal{M}_k(N) \) of weight \( k \) and tame level \( N \). Evaluated on a CM triple \( (A', t_{A'}, \omega_{A'}) \) of conductor \( p \) prime to \( p \), we then have

\[
f_p(A', t_{A'}, \omega_{A'}) = f(A', t_{A'}, \alpha_{p}^* \omega_{A'}),
\]

where \( \alpha_{p}^* : A' \rightarrow A'/[p][p] \) is the \( p \)-isogeny defined by the canonical subgroup of \( A' \) (see Section 1.5). By abuse of notation, in the following we will denote \( f_p \) also by \( f \). The map \( V \) defined in (2.11) yields an operator \( V : \mathcal{M}_k(N) \rightarrow \mathcal{M}_{k+2}(N) \) on \( p \)-adic modular forms whose effect on \( q \)-expansions is given by \( q \mapsto q^p \). Let \( a_p(f) \) be the \( U_p \)-eigenvalue of \( f \), and define the \( p \)-depletion of \( f \) by

\[
f|_p := f - a_p(f)V f.
\]

Letting \( d = q \frac{d q}{d q} : \mathcal{M}_k(N) \rightarrow \mathcal{M}_{k+2}(N) \) be the Atkin–Serre operator, for any integer \( j \) the limit

\[
d^{-1-j} f|_p := \lim_{t \to 1^-} d^{-1} f|_{p^t}
\]

is a \( p \)-adic modular form of weight \( k - 2 - j \) and tame level \( N \) (see [Ser73, Thm. 5]).

Lemma 2.9. Let the notation be as in Proposition 2.7. Then for each \( 0 \leq j \leq r \) there exists a locally analytic \( p \)-adic modular form \( G_j \) of weight \( r - j \) and tame level \( N \) such that

\[
(F_{f, \infty}(P_{A'}), \omega_{A'}^j \eta_{A'}^{r-j})_{A'} = G_{j}(A', t_{A'}, \omega_{A'}),
\]

where \( F_{f, \infty} \) is the Coleman primitive of \( \omega_f \) on \( \mathcal{W}_{\infty} \) (vanishing at \( \infty \) if \( r = 0 \)), and

\[
G_{j}(A', t_{A'}, \omega_{A'}) - \frac{a_p(f)}{p^{r-j+1}} G_{j}(p*A', t_{A'}, \omega_{A'}) = j!d^{-1-j} f|_{p^j}(A', t_{A'}, \omega_{A'}). \]
Proof. The construction of $G_j$ as the “$j$-th component” of $F_{j, \infty}$ is given in [BDP13, p.1083], and (2.22) then follows from the definition. On the other hand, (2.23) follows from the same calculations as in [loc.cit., Lemma 3.23 and Prop. 3.24]. \qed

We now relate the expression appearing in the right-hand side of Proposition 2.7 to the value of a certain $p$-adic $L$-function associated to $f$.

Recall that $(A, t_A, \omega_A)$ denotes the CM triple introduced in Section 1.5, and fix an elliptic curve $A_0/H_c$ with $\text{End}_{H_c}(A_0) \simeq \mathcal{O}_c$. The curve $A_0$ is related to $A$ by an isogeny $\varphi_0: A \to A_0$ in $\text{Isog}_c^\mathbb{A}(A)$, and we let $(A_0, t_0, \omega_0)$ be the induced triple. Since we assume that $p = \mathfrak{p}p$ splits in $K$, we may fix an isomorphism $\mu_{p^\infty} \simeq A_0[p^\infty]$ of $p$-divisible groups, where $A_0/\mathcal{O}_{C_p}$ is a good integral model of $A_0$. This amounts to the choice of an isomorphism $\iota: A_0 \to \hat{\mathbb{G}}_m$ of formal groups, and we let $\Omega_p \in C_p^\times$ be the $p$-adic period defined by the rule

$$\omega_0 = \Omega_p \cdot \omega_{\text{can}},$$

where $\omega_{\text{can}} := \frac{\text{d}t}{t}$ for the standard coordinate $t$ on $\hat{\mathbb{G}}_m$.

Finally, consider the set $\Sigma_{k,c}^+$ of algebraic Hecke characters $\chi: K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ of conductor $c$, infinity type $(k + j, -j)$ with $j \geq 0$ (with the convention in [BDP13, p.1089]), and such that

$$\chi|_{\mathbb{A}_K^\times} = N^k,$$

where $N$ is the norm character on $\mathbb{A}_K^\times$, and for every $\chi \in \Sigma_{k,c}^+$ set

$$L_p(f)(\chi) := \sum_{[a] \in \text{Pic}(\mathcal{O}_c)} \chi^{-1}(a)N(a)^{-j} \cdot d^f|_p(a \ast (A_0, t_0, \omega_{\text{can}})),$$

and define

$$L_{\text{alg}}(f, \chi^{-1}) := w(f, \chi)^{-1}C(f, \chi, c) \cdot \frac{L(f, \chi^{-1}, 0)}{\Omega^{2(k+2j)}},$$

where $w(f, \chi)$ and $C(f, \chi, c)$ are the constants defined in [BDP13, (5.1.11)] and [loc.cit., Thm. 4.6], respectively. $\Omega$ is the complex period in [loc.cit., (5.1.16)], and $L(f, \chi^{-1}, 0)$ is the central critical value of the Rankin–Selberg $L$-function $L(f \times \theta_{\chi^{-1}}, s)$ of $f$ and the theta series of $\chi^{-1}$.

As explained in [BDP13, p.1134], the set $\Sigma_{k,c}^+$ may be endowed with a natural $p$-adic topology, and we let $\hat{\Sigma}_{k,c}$ denote its completion.

Theorem 2.10. The assignment $\chi \mapsto L_p(f)(\chi)$ extends to a continuous function on $\hat{\Sigma}_{k,c}$ and satisfies the following interpolation property. If $\chi \in \Sigma_{k,c}^+$ has infinity type $(k + j, -j)$, with $j \geq 0$, then

$$\frac{L_p(f)(\chi)^2}{\Omega_p^{2(k+2j)}} = (1 - a_p(f)\chi^{-1}(\mathfrak{p}))^2 \cdot L_{\text{alg}}(f, \chi^{-1}, 0).$$

Proof. See Theorem 5.9, Proposition 5.10, and equation (5.2.4) of [BDP13], noting that $\beta_p = 0$ here, since $f$ has level divisible by $p$. \qed

Let $\Sigma_{k,c}^-$ be the set of algebraic Hecke characters of $K$ of conductor $c$ and infinity type $(k - 1 - j, 1 + j)$, with $j \geq 0$. Even though $\Sigma_{k,c}^+ \cap \Sigma_{k,c}^- = \emptyset$, any character in $\Sigma_{k,c}^-$ can be written as a limit of characters in $\Sigma_{k,c}^+$ (see [BDP13, p.1137]). Thus for any $\chi \in \Sigma_{k,c}^-$, the value $L_p(f)(\chi)$ is defined by continuity.

The next result extends the $p$-adic Gross–Zagier formula of [BDP13, Thm. 5.13] to the semistable non-crystalline setting.

Theorem 2.11. Let $f = q + \sum_{n=2}^{\infty} a_n(q^n) \in S_k(\Gamma_0(Np))$ be a $p$-new eigenform of weight $k = r+2 \geq 2$, and suppose that $\chi \in \Sigma_{k,c}^-$ has infinity type $(r + 1 - j, 1 + j)$, with $0 \leq j \leq r$. Then

$$\frac{L_p(f)(\chi)}{\Omega_p^{2(r+2j)}} = (1 - a_p(f)\chi^{-1}(\mathfrak{p})) \cdot \left(\frac{c_j}{j!} \sum_{[a] \in \text{Pic}(\mathcal{O}_c)} \chi^{-1}(a)N(a) \cdot AJ_f(\Delta_{E_{\varphi_0}})(\omega_f \wedge \omega_A^{r-j})\right).$$
we see that is given by (1)

On the other hand, by Lemma 2.9 we have

Substituting (2.26) into (2.25), summing over \([a] \in \text{Pic}(O_C)\), and noting that

we see that

Finally, since the isogeny \(\varphi_n \varphi_0 : (A, t_A, \omega_A) \to (A_0, t_0, \omega_0)\) has degree \(cN(n)\), combining Lemma 2.8 and Lemma 2.9 we have

and substituting (2.28) into (2.27), the result follows.

3. Main result

In this section we prove the main result of this paper, giving an “exceptional zero formula” for the specializations of Howard’s big Heegner points at exceptional primes in the Hida family.

3.1. Heegner points in Hida families. We begin by briefly reviewing the constructions of [How07b], which we adapt to our situation, referring the reader to [loc. cit., §2] for further details.

Recall that \(p \mid N\). Let \(f = \sum_{n=1}^{\infty} a_n(f)q^n \in \mathcal{S}_k(\Gamma_0(Np))\) be a newform, fix a finite extension \(L\) of \(\mathbb{Q}_p\) with ring of integers \(O_L\) containing the Fourier coefficients of \(f\), and let

be the Galois representation associated to \(f\). Also, let \(K = \mathbb{Q}(\sqrt{-D_K})\) be an imaginary quadratic field as in §1.5. For the rest of this paper, these will be subject to the following further hypotheses.

Assumptions 3.1. \(f\) is ordinary at \(p\), i.e., \(\nu_p(a_p(f))\) is a \(p\)-adic unit;

\(p\) is absolutely irreducible;

\(p\) ramified at every prime \(q\) dividing \((D_K, N)\);

\(p \nmid h_K := |\text{Pic}(O_K)|\), the class number of \(K\).

Note that by [How07b, Lemma 2.15], the first assumption forces the weight of \(f\) to be \(k = 2\), which will thus be assumed for the rest of this paper.

Definition 3.2. Set \(A_\mathbb{Q}_p := \mathbb{Q}_p[[1+p\mathbb{Z}_p]]\). For any \(A_\mathbb{Q}_p\)-algebra \(A\), let \(A^\times_\mathbb{Q}_p(A)\) be the set of continuous \(O_L\)-algebra homomorphisms \(\nu : A \to \overline{\mathbb{Q}}_p\) such that the composition

is given by \(\gamma \mapsto \gamma^k_{\nu-2}\), for some integer \(k_{\nu} \geq 2\) with \(k_{\nu} \equiv 2 \pmod{2(p-1)}\) called the weight of \(\nu\).

Since \(f\) is ordinary at \(p\), by [Hid86, Cor. 1.3]) there exists a local reduced finite integral extension \(\mathcal{O}_L\) of \(A_{\mathbb{Q}_p}\), and a formal \(q\)-expansion \(f = \sum_{n=1}^{\infty} a_n q^n \in \mathcal{O}_L[[q]]\) uniquely characterized by the following property. For every \(\nu \in A^\times_\mathbb{Q}_p(A)\) of weight \(k_{\nu} > 2\), there exists a newform \(f_{\nu} \in \mathcal{S}_{k_{\nu}}(\Gamma_0(N))\) such that

and we there exists a unique \(\nu_{f} \in A^\times_\mathbb{Q}_p(A)\) of weight 2 such that \(\nu_{f}(f) = f(q)\).
By [Hid86, Thm. 1.2], there is a free $\mathfrak{l}$-module $T$ of rank 2 equipped with a continuous action

$$\rho_{T} : G_{\mathbb{Q}} \rightarrow \text{Aut}_{\mathfrak{l}}(T) \cong \text{GL}_{2}(\mathfrak{l})$$

such that for every $\nu \in \mathcal{X}_{0}(\mathfrak{l})$, $\nu(\rho_{T})$ is isomorphic to the Galois representation $\rho_{f_{\nu}} : G_{\mathbb{Q}} \rightarrow \text{GL}_{2}(\overline{\mathbb{Q}}_{p})$ associated to $f_{\nu}$. Moreover, by [Wil88, Thm. 2.2.2], if $D_{p} \subset G_{\mathbb{Q}}$ is the decomposition group of any place $\mathfrak{P}$ of $\mathbb{Q}$ above $p$, there exists an exact sequence of $\mathfrak{l}[D_{p}]$-modules

$$(3.2) \quad 0 \rightarrow \mathcal{F}^{+}T \rightarrow T \rightarrow \mathcal{F}^{-}T \rightarrow 0$$

with $\mathcal{F}^{\pm}T$ free of rank 1 over $\mathfrak{l}$, and with the $D_{p}$-action on $\mathcal{F}^{-}T$ given by the unramified character sending an arithmetic Frobenius $\text{Fr}_{p}^{-1}$ to $a_{p} \in \mathfrak{l}^{\times}$.

Following [How07b, Def. 2.1.3], define the critical character $\Theta : G_{\mathbb{Q}} \rightarrow \mathfrak{l}^{\times}$ by the composite

$$(3.3) \quad \Theta : G_{\mathbb{Q}} \xrightarrow{\varepsilon_{\text{cyc}}} \mathbb{Z}_{p}^{*} \xrightarrow{(\cdot)} 1 + p\mathbb{Z}_{p} \xrightarrow{\gamma_{p}^{-1/2}} 1 + p\mathbb{Z}_{p} \twoheadrightarrow \Lambda_{\mathfrak{O}}^{\times} \twoheadrightarrow \mathfrak{l}^{\times},$$

where $\varepsilon_{\text{cyc}}$ is the $p$-adic cyclotomic character, $(\cdot)$ denotes the projection to the 1-units in $\mathbb{Z}_{p}$. Let $\mathfrak{l}^{\perp}$ be the free $\mathfrak{l}$-module of rank 1 where $G_{\mathbb{Q}}$ acts via $\Theta^{-1}$, and set

$$T^{\perp} := T \otimes_{\mathfrak{l}} \mathfrak{l}^{\perp}$$

equipped with the diagonal Galois action. Then, if for every $\nu \in \mathcal{X}_{0}(\mathfrak{l})$ we let $V_{f_{\nu}}$ be a representation space for $\rho_{f_{\nu}}$, then $\nu(T^{\perp}) := T^{\perp} \otimes_{1,\mathfrak{p}} \nu(\mathfrak{l})$ is isomorphic to a lattice in the self-dual Tate twist $V_{f_{\nu}}(k_{\nu}/2)$ of $V_{f_{\nu}}$ (see [Oht95, Thm. 1.4.3] and [NP00, (3.2.4)]).

Let $K_{\infty}$ be the anticyclotomic $\mathbb{Z}_{p}$-extension of $K$, and for each $n \geq 0$, let $K_{n}$ be the subfield of $K_{\infty}$ with $\text{Gal}(K_{n}/K) \cong \mathbb{Z}/p^{n}\mathbb{Z}$.

Theorem 3.3 (Howard). There is a system of “big Heegner points”

$$\mathfrak{H}_{\infty} := \{\mathfrak{H}_{n}\}_{n \geq 0} \in H_{1}^{\text{fr}}(K_{\infty}, T^{\perp}) := \lim \limits_{\leftarrow n} H_{1}^{\text{fr}}(K_{n}, T^{\perp})$$

with the following properties.

1. For each $n$, $\mathfrak{H}_{n}$ belongs to the Greenberg Selmer group $\text{Sel}_{\text{Gr}}(K_{n}, T^{\perp})$ of [How07b, Def. 2.4.2]. In particular, for every prime $q$ of $K$ above $p$, we have

$$\text{loc}_{q}(\mathfrak{H}_{\infty}) \in \ker \left( H_{1}^{\text{fr}}(K_{\infty}, q, T^{\perp}) \twoheadrightarrow H_{1}^{\text{fr}}(K_{n}, q, \mathcal{F}^{-}T) \right)$$

for the natural map induced by (3.2).

2. If $\mathfrak{H}^{\perp}_{\infty}$ denotes the image of $\mathfrak{H}_{\infty}$ under the action of complex conjugation, then

$$\mathfrak{H}^{\perp}_{\infty} = w \cdot \mathfrak{H}_{\infty}$$

for some $w \in \{\pm 1\}$.

Proof. In the following, all the references are to [How07b]. The construction of $\mathfrak{H}_{\infty}$ is given in §§2.2, 3.3 and the proof of (1) is given in Prop. 2.4.5. For the proof of (2), we need to briefly recall the definition of $\mathfrak{H}_{n}$. Let $H_{p^{n+1}}$ be the ring class field of $K$ of conduction $p^{n+1}$, and note that it contains $K_{n}$. By Prop. 2.3.1, the “big Heegner points” $\mathfrak{X}_{p^{n+1}} \in H_{1}(H_{p^{n+1}} T^{\perp})$ satisfy $\text{Cor}_{H_{p^{n+1}}/H_{p^{n}}}(\mathfrak{X}_{p^{n+1}}) = U_{p} \cdot \mathfrak{X}_{p^{n}}$, and hence the classes

$$(3.4) \quad \mathfrak{H}_{n} := U_{p}^{-n} \cdot \text{Cor}_{H_{p^{n+1}}/K_{n}}(\mathfrak{X}_{p^{n+1}})$$

are compatible under corestriction. Denoting by $\tau$ the image of a class under the action of complex conjugation and using Prop. 2.3.5, we find that

$$(3.5) \quad \text{Cor}_{H_{p^{n+1}}/K_{n}}(\mathfrak{X}_{p^{n+1}}) = \sum_{\sigma \in \text{Gal}(H_{p^{n+1}}/K_{n})} \mathfrak{X}_{p^{n+1}}^{\sigma}$$

for some $w \in \{\pm 1\}$. Combining (3.4) and (3.5), the result follows. □
3.2. Two-variable $p$-adic $L$-functions. As in the preceding section, let $f \in S_2(\Gamma_0(Np))$ be a newform split multiplicative at $p$, and let $f \in \mathbb{I}[[q]]$ be the Hida family passing through $f$. Recall the spaces of characters $\Sigma_{K,c}$ and $\Sigma_{K,c}$ introduced in Section 2.5. In the following, we only consider the case $c = 1$, which will henceforth be suppressed from the notation.

By [BDP13, Prop. 5.10] (see also Theorem 2.10), for every $\nu \in X_{O_L}^0(\mathbb{I})$ the assignment

$$\chi \mapsto L_p(f_\nu)(\chi) := \sum_{[a] \in \text{Pic}(O_K)} \chi^{-1}(a) N(a)^{-j} \cdot d^j f_\nu^a(a \ast (A_0, t_0, \omega_{can}))$$

extends to a continuous function on $\Sigma_{K,c}$. Using the explicit expression for these values, it is easy to show the existence of a two-variable $p$-adic $L$-function interpolating $L_p(f_\nu)$ for varying $\nu$. For the precise statement, denote by $h = h_K$ the class number of $K$ (which we assume is prime to $p$), and let $\phi_{o}$ be the unramified Hecke character defined on fractional ideals by the rule

$$(3.6) \quad \phi_{o}(a) = \alpha/\overline{\alpha}, \quad \text{where } (\alpha) = a^h.$$

Assume that $O_L$ contains the values of $\phi_{o}$, and denote by $\langle \phi_{o} \rangle$ the composition of $\phi_{o}$ with the projection onto the $Z_{p}$-free quotient of $O_L^{\times}$ which then is valued in $1 + pZ_p$, and define $\xi : K^{\times} \backslash A_{K}^{\times} \rightarrow \mathbb{I}$ by

$$(3.7) \quad \xi : K^{\times} \backslash A_{K}^{\times} \xrightarrow{\phi_{o}} O_L^{\times} \xrightarrow{(i)} 1 + pZ_p \xrightarrow{1 + pZ_p \rightarrow 1 + pZ_p} A_{O_L}^{\times} \rightarrow \mathbb{I}.$$

Similarly, recall the critical character $\Theta : G_K \rightarrow \mathbb{I}$ from (3.3), and define $\chi : K^{\times} \backslash A_{K}^{\times} \rightarrow \mathbb{I}$ by

$$\chi(x) = \Theta(\text{rec}_Q(N_{K/Q}(x))),$$

where $\text{rec}_Q : A_{K}^{\times} \rightarrow \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ is the geometrically normalized global reciprocity map. Let $\Gamma_{\infty} := \text{Gal}(K_{\infty}/K)$ be the Galois group of the anticyclotomic $Z_p$-extension of $K$, and denote by $X_{O_L}^0(\Gamma_{\infty})$ the set of continuous $O_L$-algebra homomorphisms $O_L[[\Gamma_{\infty}]] \rightarrow Q_p^{\times}$ induced by a character $\phi$ of the form $\phi = \phi_{o}^{\ell_{\phi}/h}$ for some integer $\ell_{\phi} \geq 0$ with $\ell_{\phi} \equiv 0 \pmod{p-1}$. Finally, let

$$N_K : K^{\times} \backslash A_{K}^{\times} \xrightarrow{N_{K/Q}} Q^{\times} \backslash A_{Q}^{\times} \xrightarrow{N} C^{\times}$$

be the norm character of $K$, and for every $\nu \in X_{O_L}^0(\mathbb{I})$, let $\xi_{\nu}$ and $\chi_{\nu}$ be the composition of $\xi$ and $\chi$ with $\nu$, respectively.

**Theorem 3.4.** The exists a continuous function $L_{p, \xi}(f)$ on $X_{O_L}^0(\mathbb{I}) \times X_{O_L}^0(\Gamma_{\infty})$ such that for every $\nu \in X_{O_L}^0(\mathbb{I})$ we have

$$L_{p, \xi}(f)(\nu, \phi) = L_p(f_{\nu})(\phi_{o} \chi_{\nu} N_K)$$

as functions of $\phi \in X_{O_L}^0(\Gamma_{\infty})$.

**Proof.** See [Cas14, Thm. 1.4]. (Note that if $(\nu, \phi) \in X_{O_L}^0(\mathbb{I}) \times X_{O_L}^0(\Gamma_{\infty})$, then $\phi_{o} \chi_{\nu} N_K$ is an unramified Hecke character of infinity type $(h_{\nu} + \ell_{\phi} - 1, 1 - \ell_{\phi})$, thus lying in the domain of $L_p(f_{\nu})$.) \hfill \Box

3.3. Ochiai’s big logarithm maps. By our assumption that $p \nmid h = h_K$, the extension $K_{\infty}/K$ is totally ramified at every prime $q$ above $p$; let $K_{\infty,q}$ be the completion of $K_{\infty}$ at the unique prime above $q$, and set $\Gamma_{q,\infty} = \text{Gal}(K_{\infty,q}/K_q)$. Even though $\Gamma_{q,\infty}$ may be identified with $\Gamma_{\infty}$, in the following it will be convenient to maintain the distinction between them. Write $q^a = \pi_{q} O_K$, and set $\omega_{q} = \pi_{q}/\pi_{\mathfrak{p}} \in K_{q}^{\times}$; in particular, note that $\omega_{\mathfrak{p}} = \omega_{p}^{-1}$.

Recall the $\mathfrak{p}$-adic Hecke character introduced in (3.7), and let $\xi : G_K \rightarrow \mathbb{I}$ also denote the Galois character defined by

$$\xi(\sigma) := (\hat{\phi}_{o}(\sigma))^{1/2h},$$

where $\hat{\phi}_{o} : G_{K} \rightarrow O_{L}^{\times}$ is the $p$-adic avatar of the Hecke character $\phi_{o}$ in (3.6). Finally, set

$$T_{p} := T_{p}|_{G_{K} \otimes \xi^{-1}}, \quad T_{p|_{\mathfrak{p}}} := T^{T}|_{G_{K} \otimes \xi},$$

and for every $\nu \in X_{O_L}^0(\mathbb{I})$ denote by $V_{\nu}$ the specialization of $T_{p}$ at $\nu$ (it will be clear from the context which prime $q$ above $p$ is meant).
Theorem 3.5. Let $\lambda = a_p - 1$ and set $\bar{I} := [1]\otimes \mathbb{Z}_p \hat{\mathbb{Z}}_p^{\nu}$. For each $q \in \{p, \emptyset\}$, there exists an $\mathbb{I}[[\Gamma_{q,\infty}]]$-linear map

$$L_{\mathcal{F}+T_q}^\omega : H^1_{\text{tw}}(K_{\mathcal{F}+T_q}) \rightarrow \bar{I}[[\Gamma_{q,\infty}]]$$

such for every $\mathcal{Q}_\infty \in H^1_{\text{tw}}(K_{\mathcal{F}+T_q})$ and every $(\nu, \varphi) \in \mathcal{X}^2 \cap \{\bar{I}\}$ we have

$$(1 - \left(\frac{p^{1/h}}{n}\right)^{k_1/2-1} \frac{\nu(a_p)}{p^{1/h}})^{-1} \left(\phi - \nu(a_p)b_p^{p^{-1}} \left(\log \left(\frac{\nu(\mathcal{Q}_\infty)\mathcal{Q}_\infty}{\mathcal{Q}_\emptyset}\right)\right)\right)$$

where $\log = \log_{\mathcal{F}+V_0} : H^1(K_{\mathcal{F}+V_0} \otimes \mathcal{O}) \rightarrow D_{\text{dr}}(\mathcal{F}+V_0 \otimes \mathcal{O})$ is the Bloch–Kato logarithm map, and $\nu(\mathcal{Q}_\infty)^{\phi} \in H^1(K_q, \mathcal{F}+V_0 \otimes \mathcal{O})$ is the $\phi$-specialization of $\nu(\mathcal{Q}_\infty)$.

Proof. See [Cas14, Prop. 4.3].

Remark 3.6. Fix a compatible system $\zeta_\infty = \{\zeta_r \}_{r \geq 0}$ of $p$-power roots of unity, and let $\zeta_\infty t^{-1}$ be the associated basis element of $D_{\text{dr}}(\mathcal{Q}_0(1))$. In Theorem 3.5 above, $\omega$ denotes a generator of the module

$$\mathbb{D} := (\mathcal{F}+T_q(1) \otimes \mathbb{Z}_p \hat{\mathbb{Z}}_p^{\nu})^{G_{\mathcal{K}_4}},$$

which by [Och03, Lemma 3.3] is free of rank one over $\mathbb{I}$. (Note that $\mathcal{F}+T_q(1)$ is unramified.) As explained in loc.cit., for each $\nu \in \mathcal{X}^2 \cap \{\bar{I}\}$ there is a specialization map $\nu : \mathbb{D} \rightarrow \mathbb{D} \otimes \mathbb{Z}_p \hat{\mathbb{Z}}_p^{\nu} \mathcal{Q}_p \simeq D_{\text{dr}}(\mathcal{F}+V_0(1))$.

Then, letting $\omega_\nu$ denote the image of $\omega_\nu(\omega) \otimes \zeta_\infty t^{-1}$ in $D_{\text{dr}}(\mathcal{F}+V_0(1)) \otimes D_{\text{dr}}(\mathcal{Q}_0(1)) \simeq D_{\text{dr}}(\mathcal{F}+V_0)$, the class $\hat{\omega}_\nu \in D_{\text{dr}}(\mathcal{F}^\nu_0(1))$ in the above interpolation formulæ is defined by requiring that

$$\langle \omega_\nu, \hat{\omega}_\nu \rangle = 1$$

under the de Rham pairing $\langle ., . \rangle : D_{\text{dr}}(\mathcal{F}+V_0) \times D_{\text{dr}}(\mathcal{F}^\nu_0(1)) \rightarrow \mathcal{O}$.

The big logarithm map $L_{\mathcal{F}+T_q}^\omega$ of Theorem 3.5 may not be specialized at any pair $(\nu, 1)$ with $\nu \in \mathcal{X}^2 \cap \{\bar{I}\}$ such that $\nu(\lambda) = 0$, i.e., $\nu(a_p) = 1$. Since such arithmetic primes are in fact the main concern in this paper, the following construction of an “improved” big logarithm will be useful.

Proposition 3.7. There exists an $\mathbb{I}$-linear map

$$\mathcal{L}_{\mathcal{F}+T_q}^\omega : H^1(K_p, \mathcal{F}+T_p) \rightarrow \mathbb{I} \otimes \mathbb{Z}_p \hat{\mathbb{Z}}_p$$

such for every $\mathcal{Q}_0 \in H^1(K_p, \mathcal{F}+T_p)$ and every $\nu \in \mathcal{X}^2 \cap \{\bar{I}\}$, we have

$$\nu(\mathcal{L}_{\mathcal{F}+T_q}^\omega(\mathcal{Q}_0)) = \left(1 - \left(\frac{a_p}{\Theta^{-1}(Fr_p)}\right)^{-1} \left(\log J_{\mathcal{F}}(\nu(\mathcal{Q}_0))\right)\right).$$

Proof. This can be shown by adapting the methods of Ochiai [Och03, §5]. Indeed, let

$$L_{\mathcal{F}+T_q} : H^1(K_{\mathcal{F}+T_q}) \otimes \mathcal{Q}_p \rightarrow \mathbb{D} \otimes \mathbb{Z}_p \hat{\mathbb{Z}}_p \mathcal{Q}_p$$

be the inverse of the map $\text{exp}_T$ constructed in [Ven15, Prop. 3.8] (see Remark 3.6 for the definition of $\mathbb{D}$), and define

$$L_{\mathcal{F}+T_q}^\omega : H^1(K_p, \mathcal{F}+T_p) \rightarrow \mathbb{I} \otimes \mathbb{Z}_p \hat{\mathbb{Z}}_p \mathcal{Q}_p$$

by the relation $L_{\mathcal{F}+T_q}(\cdot) = L_{\mathcal{F}+T_q}^\omega(\cdot) \cdot \omega$. Setting

$$L_{\mathcal{F}+T_q}^\omega = \left(1 - \left(\frac{a_p}{\Theta^{-1}(Fr_p)}\right)^{-1} \left(\log J_{\mathcal{F}}(\nu(\mathcal{Q}_0))\right)\right) \mathcal{L}_{\mathcal{F}+T_q}^\omega : H^1(K_p, \mathcal{F}+T_p) \rightarrow \mathbb{I} \otimes \mathbb{Z}_p \hat{\mathbb{Z}}_p \mathcal{Q}_p,$$

the result follows.

Corollary 3.8. For any $\mathcal{Q}_\emptyset = \{\mathcal{Q}_\emptyset\}_{n \geq 0} \in H^1_{\text{tw}}(K_{\mathcal{F}+T_p})$ we have the factorization in $\mathbb{I}$:

$$1 - \left(\frac{\Theta^{-1}(Fr_p)}{a_p}\right) \cdot \mathcal{V}(\mathcal{Q}_\emptyset) = \mathcal{L}_{\mathcal{F}+T_p}^\omega(\mathcal{Q}_\emptyset),$$

where $\mathcal{V} : \mathbb{I}[[\Gamma_{\infty}]] \rightarrow \mathbb{I}$ is the augmentation map.
Proof. Comparing the interpolation formulas in Theorem 3.5 and Proposition 3.7, we see that
\[
\left( 1 - \frac{\Theta_p^{-1} \xi_p(\nu_p)}{\nu(a_p)} \right) L_{\mathcal{F}^\ell + \tau_p}(\nu, 1) = \nu \left( \mathcal{L}_{\mathcal{F}^\ell + \tau_p}(\mathcal{F}_0) \right)
\]
for every \( \nu \in \mathcal{X}_G^0(\mathbb{H}) \); since these primes are dense in \( \mathbb{H} \), the corollary follows.

The proof of our main result will rely crucially on the relation found in [Cas14, §4] between the \( p \)-adic \( L \)-function \( L_{p, x}(f) \) of Theorem 3.4 and Howard’s system of big Heegner points \( \bar{f}_\infty \). We conclude this section by briefly recalling that relation.

By [How07b, Lemma 2.4.4], for every prime \( q \) of \( K \) above \( p \) the natural map
\[
H^1_{Iw}(K_{\infty, q}, \mathbb{F}^+ T^1) \rightarrow H^1_{Iw}(K_{\infty, q}, T^1)
\]
induced by (3.2) is injective. In light of Theorem 3.3, in the following we will thus view \( \text{loc}_{q}(\mathcal{F}_\infty) \) as sitting inside \( H^1_{Iw}(K_{\infty, q}, \mathbb{F}^+ T^1) \).

**Theorem 3.9.** There is a generator \( \omega = \omega_f \) of the module \( \mathbb{N} \) such that
\[
\mathcal{L}_{\mathcal{F}^\ell + \tau_p}(\text{loc}_p(\mathcal{F}_\infty^{q-1})) = L_{p, x}(f)
\]
as functions on \( \mathcal{X}_{\mathcal{O}_K}^0(\mathbb{H}) \times \mathcal{X}_{\mathcal{O}_K}^0(\Gamma_{\infty}) \).

**Proof.** The construction of the basis element \( \omega = \omega_f \) of \( \mathbb{D} \) is deduced in [KLZ15, Prop. 10.1.2] from Ohta’s work [Oht00], and it has the property that \( \langle \omega_f, \omega_f \rangle = 1 \), for all \( \nu \in \mathcal{X}_{\mathcal{O}_K}^0(\mathbb{H}) \), where \( \omega_f \) is the class in \( \text{Fil}^1 \mathcal{D}_{\text{crys}}(V_f) \simeq \mathcal{D}_{\text{crys}}(\mathbb{F}^+ V_f^1) \) associated to the \( p \)-stabilized newform (3.1); in particular,
\[
\hat{\omega}_{\nu_f} = \omega_f
\]
in the notations of Remark 3.6. The result is then the content of [Cas14, Thm. 4.4].

3.4. Exceptional zero formula. Let \( f = \sum_{n=1}^{\infty} a_n(f) q^n \in S_2(\Gamma_0(Np)) \) be an ordinary newform as in Section 3.1, and assume in addition that \( f \) is split multiplicative at \( p \), meaning that \( a_p(f) = 1 \).

Recall the CM triple \((A, t, \nu_p) \subset X(H) \) introduced in Section 1.5, which maps to the point \( P_A = (A, A[\mathbb{Q}p]) \in X_0(Np) \) under the forgetful map \( X \rightarrow X_0(Np) \). Let \( \infty \) be any cusp of \( X_0(Np) \) rational over \( \mathbb{Q} \), and let \( \kappa_f \in H^1(K, V_f) \) be the image of \( (P_A) - (\infty) \) under the composite map
\[
(3.8) \quad J_0(Np)(H) \xrightarrow{\text{Kum}} H^1(H, \text{Tor}_{p}(J_0(Np)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \rightarrow H^1(H, V_f) \xrightarrow{\text{Cor}_{H/K}} H^1(K, V_f).
\]
If \( \mathfrak{f} \in \mathbb{H}([\mathfrak{q}]) \) is the Hida family passing through \( f \), and \( \nu_f \in \mathcal{X}_{\mathcal{O}_K}^0(\mathbb{H}) \) is the arithmetic prime of \( \mathbb{H} \) such that \( \nu_f(f) = 1 \), it would be natural to expect a relation between the class \( \kappa_f \) and the specialization at \( \nu_f \) of Howard’s big Heegner point \( \bar{f}_0 \). As done in [How07a, §3], one can trace through the construction of \( \bar{f}_0 \) to deduce a relation between the generic (in the sense of [loc.cit., Def. 2]) weight 2 specializations of \( \bar{f}_0 \) and the Kummer images of certain CM points. However, the arithmetic prime \( \nu_f \) is not generic in that sense, and in fact one does not expect a similar direct relation between \( \nu_f(\bar{f}_0) \) and \( \kappa_f \) (see the discussion in [loc.cit., p.813]).

In Theorem 3.11 below we will show that in fact the localization at \( \nu_f(\bar{f}_0) \) vanishes, but that nonetheless it can be related to \( \kappa_f \) upon taking a certain “derivative” in the following sense, where we let \( \log_p: \mathbb{Q}_p^* \rightarrow \mathbb{Z}_p \) be Iwasawa’s branch of the \( p \)-adic logarithm, i.e., such that \( \log_p(p) = 0 \).

**Lemma 3.10.** Let \( T \) be a free \( \mathcal{O}_L \)-module of finite rank equipped with a linear action of \( \text{Gal}(k_\infty/\mathbb{Q}_p) \) be a \( \mathbb{Z}_p \)-extension, and let \( \gamma \in \text{Gal}(k_\infty/\mathbb{Q}_p) \) be a topological generator. Assume that \( T^{\text{Gal}(\mathbb{Q}_p)} = \{0\} \), and let \( Z_\infty = \{ Z_n \}_{n \geq 0} \in H^1_{Iw}(k_\infty, T) \) be such that \( Z_0 = 0 \). Then there exists a unique \( Z'_{\infty, \gamma} = \{ Z'_{\infty, n} \}_{n \geq 0} \in H^1_{Iw}(k_\infty, T) \) such that
\[
Z_\infty = (\gamma - 1) \cdot Z'_{\infty, \gamma}.
\]
Moreover, if \( \eta: \text{Gal}(k_\infty/\mathbb{Q}_p) \simeq \mathbb{Z}_p \) is any group isomorphism, then
\[
Z'_\mathfrak{f} := \frac{Z'_{\infty, 0}}{\log_p(\eta(\gamma))} \in H^1(\mathbb{Q}_p, T[1/p])
\]
is independent of the choice of \( \gamma \).
Proof. Consider the module \(T_{\infty} := \mathcal{T} \otimes_{\mathcal{O}_L} \mathcal{O}_L[[\text{Gal}(k_{\infty}/\mathbb{Q}_p)]]\) equipped with the diagonal Galois action, where \(G_{\mathbb{Q}_p}\) acts on the second factor via the projection \(G_{\mathbb{Q}_p} \rightarrow \text{Gal}(k_{\infty}/\mathbb{Q}_p)\). By Shapiro’s Lemma, we then have
\[
H^1(\mathbb{Q}_p, T_{\infty}) \cong H^1_W(k_{\infty}, T),
\]
and the assumption that \(T^{G_{\mathbb{Q}_p}} = \{0\}\) implies that \(H^1(\mathbb{Q}_p, T_{\infty})\) is torsion-free. Therefore, the exact sequence of \(\mathcal{O}_L[[\text{Gal}(k_{\infty}/\mathbb{Q}_p)]]\)-modules
\[
0 \rightarrow T_{\infty} \xrightarrow{\gamma^{-1}} T_{\infty} \rightarrow T \rightarrow 0
\]
duces the cohomology exact sequence
\[
0 \rightarrow H^1(\mathbb{Q}_p, T_{\infty}) \xrightarrow{\gamma^{-1}} H^1(\mathbb{Q}_p, T_{\infty}) \rightarrow H^1(\mathbb{Q}_p, T),
\]
giving the proof of the first claim, and the second follows from an immediate calculation. \(\square\)

Recall the uniformizer \(\varpi_p = \pi_p/\pi_{\mathfrak{P}} \in \mathcal{O}_{L}^p \simeq \mathbb{Q}_p^\times\) introduced in Section 3.3, and define
\[
(3.9) \quad \mathcal{L}_p(f, K) := \mathcal{L}_p(f) - \mathcal{L}_p(\chi_K),
\]
where \(\mathcal{L}_p(f)\) is the \(\mathcal{L}\)-invariant of \(f\) (as defined in [MTT86, §II.14], for example), and
\[
\mathcal{L}_p(\chi_K) := \frac{\log(\varpi_p)}{\text{ord}_p(\varpi_p)} = -\frac{2\log_p(\varpi_p)}{\mu}
\]
is the \(\mathcal{L}\)-invariant of the quadratic character \(\chi_K\) associated to \(K\) (see [Gre94, §1], for example), with \(\text{ord}_p\) the \(p\)-adic valuation on \(\mathbb{Q}_p\) with the normalization \(\text{ord}_p(p) = 1\).

The following derivative formula is the main result of this paper.

Theorem 3.11. Let \(f \in S_2(\Gamma_0(Np))\) be a newform split multiplicative at \(p\), let \(f \in \mathbb{I}[[\mathbb{I}]]\) be the Hida family passing through \(f\), let \(\mathcal{Z}_{\infty} \in H^1_{\text{IV}}(K_{\infty}, \mathbb{T})\) be Howard’s system of big Heegner points, and define \(Z_{p,f,\infty} := \{Z_{p,f,n}\}_{n \geq 0} \in H^1_{\text{IV}}(K_{\infty,p}, \mathbb{F}^+ + V_f)\) by
\[
Z_{p,f,n} := \text{loc}_p(\nu_f(\mathcal{Z}_n)),
\]
where \(\nu_f \in \mathcal{X}_{\mathcal{O}_L}^{\infty}(\mathbb{I})\) is such that \(f = \nu_f(f)\). Then \(Z_{p,f,0} = 0\) and
\[
Z_{p,f,0} = \mathcal{L}_p(f, K) \cdot \text{loc}_p(\kappa_f),
\]
where \(\mathcal{L}_p(f, K)\) is the \(\mathcal{L}\)-invariant of \((0,4)\), and \(\kappa_f \in H^1(K, V_f)\) is the image of the degree zero divisor \((A, A[\mathfrak{P}] - \infty)\) under the Kummer map (3.8).\]

Proof. By Proposition 3.7, Corollary 3.8, Theorem 3.9, and Theorem 3.4, respectively, we see that
\[
(1 - a_p(f)p^{-1}) \cdot (\log(Z_{p,f,0}), \omega_f) = \lim_{\nu \rightarrow \nu_f} \nu \left(\mathcal{L}_p^{\omega}((\text{loc}_p(\mathcal{Z}_{0}^{\omega}))) \right) = \lim_{\nu \rightarrow \nu_f} \nu \left(1 - \frac{\Theta_{\nu^{-1}}(\mathcal{L}_p^{\omega}((\text{loc}_p(\mathcal{Z}_{0}^{\omega}))))(\nu, 1)}{\nu(A_p)} \right) = \lim_{\nu \rightarrow \nu_f} \left(1 - \frac{\Theta_{\nu^{-1}}(\mathcal{L}_p^{\omega}(\nu, 1))}{\nu(A_p)} \right) = \left(1 - a_p(f)p^{-1}\right) \cdot L_p(f, N_K).
\]
Since \(a_p(f) = 1\) by hypothesis, this shows that \((\log(Z_{p,f,0}), \omega_f) = 0\), and the vanishing of \(Z_{p,f,0}\) follows. Now to the proof of the derivative formula (3.10).\]

Denote by \(L_p^{\omega}(\mathbb{I})\) the image of \(L_p^{\omega}(f)\) under the involuton of \(\mathbb{I}[[\Gamma_{\infty}]]\) induced by complex conjugation, so that \(L_p^{\omega}(\mathbb{I})(\chi) = L_p^{\omega}(\mathbb{I})(\chi^{-1})\) for every character \(\chi\) of \(\Gamma_{\infty}\). One immediately checks the commutativity of the diagram
\[
\begin{aligned}
H^1_{\text{IV}}(K_{\infty}, \mathbb{F}^+ + V_p) &\xrightarrow{\text{loc}_p} H^1_{\text{IV}}(K_{\infty}, \mathbb{F}^+ + V_p) \\
\quad \xrightarrow{i} &\xrightarrow{i} \quad H^1_{\text{IV}}(K_{\infty}, \mathbb{F}^+ + V_p) \xrightarrow{\text{loc}_p} H^1_{\text{IV}}(K_{\infty}, \mathbb{F}^+ + V_p)
\end{aligned}
\]
where the left and middle vertical arrows denote the action of complex conjugation.
By the discussion in [GS93, §2.6], we may find a disc $U \subset \mathbb{Z}/2(p-1) \mathbb{Z} \times \mathbb{Z}_p$ contained in the residue class of 2, and a unique morphism of $\Lambda_{O_L}$-modules

$$\mathcal{M} = \mathcal{M}_f : I \longrightarrow \mathcal{S}_U$$

such that $\mathcal{M}(r)|_{k=2} = \nu_f(r)$ for every $r \in \mathbb{Z}$, where $\mathcal{S}_U \subset L([k-2])$ denotes the subring of power series convergent for $k \in U$ endowed with the $\Lambda_{O_L}$-algebra structure induced by the character $1 + p\mathbb{Z}_p \longrightarrow \mathcal{S}_U$ which sends $\gamma \in 1 + p\mathbb{Z}_p$ to the power series $\gamma^{k-2} := \exp((k-2) \log_p(\gamma))$.

For every $k \in U \cap \mathbb{Z}_{\geq 2}$ the composition of $\mathcal{M}$ with the evaluation map at $k$ defines an element $\nu_k \in \mathcal{X}_{O_L}(I)$. Set $a_p(k) := \mathcal{M}(a_p)$, and for $(k, \phi_k^H) \in (U \cap \mathbb{Z}_{\geq 2}) \times \mathcal{X}_{O_L}(\Gamma_{\infty})$, define the functions

$$\mathcal{L}_p(k, t) := \left(1 - \frac{(p\varpi_p^{1/k})^{k-1}}{p^2(1-1/k)}\right) \mathcal{L}_{\mathcal{M}_f}(\mathcal{F})(\nu_k, \phi_k^H), \quad \mathcal{L}_{\mathcal{M}}(k, t) := \left(1 - \frac{(p\varpi_p^{1/h})^{k-1}}{a_p(k)\varpi_p^{1/h}}\right) \mathcal{L}_{\mathcal{M}_f}(\mathcal{F})(\nu_k, \phi_k^{H^*}).$$

By the combination of Theorem 3.5 and Theorem 3.9, we then have

$$\mathcal{L}_p(k, t) = \frac{1}{t!} \left(1 - \frac{a_p(k)\varpi_p^H}{p(p\varpi_p^{1/k})^{k-1}}\right) \langle \log(\text{loc}_p(\nu_k(3\infty)^{\phi(1-k/2+1/\lambda)})), \hat{\omega}_{\nu_k}\rangle,$$

and by the above diagram we also have

$$\mathcal{L}_{\mathcal{M}}(k, t) = \frac{1}{t!} \left(1 - \frac{a_p(k)\varpi_p^H}{p(p\varpi_p^{1/k})^{k-1}}\right) \langle \log(\text{loc}_p(\nu_k(3\infty)^{\phi(k/2-1+1/\lambda)})), \hat{\omega}_{\nu_k}\rangle.$$

By the “functional equation” satisfied by $3_{\infty}$ (see Theorem 3.3), it follows that the function

$$\mathcal{L}_p(k, t) := \mathcal{L}_p(k, t) - wL_p(k, k-2 - t)$$

vanishes identically along the “line” $t = k/2 - 1$. By [How07b, Prop. 2.3.6], the sign $w$ is the opposite of the sign in the functional equation for the $p$-adic $L$-function $L_p(f, s)$ associated to $f$ in [MTT86]. Thus, if $w = 1$, then $\text{ord}_{s=1}L_p(f, s) > 2$, and by [Ven15, Lemma 6.1] the right-hand side of (3.10) vanishes; since the vanishing of the left-hand side follows easily from the construction of $Z_{\infty}^{s_0}$ in Lemma 3.10, we conclude that (3.10) reduces to the identity “$0 = 0$” when $w = 1$. As a consequence, in the following we shall assume that $w = -1$.

Using the formula for the $\mathcal{L}$-invariant of $f$ as the logarithmic derivative of $a_p(k)$ at $k = 2$ (see [GS93, Thm. 3.18], for example) and noting that $(p\varpi_p^{1/h})^{k/2-1} = \pi_p^{(k-2)/h}$ by definition, we find

$$\frac{\partial}{\partial k} \mathcal{L}_p(k, t) \mid_{(2, 0)} = \left[\frac{d}{dk}a_p(k) \mid_{k=2} - \frac{\log_p(\pi_p)}{h} - w \left(\frac{d}{dk}a_p(k) \mid_{k=2} - \frac{\log_p(\pi_p)}{h}\right)\right] L_p(f)(N_K)$$

$$= -\frac{(1-w)}{2} \left(\mathcal{L}_p(f) - \mathcal{L}_p(\chi_K)\right) L_p(f)(N_K)$$

$$= -\mathcal{L}_p(f, K) \cdot L_p(f)(N_K).$$

Using the aforementioned vanishing of $\mathcal{L}_p(k, k-2 - 1)$ for the first equality, we also find

$$\frac{\partial}{\partial k} \mathcal{L}_p(k, t) \mid_{(2, 0)} = -\frac{1}{2} \frac{\partial}{\partial t} \mathcal{L}_p(k, t) \mid_{(2, 0)} = -\frac{(1-w)}{2} (1 - a_p(f)p^{-1}) \langle \log(Z_{p, f, 0}), \omega_f\rangle$$

$$= -(1-p^{-1}) \cdot \langle \log(Z_{p, f, 0}), \omega_f\rangle,$$

and comparing (3.11) and (3.12), we arrive at the equality

$$\langle 1 - p^{-1} \cdot \langle \log(Z_{p, f, 0}), \omega_f\rangle, \mathcal{L}_p(f, K) \rangle = \mathcal{L}_p(f, K) \cdot L_p(f)(N_K).$$

On the other hand, letting $\varphi : A \longrightarrow A$ be the identity isogeny, by Theorem 2.11 we have

$$L_p(f)(N_K) = (1 - a_p(f)p^{-1}) \sum_{[a] \in \text{Pic}(O_K)} \langle A\mathcal{J}_F(\Delta_{\varphi, \varphi}), \omega_f\rangle$$

$$= (1-p^{-1}) \cdot \langle \log(\text{loc}_p(s_f)), \omega_f\rangle,$$

which combined with (3.13) concludes the proof of Theorem 3.11. \qed