PROJECTS: PROPAGATING THE IWASAWA MAIN CONJECTURE VIA CONGRUENCES

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ABSTRACT. We describe projects for the course by Christopher Skinner at AWS 2018.

1. GOAL OF THESE PROJECTS

Let \( f, g \in \mathcal{S}_k(\Gamma_0(N)) \) be normalized eigenforms (not necessarily newforms) of weight \( k \geq 2 \), say with rational Fourier coefficients \( a_n, b_n \in \mathbb{Q} \) for simplicity, and assume that

\[
f \equiv g \pmod{p}
\]

in the sense that \( a_n \equiv b_n \pmod{p} \) for all \( n > 0 \). Roughly speaking, the goal of these projects is to study how knowledge of the Iwasawa main conjecture for \( f \) can be “transferred” to \( g \).

For \( k = 2 \) and primes \( p \nmid N \) of ordinary reduction, such study was pioneered by Greenberg–Vatsal [GV00], and in these projects we will aim to extend some of their results to:

- non-ordinary primes;
- certain anticyclotomic settings;
- (more ambitiously) some of the “residually reducible” cases which eluded the methods of [GV00], with applications to the \( p \)-part of the BSD formula in ranks 0 and 1.

2. THE METHOD OF GREENBERG–VATSAL

Before jumping into the specifics of each of those settings, let us begin with a brief outline of the method of Greenberg–Vatsal (which is beautifully explained in [GV00, §1]). Let \( F_\infty/F \) be a \( \mathbb{Z}_p \)-extension of a number field \( F \), and identify the Iwasawa algebra \( \mathbb{Z}_p[[\text{Gal}(F_\infty/F)]] \) with the one-variable power series ring \( \Lambda = \mathbb{Z}_p[[T]] \) in the usual fashion.

Recall that Iwasawa’s main conjecture for \( f \) over \( F_\infty/F \) posits the following equality between principal ideals of \( \Lambda \):

\[
(L_{p}^{alg}(f)) \supseteq (L_{p}^{an}(f)),
\]

where

- \( L_{p}^{alg}(f) \in \Lambda \) is a characteristic power series of a Selmer group for \( f \) over \( F_\infty/F \).
- \( L_{p}^{an}(f) \in \Lambda \) is a \( p \)-adic \( L \)-function interpolating critical values for \( L(f/F, s) \) twisted by certain characters of \( \text{Gal}(F_\infty/F) \).

By the Weierstrass preparation theorem, assuming \( L_{p}^{alg}(f) \) is nonzero, we may uniquely write

\[
L_{p}^{alg}(f) = p^{\mu^{alg}(f)} \cdot Q^{alg}(f) \cdot U,
\]

with \( \mu^{alg}(f) \in \mathbb{Z}_{\geq 0} \), \( Q^{alg}(f) \in \mathbb{Z}_p[T] \) a distinguished polynomial, and \( U \in \Lambda^\times \) an invertible power series. Letting

\[
\lambda^{alg}(f) := \deg Q^{alg}(f),
\]

and similarly defining \( \mu^{an}(f) \) and \( \lambda^{an}(f) \) in terms \( L_{p}^{an}(f) \), the strategy of [GV00] is based on the following three observations:

**O1.** The equality (2.1) amounts to having:

\[
(1) \ (L_{p}^{alg}(f)) \supseteq (L_{p}^{an}(f)),
\]
(2) $\mu^\text{alg}(f) = \mu^\text{an}(f)$,

(3) $\lambda^\text{alg}(f) = \lambda^\text{an}(f)$.

We shall place ourselves in a situation where one expects that $\mu^\text{alg}(f) = \mu^\text{an}(f) = 0$.

**O2.** For $\Sigma$ any finite set of primes $\ell \neq p, \infty$, the equality (2.1) is equivalent to the equality

\[(2.2) \quad (L^\Sigma_{p,\text{alg}}(f)) \equiv (L^\Sigma_{p,\text{an}}(f)),\]

where $L^\Sigma_{p,\text{alg}}(f)$ and $L^\Sigma_{p,\text{an}}(f)$ are the “imprimitive” counterparts of $L^\text{alg}_p(f)$ and $L^\text{an}_p(f)$ obtained (roughly speaking) by relaxing the local conditions/removing the Euler factors at the primes $\ell \in \Sigma$.

**O3.** For appropriate $\Sigma$, the objects involved in (2.2) are well-behaved under congruences. Letting $\mu^\Sigma_{\text{alg}}(f)$, $\lambda^\Sigma_{\text{alg}}(f)$, etc. be the obvious invariants from the above discussion, this translates into:

**Expectation 1.** Assume that $f \equiv g \pmod{p}$, and let $* \in \{\text{alg, an}\}$. If $\mu^\Sigma_*(f) = 0$, then $\mu^\Sigma_*(g) = 0$ and $\lambda^\Sigma_*(f) = \lambda^\Sigma_*(g)$.

Now, if we are given $f \equiv g \pmod{p}$ and the divisibilities

\[(2.3) \quad (L^\text{alg}_p(f)) \supseteq (L^\text{an}_p(f)) \quad \text{and} \quad (L^\text{alg}_p(g)) \supseteq (L^\text{an}_p(g)),\]

we see that the equivalence of **O2** combined with **Expectation 1** yields the implication

\[(2.4) \quad (L^\text{alg}_p(f)) = (L^\text{an}_p(f)) \implies (L^\text{alg}_p(g)) = (L^\text{an}_p(g)).\]

Note that this has interesting applications. Indeed, if for example the residual representation $\bar{\rho}_f$ is absolutely irreducible, then one can hope to establish (2.3) by an Euler/Kolyvagin system argument. Proving the opposite divisibility (either via Eisenstein congruences, or via a refined Euler/Kolyvagin system argument) often requires additional ramification hypotheses on $\bar{\rho}_f$ relative to the level of $f$ (see below for specific examples), a restriction that could be ultimately removed thanks to (2.4).

### 3. ON THE CYCLOTOMIC MAIN CONJECTURES FOR NON-ORDINARY PRIMES

Here we let $F_{\text{alg}}/F$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, let $p \nmid N$ be a non-ordinary prime for $f \in S_k(\Gamma_0(N))$, and let $\alpha, \beta$ be the roots of the $p$-th Hecke polynomial of $f$. In this setting, Lei–Loeffler–Zerbes [LLZ10], [LLZ11], formulated\(^1\) “signed” main conjectures:

\[(3.1) \quad (L^\sigma_p(f)) \equiv \text{Char}_\Lambda(\text{Sel}_f(f)^\vee), \quad (L^\beta_p(f)) \equiv \text{Char}_\Lambda(\text{Sel}_f(f)^\vee),\]

where $\text{Sel}_f(f)$ and $\text{Sel}_f(f)$ are Selmer groups cut out by local condition at $p$ more stringent that the usual ones, and $L^\sigma_p(f), L^\beta_p(f) \in \Lambda$ are related to the $p$-adic $L$-functions $L^\sigma_p(f), L^\beta_p(f)$ of Amice–Vélu and Vishik in the following manner:

\[(3.2) \quad \begin{pmatrix} L^\alpha_p(f) \\ L^\beta_p(f) \end{pmatrix} = Q_{\alpha,\beta}^{-1}M_{\text{log}} \cdot \begin{pmatrix} L^\sigma_p(f) \\ L^\beta_p(f) \end{pmatrix},\]

where $Q_{\alpha,\beta} = \begin{pmatrix} \alpha & -\beta \\ -p & p \end{pmatrix}$ and $M_{\text{log}}$ is a certain “logarithm matrix”.

**Project A.** Show **Expectation 1** for the signed $p$-adic $L$-functions. More precisely, for each $ullet \in \{a, b\}$, show that if $f \equiv g \pmod{p}$, then

$$\mu(L^\bullet_p(f)) = 0 \implies \mu(L^\bullet_p(g)) = 0$$

and the $\lambda$-invariants of $\Sigma$-imprimitive versions of $L^\bullet_p(f)$ and $L^\bullet_p(g)$ are equal.

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\(^1\)Extending earlier work of Kobayashi, Pollack, Lei, and Sprung
Say $k = 2$ for simplicity. Similarly as in [GV00], the proof of this result would follow from the equality

$$L_p^{\Sigma, \ast}(f) \equiv u L_p^{\Sigma, \ast}(g) \pmod{p\Lambda},$$

for some unit $u \in \mathbb{Z}_p^\times$, which in turn would follow from establishing the congruence

$$(3.3) \quad L_p^{\Sigma, \ast}(f, \zeta - 1) \equiv u L_p^{\Sigma, \ast}(g, \zeta - 1) \pmod{p\mathbb{Z}_p[\zeta]},$$

for all $\zeta \in \mu_p^\infty$ and some $u \in \mathbb{Z}_p^\times$ independent of $\zeta$. However, a point of departure here from the $p$-ordinary setting is that (unless $a_p = b_p = 0$) the signed $p$-adic $L$-functions $L_p^{\ast}(f)$, $L_p^{\ast}(g)$ are not directly related to twisted $L$-values, and so the arguments of [GV00, §3] do not suffice to cover this case. Nonetheless, it should be possible to exploit the result of [Vat99, Prop. 1.7], which amounts to the congruence

$$L_p^{\Sigma, \ast}(f, \zeta - 1) \equiv u L_p^{\Sigma, \ast}(g, \zeta - 1) \pmod{p\mathbb{Z}_p[\zeta]}$$

for both $\ast \in \{\alpha, \beta\}$, together with (3.2) to establish (3.3). This will involve a detailed analysis of the values of $M_{\text{low}}$ at $p$-power roots of unity, for which some of the calculations in [LLZ17] (see esp. [loc.cit., Lem. 3.7]) might be useful.

**Remark 3.1.** The algebraic analogue of Project A has recently been established by Hatley–Lei (see [HL16, Thm. 4.6]). On the other hand, as shown in [LLZ11, Cor. 6.6], either of the main conjectures (3.1) is equivalent to Kato’s main conjecture (see [LLZ11, Conj. 6.2]). Thus from the discussion of §2 and the main result of [KKS17], we see that a successful completion of Project A would yield cases of the signed main conjectures beyond those covered by [Wan14] or [CCSS17, Thm. B], where the following hypothesis is needed:

there exists a prime $\ell \neq p$ with $\ell | N$ such that $\tilde{\rho}_f$ is ramified at $\ell$.

(cf. [KKS17, §1.2.3]).

**4. ON THE ANTICYCLOTOMIC MAIN CONJECTURE OF BERTOLINI–DARMON–PRASANNA**

Here we let $F_\infty / F$ be the anticyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field $K$ in which

$$p = \mathfrak{p}\mathfrak{p}$$

splits,

let $f \in S_2(\Gamma_0(N))$, and let $p \nmid N$ be a prime. Assume also that every prime factor of $N$ splits in $K$; so $K$ satisfies the Heegner hypothesis, and $N^{-} = 1$ with the standard notation.

The Iwasawa–Greenberg main conjecture for the $p$-adic $L$-function $L_p(f) \in \mathbb{Z}_p[[\text{Gal}(F_\infty / F)]]$ introduced in [BDP13] predicts that

$$\text{Char}_A(\text{Sel}_p(f)^\vee) \Lambda_{\mathbb{Z}_p}^G \equiv (L_p(f)),$$

where $\Lambda_{\mathbb{Z}_p} = \mathbb{Z}_p[[T]]$ and $\text{Sel}_p(f)$ is a Selmer group defined by imposing local triviality (resp. no condition) at the primes above $p$ (resp. $\mathfrak{p}$).

**Project B.** Show Expectation 1 for the $p$-adic $L$-functions of [BDP13]. That is, if $f \equiv g \pmod{p}$, then $\mu(L_p(f)) = \mu(L_p(g)) = 0^3$ and the $\lambda$-invariants of $\Sigma$-imprimitive versions of $L_p(f)$ and $L_p(g)$ are equal.

Similarly as for Project A, in weight $k = 2$ this problem can be reduced to establishing the congruence

$$L_p^{\Sigma}(f, \zeta - 1) \equiv u L_p^{\Sigma}(g, \zeta - 1) \pmod{p\mathbb{Z}_p[\zeta]}$$

\footnote{Subject to the nonvanishing mod $p$ of some “Kurihara number”}

\footnote{Note that in this case the vanishing of $\mu$-invariants is known under mild hypotheses by [Hsi14, Thm. B] and [Bur17, Thm. B]}
for all $\zeta \in \mu_p^\infty$ and some $u \in \mathbb{Z}_p^\times$ independent of $\zeta$. Now, by the $p$-adic Waldspurger formula of [BDP13, Thm. 5.13], the congruence of [KL16, Thm. 2.9] amounts to (4.2) for $\zeta = 1$, and so a promising approach to Project B would be based on extending the result of [KL16, Thm. 2.9] to ramified characters.

**Remark 4.1.** When $p$ is a good ordinary prime, the algebraic analogue of Project B has recently been established by Hatley–Lei (see [HL17, Prop. 4.2 and Thm. 5.4]). On the other hand, one can show that Howard’s divisibility towards Perrin-Riou’s Heegner point main conjecture implies one of the divisibilities predicted by (4.1) (see [How04, Thm. B] and [Cas17b, App. A]). Similarly as in [KKS17], it should be possible to show (this is work in progress) that a suitable refinement of the Kolyvagin system arguments of [How04] combined with Wei Zhang’s proof of Kolyvagin’s conjecture [Zha14] yields the full equality (4.1). In particular, this would yield new cases of conjecture (4.1) with $N^- = 1$ (not currently available in the literature), and even more cases (under a somewhat weaker version of Hypothesis ♠ in [Zha14], still with $N^- = 1$) after a successful completion of Project B.

Finally, in line with the previous remark, we note that the following should be possible:

**Project C.** Extend the results of [HL17] to the non-ordinary case.

5. **On the $p$-part of the Birch–Swinnerton-Dyer formula for residually reducible primes**

Here we consider the primes $p > 2$ for which the associated residual representation $\bar{\rho}_f$ is reducible. For simplicity, assume that $f$ corresponds to an elliptic curve $E/\mathbb{Q}$ (admitting a rational $p$-isogeny with kernel $\Phi$). The combination of [GV00, Thm. 3.12] (with a key input from [Kat04, Thm. 17.4]) and [Gre99, Thm. 4.1] yields the $p$-part of the BSD formula for $E$ in analytic rank 0, i.e., when $L(E, 1) \neq 1$, provided the following holds:

$$
(GV) \quad \text{the } G_{\mathbb{Q}}\text{-action on } \Phi \subset E[p] \text{ is either } \begin{cases} \text{ramified at } p \text{ and even, or} \\ \text{unramified at } p \text{ and odd.} \end{cases}
$$

Similarly as in the residually irreducible cases considered in [JSW17], the above result (applied to a suitable quadratic twist of $E$) would be an important ingredient in the following:

**Project D.** Prove the $p$-part of the BSD formula in analytic rank 1 for elliptic curves $E$ and primes $p > 2$ for which $(GV)$ does not hold.

Following the strategy of [JSW17] and [Cas17a], a key ingredient toward this\footnote{Which can be seen as proving “primitivity” in the sense of [MR04] of the Heeger point Kolyvagin system} would be the proof of the relevant cases of the anticyclotomic main conjecture (4.1). By the discussion in §2, this could be approached in the following steps:

1. establish the divisibility “⊇” in (4.1) (possibly after inverting $p$), based on a suitable refinement of the Kolyvagin system argument in [How04].
2. show that $\mu(L_p(f)) = 0$ based on the congruence of [Kri16, Thm. 3] between $L_p(f)$ and an anticyclotomic Katz $p$-adic $L$-function, and Hida’s results on the vanishing of $\mu$ for the latter.
3. letting $L_p^{\text{alg}}(f)$ be a generator of the characteristic ideal in (4.1), show that $\mu(L_p^{\text{alg}}(f)) = 0$ and $\lambda(L_p^{\text{alg}}(f)) = \lambda(L_p(f))$ based on an algebraic counterpart of [Kri16, Thm. 3] and the known cases of the main conjecture for the anticyclotomic Katz $p$-adic $L$-function.

After this is carried out, we could try to study the missing cases:

\footnote{Note that there are other points where the residually irreducible hypothesis is used in [JSW17], e.g. in the “anticyclotomic control theorem” of [loc.cit., §3.3], but handling these should be relatively easy.}
**Project E.** Prove the $p$-part of the BSD formula for elliptic curves $E/\mathbb{Q}$ at residually reducible primes $p > 2$ when:

- $L(E, 1) \neq 0$ and (GV) doesn’t hold (complementing the cases that follow from [GV00]).
- $\text{ord}_{s=1}L(E, s) = 1$ and (GV) holds (complementing the cases covered by Project D).

Finally, we should note that $p = 2$ has been neglected throughout the above discussion, but one would of course like to understand this case as well. (See e.g. [CLZ17] for recent results in this direction.)

**References**


