p-ADIC HEIGHTS OF HEEGNER POINTS AND BEILINSON–FLACH CLASSES

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Abstract. We give a new proof of Howard's Λ-adic Gross–Zagier formula [How05], which we extend to the context of indefinite Shimura curves over $\mathbf{Q}$ attached to nonsplit quaternion algebras. This formula relates the cyclotomic derivative of a two-variable p-adic L-function restricted to the anticyclotomic line to the cyclotomic p-adic heights of Heegner points over the anticyclotomic tower, and our proof, rather than inspired by the influential approaches of Gross–Zagier [GZ86] and Perrin-Riou [PR87b], is via Iwasawa theory, based on the connection between Heegner points, Beilinson–Flach elements, and their explicit reciprocity laws. This novel approach is expected to yield an extension of Howard's formula to the setting of Heegner points in Hida families [How07], whose applications will be explored in forthcoming work.

Contents

Introduction

Let $E/\mathbf{Q}$ be an elliptic curve of conductor N, let $K/\mathbf{Q}$ be an imaginary quadratic field of discriminant $-D_K < 0$ prime to N, and let $f = \sum_{n=1}^{\infty} a_n(f)q^n \in S_2(\Gamma_0(N))$ be the normalized newform associated with E. The field K determines a factorization $N = N^+N^-$ such that a prime ℓ divides N^+ if ℓ is split in K, and divides N^- if ℓ is inert in K. Throughout this paper, we shall assume that

$$N \text{ is square-free,} \qquad (\square\text{-free})$$

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and that K satisfies the following *generalized Heegner hypothesis* relative to N:

(Heeg) N^- is the product of an *even* number of primes.

Under the latter hypothesis, E can be realized as a quotient of the Jacobian $\mathrm{Jac}(X_{N^+,N^-})$ of a Shimura curve X_{N^+,N^-} attached to the pair (N^+,N^-). More precisely, after possibly replacing E by a $\mathbf{Q}$-isogenous elliptic curve, we may fix a parametrization

$$\Phi_E : \mathrm{Jac}(X_{N^+,N^-}) \longrightarrow E.$$

Fix also a prime $p \geqslant 5$ of good ordinary reduction for E, let K_∞^{cyc} and K_∞^{ac} be the cyclotomic and anticyclotomic $\mathbf{Z}_p$-extensions of K, respectively, and set $K_\infty = K_\infty^{\mathrm{ac}} K_\infty^{\mathrm{cyc}}$. Then, by work of Hida and Perrin-Riou (see [Hid85], [PR88], [PR87b]), there is a p-adic L-function

$$L_p(f/K) \in \mathbf{Z}_p[[\mathrm{Gal}(K_\infty/K)]] \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$$

interpolating the central values for the Rankin–Selberg convolution of f with the theta series attached to finite order characters of $\mathrm{Gal}(K_\infty/K)$. (This is denoted $L_p(f/K,\Sigma^{(1)})$ in the body of the paper.)

Letting $\Lambda^{\mathrm{ac}} := \mathbf{Z}_p[[\mathrm{Gal}(K_\infty^{\mathrm{ac}}/K)]]$, we may identify

$$\mathbf{Z}_p[[\mathrm{Gal}(K_\infty/K)]] \simeq \Lambda^{\mathrm{ac}}[[\mathrm{Gal}(K_\infty^{\mathrm{cyc}}/K)]],$$

and hence upon the choice of a topological generator $\gamma^{\mathrm{cyc}} \in \mathrm{Gal}(K_\infty^{\mathrm{cyc}}/K)$, we may expand

$$L_p(f/K) = L_{p,0}(f/K) + L_{p,1}^{\mathrm{cyc}}(f/K)(\gamma^{\mathrm{cyc}}-1) + \cdots$$

as a power series in $(\gamma^{\mathrm{cyc}}-1)$ with coefficients in $\Lambda^{\mathrm{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. As a consequence of hypothesis (Heeg), the sign in the functional equation satisfied by $L_p(f/K)$ forces the vanishing of $L_{p,0}(f/K)$, and we are led to consider the 'linear term' $L_{p,1}^{\mathrm{cyc}}(f/K)$ in the above expansion.

On the other hand, taking certain linear combinations of Heegner points on X_{N^+,N^-} defined over ring class fields of K of p-power conductor, and mapping them onto E via Φ_E, it is possible to define a class $\mathbf{z}_f$ in the Selmer group

$$\mathrm{Sel}(K,\mathbf{T}^{\mathrm{ac}}) := \varprojlim_n \varprojlim_m \mathrm{Sel}_{p^m}(E/K_n^{\mathrm{ac}})$$

attached to the E over the tower K_∞^{ac}/K, where $\mathrm{Sel}_{p^m}(E/K_n^{\mathrm{ac}}) \subseteq H^1(K_n^{\mathrm{ac}}, E[p^m])$ is the usual p^m-th Selmer group. Let $T_pE = \varprojlim_m E[p^m]$ be the p-adic Tate module of E. In Section 3.2 we recall the definition of a cyclotomic Λ^{ac}-adic height pairing

$$\langle\ ,\ \rangle_{K_\infty^{\mathrm{ac}}}^{\mathrm{cyc}} : \mathrm{Sel}(K,\mathbf{T}^{\mathrm{ac}}) \times \mathrm{Sel}(K,\mathbf{T}^{\mathrm{ac}}) \longrightarrow \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \Lambda^{\mathrm{ac}} \otimes_{\mathbf{Z}_p} \mathcal{I}/\mathcal{I}^2,$$

where $\mathcal{I}$ is the augmentation ideal of $\mathbf{Z}_p[[\mathrm{Gal}(K_\infty^{\mathrm{cyc}}/K)]]$. Using our fixed topological generator γ^{cyc}, we may view $\langle\ ,\ \rangle_{K_\infty^{\mathrm{ac}}}^{\mathrm{cyc}}$ as taking values in $\Lambda^{\mathrm{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

The main result of this paper is then the following Λ^{ac}-adic Gross–Zagier formula.

Theorem A. *In addition to* (Heeg) *and* ($\square$-free)*, assume that:*

- $p = \mathfrak{p}\overline{\mathfrak{p}}$ *splits in* K*,*
- $N^- \neq 1$*,*
- $E[p]$ *is ramified at every prime* $\ell \mid N^-$*,*
- $\mathrm{Gal}(\overline{\mathbf{Q}}/K) \rightarrow \mathrm{Aut}_{\mathbf{Z}_p}(T_pE)$ *is surjective.*

Then we have the equality

$$(L_{p,1}^{\mathrm{cyc}}(f/K)) = (\langle \mathbf{z}_f, \mathbf{z}_f \rangle_{K_\infty^{\mathrm{ac}}}^{\mathrm{cyc}})$$

as ideals of $\Lambda^{\mathrm{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

For $N^- = 1$, Theorem A follows from Howard's extension in the anticyclotomic direction of Perrin-Riou's p-adic Gross–Zagier formula (see [How05]). Even though it should be possible to extend Howard's calculations to also cover the cases considered here[1], our proof of Theorem A is completely different. In fact, as we shall explain in the remaining part of this Introduction, rather than inspired by the work of Gross–Zagier [GZ86] and Perrin-Riou [PR87b], our proof is via Iwasawa theory. A key advantage (and arguably the main interest) of this new alternative approach is that it seems amenable to generalization to higher weights[2], to the supersingular case (where an analogue of Howard's Λ-adic Gross–Zagier formula was not known[3]), and even to the setting of 'big Heegner points' in Hida families [How07]. The application of the latter extension in connection to well-known nonvanishing conjectures due to Howard [How07] and Greenberg [Gre94] will be explained in forthcoming work [CW17].

Our starting point is the observation (already pointed out by Howard in the introduction to [How05]) that the content of Theorem A can be parlayed into an equality without reference to Heegner points. Indeed, let K_n^{ac}/K be the unique subextension of K_∞^{ac}/K with $\mathrm{Gal}(K_n^{\mathrm{ac}}/K) \simeq \mathbf{Z}/p^n\mathbf{Z}$, and let $\mathrm{Sel}_{p^\infty}(E/K_n^{\mathrm{ac}}) \subseteq H^1(K_n^{\mathrm{ac}}, E[p^\infty])$ be the usual p-power Selmer group. Under hypothesis (Heeg), and as a reflection of the above vanishing of $L_{p,0}(f/K)$, the Selmer group

$$\mathrm{Sel}(K, \mathbf{A}^{\mathrm{ac}}) := \varinjlim_n \varinjlim_m \mathrm{Sel}_{p^m}(E/K_n^{\mathrm{ac}})$$

can be shown to have positive corank over the anticyclotomic Iwasawa algebra Λ^{ac}. Moreover, the 'Heegner point main conjecture' formulated by Perrin-Riou [PR87a] predicts that both $\mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}})$ and the Pontryagin dual $X(K, \mathbf{A}^{\mathrm{ac}}) := \mathrm{Sel}(K, \mathbf{A}^{\mathrm{ac}})^\vee$ have Λ^{ac}-rank one, and that letting the subscript 'tors' denote the Λ^{ac}-torsion submodule, we have the equality

$$\text{(HP)} \qquad Ch_{\Lambda^{\mathrm{ac}}}(X(K, \mathbf{A}^{\mathrm{ac}})_{\mathrm{tors}}) = Ch_{\Lambda^{\mathrm{ac}}}\left(\frac{\mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}})}{\Lambda^{\mathrm{ac}} \cdot \mathbf{z}_f}\right)^2$$

as ideals in $\Lambda^{\mathrm{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

Building on the work of Cornut–Vatsal [CV05] (which implies the nontriviality of $\mathbf{z}_f$) and an extension to the anticyclotomic setting of Mazur–Rubin's theory of Kolyvagin systems [MR04], Howard established in [How04a] one of the divisibilities in (HP), while the converse divisibility has been more recently obtained by Wan [Wan14a]. Letting $\mathcal{R}_{\mathrm{cyc}}$ be the characteristic ideal of the cokernel of $\langle\ ,\ \rangle_{K_\infty^{\mathrm{ac}}}^{\mathrm{cyc}}$, one sees without difficulty that Theorem A is a consequence of the following Λ^{ac}-adic analog of the Birch and Swinnerton-Dyer conjecture.

Theorem B. *Under the hypotheses of Theorem A, we have*

$$(L_{p,1}^{\mathrm{cyc}}(f/K)) = \mathcal{R}_{\mathrm{cyc}} \cdot Ch_{\Lambda^{\mathrm{ac}}}(X(K, \mathbf{A}^{\mathrm{ac}})_{\mathrm{tors}})$$

as ideals in $\Lambda^{\mathrm{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

For rational elliptic curves with complex multiplication by K, an analog of Theorem B was obtained by Agboola–Howard [AH06] using the Euler system of elliptic units, Rubin's proof of the Iwasawa main conjecture for K, and a variant of Rubin's p-adic height formula [Rub94]. By ($\square$-free), the CM setting is excluded here, but nonetheless we are able to prove Theorem B by a strategy similar to theirs, using the Euler system of Beilinson–Flach elements developed in a remarkable series of works [LLZ14, LLZ15, KLZ17, KLZ15] by Lei–Loeffler–Zerbes and Kings–Loeffler–Zerbes which refined and generalized earlier work of Bertolini–Darmon–Rotger [BDR15a, BDR15b].

[1]See [Dis15] for an approach along these lines.

[2]After the first version of this paper was released, this has largely been carried out by the combined works of Büyükboduk–Lei [BL16b] and Longo–Vigni [LV17].

[3]See the recent preprints [CW16] and [BL16a].

Notwhithstanding this similarity, a notable difference between the approach in this paper and that in [AH06] is that, even though we essentially rely on the explicit reciprocity laws for Beilinson–Flach elements [KLZ17], we do not exploit their Euler system properties; instead, we build on Howard's results [How04a, How04b] on the anticyclotomic Euler system of classical Heegner points. This appears to be an important technical point, since it allows us to dispense with the twist of E by a rather undesirable p-distinguished character.

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1. p-adic L-functions

Throughout this section, we let $f=\sum_{n=1}^{\infty}a_n(f)q^n\in S_2(\Gamma_0(N_f))$ be a normalized newform of conductor N_f, and let K be an imaginary quadratic field of discriminant $-D_K<0$ prime to N_f. We denote by $\mathcal{O}_K$ the ring of integers of K. Fix a prime $p\nmid 6N_fD_K$, and assume that

(spl) $$p=\mathfrak{p}\overline{\mathfrak{p}} \text{ splits in } K.$$

Fix complex and p-adic embeddings $\overline{\mathbf{Q}}\overset{\imath_\infty}{\hookrightarrow}\mathbf{C}$ and $\overline{\mathbf{Q}}\overset{\imath_p}{\hookrightarrow}\mathbf{C}_p$, and let $\mathfrak{p}$ be the prime of K above p induced by $\imath_p$. Finally, since it will suffice for the applications in this paper, we assume that the number field generated by the coefficients $a_n(f)$ embeds into $\mathbf{Q}_p$.

1.1. Hida's p-adic Rankin L-series, I.

Let Ξ_K denote the set of algebraic Hecke characters $\psi:K^\times\backslash\mathbb{A}_K^\times\to\mathbf{C}^\times$. We say that $\psi\in\Xi_K$ has infinity type $(\ell_1,\ell_2)\in\mathbf{Z}^2$ if

$$\psi_\infty(z)=z^{\ell_1}\overline{z}^{\ell_2},$$

where for each place v of K we let $\psi_v:K_v^\times\to\mathbf{C}^\times$ be the v-component of ψ. The conductor of ψ is the largest ideal $\mathfrak{c}\subseteq\mathcal{O}_K$ such that $\psi_{\mathfrak{q}}(u)=1$ for all $u\in(1+\mathfrak{c}\mathcal{O}_{K,\mathfrak{q}})^\times\subseteq K_{\mathfrak{q}}^\times$. If ψ has conductor $\mathfrak{c}_\psi$ and $\mathfrak{a}$ is any fractional ideal of K prime to $\mathfrak{c}_\psi$, we write $\psi(\mathfrak{a})$ for $\psi(a)$, where a is any idele satisfying $a\hat{\mathcal{O}}_K\cap K=\mathfrak{a}$ and $a_{\mathfrak{q}}=1$ for all primes $\mathfrak{q}$ dividing $\mathfrak{c}_\psi$. As a function on fractional ideals, then ψ satisfies

$$\psi((\alpha))=\alpha^{-\ell_1}\overline{\alpha}^{-\ell_2}$$

for all $\alpha\in K^\times$ with $\alpha\equiv 1 \pmod{\mathfrak{c}_\psi}$.

We say that a Hecke character ψ of infinity type (ℓ_1,ℓ_2) is *critical* (for f) if the value $s=1$ is a critical value in the sense of Deligne for

$$L(f/K,\psi,s)=L\left(\pi_f\times\pi_\psi,s+\frac{\ell_1+\ell_2-1}{2}\right),$$

where $L(\pi_f\times\pi_\psi,s)$ is the L-function for the Rankin–Selberg convolution of the automorphic representations of $\mathbf{GL}_2(\mathbb{A})$ corresponding to f and the theta series θ_ψ associated to ψ. The set of infinity types of critical characters can be written as the disjoint union

$$\Sigma=\Sigma^{(1)}\sqcup\Sigma^{(2)}\sqcup\Sigma^{(2')},$$

with $\Sigma^{(1)}=\{(0,0)\}$, $\Sigma^{(2)}=\{(\ell_1,\ell_2)\colon \ell_1\leqslant-1,\ell_2\geqslant1\}$, $\Sigma^{(2')}=\{(\ell_1,\ell_2)\colon \ell_2\leqslant-1,\ell_1\geqslant1\}$.

Denote by ψ^ρ the character obtained by composing $\psi:K^\times\backslash\mathbb{A}_K^\times\to\mathbf{C}^\times$ with the complex conjugation on $\mathbb{A}_K^\times$. The involution $\psi\mapsto\psi^\rho$ on Ξ_K has the effect on infinity types of interchanging the regions $\Sigma^{(2)}$ and $\Sigma^{(2')}$ (while leaving $\Sigma^{(1)}$ stable). Since the values $L(f/K,\psi,1)$ and $L(f/K,\psi^\rho,1)$ are the same, for the purposes of p-adic interpolation we may restrict our attention to the first two subsets in the above the decomposition of Σ.

Definition 1.1. Let $\psi = \psi^\infty\psi_\infty \in \Xi_K$ be an algebraic Hecke character of infinity type (ℓ_1, ℓ_2). The *p-adic avatar* of ψ is the character $\hat{\psi} : K^\times\backslash\hat{K}^\times \to \mathbf{C}_p^\times$ defined by

$$\hat{\psi}(z) = \imath_p\imath_\infty^{-1}(\psi^\infty(z))z_{\mathfrak{p}}^{\ell_1}z_{\overline{\mathfrak{p}}}^{\ell_2}.$$

For each ideal $\mathfrak{m}$ of K, let $H_{\mathfrak{m}} \simeq \mathrm{Gal}(K(\mathfrak{m})/K)$ be the ray class group of K modulo $\mathfrak{m}$, and set $H_{p^\infty} = \varprojlim_r H_{p^r}$. By the Artin reciprocity map, the correspondence $\psi \mapsto \hat{\psi}$ establishes a bijection between the set of algebraic Hecke characters of K of conductor dividing p^∞ and the set of locally algebraic $\overline{\mathbf{Q}}_p$-valued characters of H_{p^∞}.

For the next theorem, let N be a positive integer divisible by D_KN_f and with the same prime factors as D_KN_f.

Theorem 1.2. *Let α and β be the roots of $x^2 - a_p(f)x + p$, ordered so that $0 \leqslant v_p(\alpha) \leqslant v_p(\beta)$.*

(i) There exists a p-adic L-function $L_p(f/K, \Sigma^{(2)}) \in \mathrm{Frac}(\mathbf{Z}_p[[H_{p^\infty}]])$ such that for every character $\chi \in \Xi_K$ of trivial conductor and infinity type $(\ell_1, \ell_2) \in \Sigma^{(2)}$, we have

$$L_p(f/K, \Sigma^{(2)})(\hat{\psi}) = \frac{\Gamma(\ell_2)\Gamma(\ell_2+1)\cdot\mathcal{E}(\psi, f)}{(1-\psi^{1-\rho}(\mathfrak{p}))(1-p^{-1}\psi^{1-\rho}(\mathfrak{p}))}\cdot\frac{L(f/K,\psi,1)}{(2\pi)^{2\ell_2+1}\cdot\langle\theta_{\psi_{\ell_2}},\theta_{\psi_{\ell_2}}\rangle_N},$$

where $\theta_{\psi_{\ell_2}}$ is the theta series of weight $\ell_2 - \ell_1 + 1 \geqslant 3$ associated to the Hecke character $\psi_{\ell_2} := \psi\mathbf{N}_K^{\ell_2}$ of infinity type $(\ell_1 - \ell_2, 0)$, and

$$\mathcal{E}(\psi, f) = (1 - p^{-1}\psi(\mathfrak{p})\alpha)(1 - p^{-1}\psi(\mathfrak{p})\beta)(1 - \psi^{-1}(\overline{\mathfrak{p}})\alpha^{-1})(1 - \psi^{-1}(\overline{\mathfrak{p}})\beta^{-1}).$$

(ii) If $v_p(\alpha) = 0$, there exists a p-adic L-function $L_p(f/K, \Sigma^{(1)}) \in \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[[H_{p^\infty}]]$ such that for every finite order character $\psi \in \Xi_K$ of conductor $\mathfrak{p}^m\overline{\mathfrak{p}}^n$, we have

$$L_p(f/K, \Sigma^{(1)})(\hat{\psi}) = \frac{\prod_{\mathfrak{q}|p}(1 - p^{-1}\beta\psi(\mathfrak{q}))(1 - \alpha^{-1}\psi^{-1}(\mathfrak{q}))}{(1 - \beta\alpha^{-1})(1 - \beta p^{-1}\alpha^{-1})}\cdot C(f,\psi)\cdot\frac{L(f/K,\psi,1)}{8\pi^2\cdot\langle f, f\rangle_N},$$

where

$$C(f,\psi) = \begin{cases} -iN & \text{if } m = n = 0; \\ \tau(\psi)\alpha^{-m-n+1}(Np^{n+m})^{1/2} & \text{if } m + n > 0, \end{cases}$$

with $\tau(\psi)$ denoting the root number of ψ.

Proof. The first part is a reformulation of [LLZ15, Thm. 6.1.3(ii)], and the second part follows from an extension of the calculations in [LLZ15, Thm. 6.1.3(i)]) (see e.g. [BL16b, Thm. 2.1] for details). ☐

Remark 1.3. The p-adic L-functions of Theorem 1.2 are nonzero. Indeed, as shown in the proof of [KLZ14, Prop. 9.2.1], the nonvanishing of $L_p(f/K, \Sigma^{(2)})$ follows immediately from the existence of infinitely many characters ψ with infinity type $(\ell_1, \ell_2) \in \Sigma^{(2)}$ for which the Euler product defining $L(f/K, \psi, s)$ converges at $s = 1$, while the nonvanishing of $L_p(f/K, \Sigma^{(1)})$ (in fact, even of the "cyclotomic" restriction of $L_p(f/K, \Sigma^{(1)})$ to characters of H_{p^∞} factoring through the norm map) follows from the nonvanishing results of [Roh88].

1.2. **Anticyclotomic p-adic L-functions.**

We keep the notations from §1.1, and write

$$N_f = N^+N^-$$

with N^+ (resp. N^-) equal to the product of the prime factors of N_f split (resp. inert) in K. As in the Introduction, we say that the pair (f, K) satisfies the *generalized Heegner hypothesis* if

(Heeg) N^- is the square-free product of an *even* number of primes.

Let K_∞/K be the $\mathbf{Z}_p^2$-extension of K, and set $\Gamma_K = \mathrm{Gal}(K_\infty/K)$. We may decompose

$$H_{p^\infty} \simeq \Delta \times \Gamma_K$$

for a finite group Δ. The Galois group $\mathrm{Gal}(K/\mathbf{Q})$ acts naturally on Γ_K; let $\Gamma^{\mathrm{cyc}} \subseteq \Gamma_K$ be the fixed part by this action, and set $\Gamma^{\mathrm{ac}} := \Gamma_K/\Gamma^{\mathrm{cyc}}$. Then $\Gamma^{\mathrm{ac}} \simeq \mathrm{Gal}(K_\infty^{\mathrm{ac}}/K)$ is identified with the Galois group of the *anticyclotomic* $\mathbf{Z}_p$-extension of K, on which we have $\tau\sigma\tau^{-1} = \sigma^{-1}$ for the non-trivial element $\tau \in \mathrm{Gal}(K/\mathbf{Q})$. Similarly, we say that a character ψ is *anticyclotomic* if $\psi^\rho = \psi^{-1}$.

Assume that $a_p(f) \in \mathbf{Z}_p^\times$, so that the p-adic L-function $L_p(f/K, \Sigma^{(1)})$ introduced in Theorem 1.2 is defined, and let $L_p^{\mathrm{ac}}(f/K, \Sigma^{(1)})$ denote its image under the map $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[[H_{p^\infty}]] \to \mathbf{Q}_p \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[[\Gamma^{\mathrm{ac}}]]$ induced by the natural projection $H_{p^\infty} \twoheadrightarrow \Gamma^{\mathrm{ac}}$.

Proposition 1.4. *If the generalized Heegner hypothesis* (Heeg) *holds, then* $L_p^{\mathrm{ac}}(f/K, \Sigma^{(1)})$ *is identically zero.*

Proof. If ψ is any finite order character of K as in the range of interpolation for $L_p(f/K, \Sigma^{(1)})$ which is anticyclotomic, then the Rankin–Selberg L-function $L(f/K, \psi, s)$ is self-dual, with a functional equation relating its values at s and $2 - s$. By hypothesis (Heeg), the sign in this functional equation is -1 (see e.g. [Zha01a, §3]), and so $L(f/K, \psi, 1) = 0$. The result thus follows from the interpolation property of $L_p(f/K, \Sigma^{(1)})$. $\square$

In contrast with the immediate proof of Proposition 1.4, the understanding of $L_p^{\mathrm{ac}}(f/K, \Sigma^{(2)})$, defined as the image of the p-adic L-function $L_p(f/K, \Sigma^{(2)})$ of Theorem 1.2 under the natural projection $\mathrm{Frac}(\mathbf{Z}_p[[H_{p^\infty}]]) \to \mathrm{Frac}(\mathbf{Z}_p[[\Gamma^{\mathrm{ac}}]])$, requires more work.

To proceed, we need to recall the properties of another anticyclotomic p-adic L-function. Let K_Δ be the fixed field of Γ_K in $K(p^\infty)$, so that $\mathrm{Gal}(K_\Delta/K) \simeq \Delta$. The compositum $K_\infty^{\mathrm{ac}} K_\Delta$ contains $K[p^\infty] = \cup_{n \geqslant 0} K[p^n]$, where $K[m]$ denotes the ring class field of K of conductor m. Let $\mathbf{Q}_p^{\mathrm{nr}}$ be the maximal unramified extension of $\mathbf{Q}_p$, and denote by R_0 the completion of its ring of integers. Also, let

$$\rho_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}_2(\mathbf{Q}_p)$$

be the Galois representation associated with f, and denote by $\bar{\rho}_f$ the corresponding semisimple residual representation.

Theorem 1.5. *Assume that* $p = \mathfrak{p}\overline{\mathfrak{p}}$ *splits in* K *and that hypothesis* (Heeg) *holds. There exists a* p*-adic* L*-function* $\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K) \in R_0[[\Gamma^{\mathrm{ac}}]]$ *such that if* $\hat{\psi} : \Gamma^{\mathrm{ac}} \to \mathbf{C}_p^\times$ *is the* p*-adic avatar of an unramified Hecke character* ψ *with infinity type* $(-\ell, \ell)$ *with* $\ell \geqslant 1$*, then*

$$\left(\frac{\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K)(\hat{\psi})}{\Omega_p^{2\ell}}\right)^2 = \Gamma(\ell)\Gamma(\ell+1) \cdot (1 - p^{-1}\psi(\mathfrak{p})\alpha)^2 \cdot (1 - p^{-1}\psi(\mathfrak{p})\beta)^2 \cdot \frac{L(f/K, \psi, 1)}{\pi^{2\ell+1} \cdot \Omega_K^{4\ell}},$$

where $\Omega_p \in R_0^\times$ *and* $\Omega_K \in \mathbf{C}^\times$ *are CM periods. Moreover, if* $\bar{\rho}_f$ *is absolutely irreducible, then* $\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K)$ *does not vanish identically.*

Proof. See [CH17, §3.3] for the construction of $\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K)$ (see also [Bur17, §5.2] for the case $N^- \neq 1$), and [CH17, Thm. 3.9] for the proof of its nontriviality. $\square$

Note that the projection $L_p^{\mathrm{ac}}(f/K, \Sigma^{(2)})$ and the square of the p-adic L-function $\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K)$ are defined by the interpolation of the same L-values. However, the archimedean periods used in their normalization are different, and therefore these p-adic L-functions need not be equal. In fact, as we shall see below, the ratio between these different periods is interpolated by an anticyclotomic projection of a Katz p-adic L-function.

Recall that the Hecke L-function of $\psi \in \Xi_K$ is defined by (the analytic continuation of) the Euler product

$$L(\psi, s) = \prod_{\mathfrak{l}} \left(1 - \frac{\psi(\mathfrak{l})}{N(\mathfrak{l})^s}\right)^{-1},$$

where $\mathfrak{l}$ runs over all prime ideals of K, with the convention that $\psi(\mathfrak{l})=0$ for $\mathfrak{l}\mid\mathfrak{c}_\psi$. The set of infinity types of $\psi\in\Xi_K$ for which $s=0$ is a critical value of $L(\psi,s)$ can be written as the disjoint union $\Sigma_K\sqcup\Sigma_K'$, where $\Sigma_K=\{(\ell_1,\ell_2):0<\ell_1\leqslant\ell_2\}$ and $\Sigma_K=\{(\ell_1,\ell_2):0<\ell_2\leqslant\ell_1\}$.

Theorem 1.6 (Katz)**.** *Assume that $p=\mathfrak{p}\overline{\mathfrak{p}}$ splits in K. Then there exists a p-adic L-function $\mathscr{L}_{\mathfrak{p}}(K)\in R_0[[H_{p^\infty}]]$ such that if $\psi\in\Xi_K$ has trivial conductor and infinity type $(\ell_1,\ell_2)\in\Sigma_K$, then*

$$\mathscr{L}_{\mathfrak{p}}(K)(\hat\psi)=\left(\frac{\sqrt{D_K}}{2\pi}\right)^{\ell_1}\cdot\Gamma(\ell_2)\cdot(1-\psi(\mathfrak{p}))\cdot(1-p^{-1}\psi^{-1}(\overline{\mathfrak{p}}))\cdot\frac{\Omega_p^{\ell_2-\ell_1}}{\Omega_K^{\ell_2-\ell_1}}\cdot L(\psi,0),$$

where Ω_K and Ω_p are as in Theorem 1.5. Moreover, it satisfies the functional equation

$$\mathscr{L}_{\mathfrak{p}}(K)(\hat\psi^{\rho})=\mathscr{L}_{\mathfrak{p}}(K)(\hat\psi^{-1}\mathbf{N}_K).$$

Proof. See [Kat78, §5.3.0] or [dS87, Thm. II.4.14] for the construction, and [Kat78, §5.3.7] or [dS87, Thm. II.6.4] for the functional equation. □

Let $\mathscr{L}_{\mathfrak{p}}^{\rm ac}(K)$ be the image of the Katz p-adic L-function $\mathscr{L}_{\mathfrak{p}}(K)$ under the natural projection $R_0[[H_{p^\infty}]]\to R_0[[\Gamma^{\rm ac}]]$.

Theorem 1.7. *Assume that $p=\mathfrak{p}\overline{\mathfrak{p}}$ splits in K and that hypothesis* (Heeg) *hold. Then*

$$L_p^{\rm ac}(f/K,\Sigma^{(2)})(\hat\psi)=\frac{w_K}{h_K}\cdot\frac{\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K)^2(\hat\psi)}{\mathscr{L}_{\mathfrak{p}}^{\rm ac}(K)(\hat\psi^{\rho-1})}$$

up to a unit in $\mathbf{Z}_p[[\Gamma^{\rm ac}]]^\times$, where $w_K=|\mathcal{O}_K^\times|$ and h_K is the class number of K. In particular, if $\bar\rho_f$ is absolutely irreducible, then $L_p^{\rm ac}(f/K,\Sigma^{(2)})$ does not vanish identically.

Proof. In the following, for any two $\mathbf{C}_p$-valued functions f_1 and f_2 defined on the characters of $\Gamma^{\rm ac}$, we shall write $f_1\sim f_2$ to indicate that their ratio is interpolated by an invertible Iwasawa function in $\mathbf{Z}_p[[\Gamma^{\rm ac}]]^\times$. We first note that, when restricted to anticyclotomic characters, the Euler-like factor $\mathcal{E}(\psi,f)$ in Theorem 1.2 becomes a square; in fact, if $\hat\psi:\Gamma^{\rm ac}\to\mathbf{C}_p^\times$ has trivial conductor and infinity type $(-\ell,\ell)$ with $\ell\geqslant 1$, we see that

$$\begin{aligned}L_p(f/K,\Sigma^{(2)})(\hat\psi)\ &\sim\Gamma(\ell)\Gamma(\ell+1)\cdot\frac{(1-p^{-1}\psi(\mathfrak{p})\alpha)^2\cdot(1-p^{-1}\psi(\mathfrak{p})\beta)^2}{(1-\psi^{1-\rho}(\mathfrak{p}))\cdot(1-p^{-1}\psi^{1-\rho}(\mathfrak{p}))}\\ &\qquad\times\frac{L(f/K,\psi,1)}{\pi^{2\ell+1}\cdot\langle\theta_{\psi_\ell},\theta_{\psi_\ell}\rangle_M}.\end{aligned}\tag{1.1}$$

We will have use for the following result.

Lemma 1.8. *We have*

$$\pi^{2\ell+1}\cdot\langle\theta_{\psi_\ell},\theta_{\psi_\ell}\rangle_M\ \sim\ \frac{h_K}{w_K}\cdot\Gamma(2\ell+1)\cdot L(\psi_\ell^{1-\rho},2\ell+1).$$

Proof. Denote by ε_K the quadratic character associated with K. Since $\psi_\ell=\psi\mathbf{N}_K^\ell$ has infinity type $(-2\ell,0)$, the result follows immediately after setting $s=2\ell+1$ in the factorization

$$L(\mathrm{Ad}^0(\theta_{\psi_\ell}),s)=L(\varepsilon_K,s-2\ell)\cdot L(\psi_\ell^{1-\rho},s),$$

and using [Hid81, Thm. 5.1] and Dirichlet's class number formula (see [DLR15, Lem. 3.7]). □

Continuing with the proof of Theorem 1.7, we see that $L(\psi_\ell^{1-\rho},2\ell+1)=L(\psi^{1-\rho},1)$ is interpolated by the value $\mathscr{L}_{\mathfrak{p}}(K)(\hat\psi^{1-\rho}\mathbf{N}_K)$ of the Katz p-adic L-function. Using the functional equation of $\mathscr{L}_{\mathfrak{p}}(K)$, and combining (1.1) and Lemma 1.8 with the interpolation property in Theorem 1.6, we thus obtain

$$\begin{aligned} L_p(f/K,\Sigma^{(2)})(\hat{\psi})\cdot\mathscr{L}_{\mathfrak{p}}(K)(\hat{\psi}^{\rho-1})\cdot\frac{h_K}{w_K} &\sim \frac{\Gamma(\ell)\Gamma(\ell+1)}{\pi^{2\ell+1}}\cdot(1-p^{-1}\psi(\mathfrak{p})\alpha)^2\cdot(1-p^{-1}\psi(\mathfrak{p})\beta)^2\\ &\times\frac{\Omega_p^{4\ell}}{\Omega_K^{4\ell}}\cdot L(f/K,\psi,1), \end{aligned}$$

which compared with the interpolation property in Theorem 1.5 yields the result. $\square$

1.3. **Hida's p-adic Rankin L-series, II.** Recall from §1.2 the decomposition $H_{p^\infty}\simeq\Delta\times\Gamma_K$, set $\Lambda=\mathbf{Z}_p[[\Gamma_K]]$ and $\Lambda_{R_0}=R_0[[\Gamma_K]]$, and continue to denote by

$$L_p(f/K,\Sigma^{(2)})\in\operatorname{Frac}(\Lambda)\qquad\text{and}\qquad\mathscr{L}_{\mathfrak{p}}(K)\in\Lambda_{R_0}$$

the natural projections of the p-adic L-functions $L_p(f/K,\Sigma^{(2)})$ and $\mathscr{L}_{\mathfrak{p}}(K)$ of Theorem 1.2 and Theorem 1.6, respectively.

Theorem 1.9. *Assume that $p=\mathfrak{p}\overline{\mathfrak{p}}$ splits in K. There exists a p-adic L-function $\mathscr{L}_{\mathfrak{p}}(f/K)\in\operatorname{Frac}(\Lambda_{R_0})$ such that if $\hat{\psi}:\Gamma\to\mathbf{C}_p^\times$ has trivial conductor and infinity type $(\ell_1,\ell_2)\in\Sigma^{(2)}$, then*

$$\mathscr{L}_{\mathfrak{p}}(f/K)(\hat{\psi})=\frac{\Gamma(\ell_2)\Gamma(\ell_2+1)}{\pi^{2\ell_2+1}}\cdot\mathcal{E}(\psi,f)\cdot\frac{\Omega_p^{2(\ell_2-\ell_1)}}{\Omega_K^{2(\ell_2-\ell_1)}}\cdot L(f/K,\psi,1),$$

where

$$\mathcal{E}(\psi,f)=(1-p^{-1}\psi(\mathfrak{p})\alpha)(1-p^{-1}\psi(\mathfrak{p})\beta)(1-\psi^{-1}(\overline{\mathfrak{p}})\alpha^{-1})(1-\psi^{-1}(\overline{\mathfrak{p}})\beta^{-1}),$$

and Ω_K and Ω_p are as in Theorem 1.5. Moreover, it differs from the product

$$L_p(f/K,\Sigma^{(2)})(\hat{\psi})\cdot\frac{h_K}{w_K}\cdot\mathscr{L}_{\mathfrak{p}}^{\mathrm{ac}}(K)(\hat{\psi}^{\rho-1})$$

by a unit in $\Lambda_{R_0}^\times$.

Proof. This follows from exactly the same calculation as in the proof of Theorem 1.7. $\square$

Corollary 1.10. *Assume that $p=\mathfrak{p}\overline{\mathfrak{p}}$ splits in K and hypothesis* (Heeg) *holds, and denote by $\mathscr{L}_{\mathfrak{p}}^{\mathrm{ac}}(f/K)$ the image of the p-adic L-function $\mathscr{L}_{\mathfrak{p}}(f/K)$ of Theorem 1.9 under the natural projection $\Lambda_{R_0}\to\Lambda_{R_0}^{\mathrm{ac}}$. Then*

$$\mathscr{L}_{\mathfrak{p}}^{\mathrm{ac}}(f/K)=\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K)^2$$

up to a unit in $(\Lambda_{R_0}^{\mathrm{ac}})^\times$.

Proof. This is clear from Theorem 1.7 and the last claim in Theorem 1.9. $\square$

2. Iwasawa theory

Throughout this section, we let $f\in S_2(\Gamma_0(N_f))$, $K/\mathbf{Q}$, and $p\geqslant 5$ be as in §1. In particular, $p=\mathfrak{p}\overline{\mathfrak{p}}$ splits in K, and we assume in addition that p is ordinary for f, i.e., $a_p(f)\in\mathbf{Z}_p^\times$.

Let V be a $\mathbf{Q}_p$-vector space affording the Galois representation $\rho_f:\operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})\to\operatorname{GL}_2(\mathbf{Q}_p)$ attached to f, fix a Galois-stable $\mathbf{Z}_p$-lattice $T\subseteq V$, and set $A:=V/T$. By p-ordinarity, there exists a $\operatorname{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$-stable filtration

$$0\longrightarrow\mathscr{F}^+V\longrightarrow V\longrightarrow\mathscr{F}^-V\longrightarrow 0$$

with both $\mathscr{F}^+V$ and $\mathscr{F}^-V:=V/\mathscr{F}^+V$ one-dimensional over $\mathbf{Q}_p$ and with the Galois action on $\mathscr{F}^-V$ being unramified. Set $\mathscr{F}^+T:=T\cap\mathscr{F}^+T$, $\mathscr{F}^-T:=T/\mathscr{F}^+T$, $\mathscr{F}^+A:=\mathscr{F}^+V/\mathscr{F}^+T$, and $\mathscr{F}^-A:=A/\mathscr{F}^+A$.

2.1. **Selmer groups.** Let Σ be a finite set of places of K containing the places where V is ramified and the places dividing $p\infty$, and for any finite extension F of K, let $\mathfrak{G}_{F,\Sigma}$ denote the Galois group of the maximal extension of F unramified outside the places above Σ.

In the next two definitions, we let M denote either V, T, or A.

Definition 2.1. The *Greenberg Selmer group* of M over F is

$$\mathrm{Sel}(F, M) = \ker\left\{ H^1(\mathfrak{G}_{F,\Sigma}, M) \longrightarrow \prod_{v\in\Sigma} \frac{H^1(F_v, M)}{H^1_{\mathrm{Gr}}(F_v, M)} \right\},$$

where

$$H^1_{\mathrm{Gr}}(F_v, M) = \begin{cases} \ker\{H^1(F_v, M) \longrightarrow H^1(F_v^{\mathrm{nr}}, \mathscr{F}^- M)\} & \text{if } v \mid p; \\ \ker\{H^1(F_v, M) \longrightarrow H^1(F_v^{\mathrm{nr}}, M\} & \text{else.} \end{cases}$$

We will also have use for certain modified Selmer groups cut out by different local conditions at the places above p.

Definition 2.2. For $v \mid p$ and $\mathscr{L}_v \in \{\emptyset, \mathrm{Gr}, 0\}$, set

$$H^1_{\mathscr{L}_v}(F_v, M) := \begin{cases} H^1(F_v, M) & \text{if } \mathscr{L}_v = \emptyset; \\ H^1_{\mathrm{Gr}}(F_v, M) & \text{if } \mathscr{L}_v = \mathrm{Gr}; \\ \{0\} & \text{if } \mathscr{L}_v = 0, \end{cases}$$

and for $\mathscr{L} = \{\mathscr{L}_v\}_{v|p}$, define

$$\mathrm{Sel}_{\mathscr{L}}(F, M) := \ker\left\{ H^1(\mathfrak{G}_{F,\Sigma}, M) \longrightarrow \prod_{\substack{v\in\Sigma \\ v\nmid p}} \frac{H^1(F_v, M)}{H^1_{\mathrm{Gr}}(F_v, M)} \times \prod_{v|p} \frac{H^1(F_v, M)}{H^1_{\mathscr{L}_v}(F_v, M)} \right\}.$$

Of course, when $\mathscr{L}$ is given by $\mathscr{L}_v = \mathrm{Gr}$ for all $v \mid p$ we recover the previous definition, in which case $\mathscr{L}$ will be omitted from the notation.

Two-variable Selmer groups. Recall that $\Gamma_K = \mathrm{Gal}(K_\infty/K)$ denotes the Galois group of the $\mathbf{Z}_p^2$-extension of K, let $\Lambda = \mathbf{Z}_p[[\Gamma_K]]$ be the associated Iwasawa algebra, and define the Λ-modules

$$\mathbf{T} := T \otimes_{\mathbf{Z}_p} \Lambda, \qquad \mathbf{A} := T \otimes_{\mathbf{Z}_p} \Lambda^*,$$

where $\Lambda^* = \mathrm{Hom}_{\mathbf{Z}_p}(\Lambda, \mathbf{Q}_p/\mathbf{Z}_p)$ is the Pontrjagin dual of Λ. We equip these modules with the G_K-action given by $\rho_f \otimes \Psi^{-1}$, where $\Psi : G_K \twoheadrightarrow \Gamma_K \hookrightarrow \Lambda^\times$ is the natural character.

Setting

$$\mathscr{F}^\pm\mathbf{T} := \mathscr{F}^\pm T \otimes_{\mathbf{Z}_p} \Lambda, \qquad \mathscr{F}^\pm\mathbf{A} := \mathscr{F}^\pm T \otimes_{\mathbf{Z}_p} \Lambda^*,$$

the Selmer groups $\mathrm{Sel}_{\mathscr{L}}(K, \mathbf{T})$ and $\mathrm{Sel}_{\mathscr{L}}(K, \mathbf{A})$ are defined as before, and by Shapiro's lemma we then have canonical Λ-module isomorphisms

$$\mathrm{Sel}_{\mathscr{L}}(K, \mathbf{T}) \simeq \varprojlim_{K\subseteq F\subseteq K_\infty} \mathrm{Sel}_{\mathscr{L}}(F, T), \qquad \mathrm{Sel}_{\mathscr{L}}(K, \mathbf{A}) \simeq \varinjlim_{K\subseteq F\subseteq K_\infty} \mathrm{Sel}_{\mathscr{L}}(F, A), \tag{2.1}$$

where the limits are with respect to the corestriction and restriction map, respectively, as F runs over the finite extensions of K contained in K_∞ (see e.g. [SU14, Prop. 3.4]). Finally, set

$$X_{\mathscr{L}}(K, \mathbf{A}) := \mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Sel}_{\mathscr{L}}(K, \mathbf{A}), \mathbf{Q}_p/\mathbf{Z}_p),$$

omitting $\mathscr{L}$ from the notation in the same case as before.

Anticyclotomic Selmer groups. For the ease of notation, set

$$\Lambda^{\mathrm{ac}} = \mathbf{Z}_p[[\Gamma^{\mathrm{ac}}]].$$

The Selmer groups $\mathrm{Sel}_{\mathscr{L}}(K, \mathbf{T}^{\mathrm{ac}})$ and $X_{\mathscr{L}}(K, \mathbf{A}^{\mathrm{ac}})$ are defined by replacing Λ with Λ^{ac} in the above definitions. For any Λ^{ac}-module M, let M^{ι} be the underlying abelian group M with the original Λ^{ac}-module structure composed with the involution $\iota : \Lambda^{\mathrm{ac}} \to \Lambda^{\mathrm{ac}}$ given by inversion on group-like elements, and let M_{tors} denote the Λ^{ac}-torsion submodule of M. Also, let M_{div} be the maximal divisible submodule of M.

Lemma 2.3.

(1) $\mathrm{rank}_{\Lambda^{\mathrm{ac}}}(\mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}})) = \mathrm{rank}_{\Lambda^{\mathrm{ac}}}(X(K, \mathbf{A}^{\mathrm{ac}}))$.
(2) $\mathrm{rank}_{\Lambda^{\mathrm{ac}}}(X_{\mathrm{Gr},\emptyset}(K, \mathbf{A}^{\mathrm{ac}})) = 1 + \mathrm{rank}_{\Lambda^{\mathrm{ac}}}(X_{\mathrm{Gr},0}(K, \mathbf{A}^{\mathrm{ac}}))$.
(3) $Ch_{\Lambda^{\mathrm{ac}}}(X_{\mathrm{Gr},\emptyset}(K, \mathbf{A}^{\mathrm{ac}})_{\mathrm{tors}}) = Ch_{\Lambda^{\mathrm{ac}}}(X_{\mathrm{Gr},0}(K, \mathbf{A}^{\mathrm{ac}})_{\mathrm{tors}})$.

Proof. The first statement follows easily from [How04a, Prop. 2.2.8] (see [Wan14a, Lem. 3.5]), so we just need to prove the other two, for which we will adapt the arguments in [AH06, §1.2].

For any finite order character $\psi : \Gamma^{\mathrm{ac}} \to \mathfrak{O}_L^\times$ with values in the ring of integers of a finite extension $L/\mathbf{Q}_p$, we continue to denote by ψ its natural linear extension $\Lambda^{\mathrm{ac}} \to \mathfrak{O}_L$, and set $A(\psi) := \mathbf{A}^{\mathrm{ac}} \otimes_{\Lambda^{\mathrm{ac}},\psi} \mathfrak{O}_L$ equipped with the diagonal G_K-action. Then, by [MR04, Lem. 3.5.3, Thm. 4.1.13] there is a non-canonical isomorphism

$$H^1_{\mathrm{Gr},\emptyset}(K, A(\psi))[p^i] \simeq (L/\mathfrak{O}_L)^r[p^i] \oplus H^1_{\mathrm{Gr},0}(K, A(\psi^{-1}))[p^i], \tag{2.2}$$

where

- $H^1_{\mathrm{Gr},\emptyset}(K, A(\psi))$ is the subgroup $\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, A(\psi))$ consisting of classes whose restriction at $\mathfrak{p}$ (resp. $\overline{\mathfrak{p}}$) lies in $H^1(K_{\mathfrak{p}}, A(\psi))_{\mathrm{div}}$ (resp. $H^1_{\mathrm{Gr}}(K_{\overline{\mathfrak{p}}}, A(\psi))_{\mathrm{div}}$);
- $H^1_{\mathrm{Gr},0}(K, A(\psi^{-1}))$ is the subgroup of $\mathrm{Sel}_{\mathrm{Gr},0}(K, A(\psi^{-1}))$ consisting of classes whose restriction to $\mathfrak{p}$ lies in $H^1_{\mathrm{Gr}}(K_{\mathfrak{p}}, A(\psi))_{\mathrm{div}}$;
- and $r \geqslant 0$ is the *core rank* (see [MR04, Def. 4.1.11]) of the Selmer conditions defining $H^1_{\mathrm{Gr},\emptyset}(K, A(\psi))$.

By [DDT94, Thm. 2.18], the value of r is given by

$$\mathrm{corank}_{\mathbf{Z}_p} H^1(K_{\mathfrak{p}}, \mathscr{F}^+ A(\psi)) + \mathrm{corank}_{\mathbf{Z}_p} H^1(K_{\overline{\mathfrak{p}}}, A(\psi)) - \mathrm{corank}_{\mathbf{Z}_p} H^0(K_v, A(\psi)), \tag{2.3}$$

where v denotes the unique archimedean place of K. By the local Euler characteristic formula, the first two terms in (2.3) are equal to 1 and 2, respectively, while the third one clearly equals 2. Thus $r = 1$ in (2.2) and letting $i \to \infty$ we conclude that

$$H^1_{\mathrm{Gr},\emptyset}(K, A(\psi)) \simeq (L/\mathfrak{O}_L) \oplus H^1_{\mathrm{Gr},0}(K, A(\psi^{-1})). \tag{2.4}$$

Now, it is easy to show that the natural restriction maps

$$H^1_{\mathrm{Gr},\emptyset}(K, A(\psi)) \longrightarrow \mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, \mathbf{A}^{\mathrm{ac}})(\psi)^{\Gamma^{\mathrm{ac}}}$$
$$H^1_{\mathrm{Gr},0}(K, A(\psi^{-1})) \longrightarrow \mathrm{Sel}_{\mathrm{Gr},0}(K, \mathbf{A}^{\mathrm{ac}})(\psi^{-1})^{\Gamma^{\mathrm{ac}}}$$

are injective with finite bounded cokernel as ψ varies (cf. [AH06, Lem. 1.2.4]), and since

$$\mathrm{Sel}_{\mathrm{Gr},0}(K, \mathbf{A}^{\mathrm{ac}})(\psi^{-1})^{\Gamma^{\mathrm{ac}}} \simeq \mathrm{Sel}_{\mathrm{Gr},0}(K, \mathbf{A}^{\mathrm{ac}})(\psi)^{\Gamma^{\mathrm{ac}}}$$

by the action of complex conjugation, we see that statements (2) and (3) follow from (2.4) by the same argument as in [AH06, Lem. 1.2.6]. (For (3), note that in the proof of *loc.cit.* the prime $p\Lambda^{\mathrm{ac}}$ is excluded, but this can be dealt with similarly as in [How04a, Thm. 2.2.10].) $\square$

2.2. **Beilinson–Flach classes and explicit reciprocity laws.** In this subsection, we briefly recall the special type of Beilinson–Flach classes from [KLZ17] that we will need in this paper, and the "explicit reciprocity laws" relating them to p-adic L-functions.

For any Hida family $\mathbf{f}$, we let $M(\mathbf{f})^*$ be the associated Galois representation as in [KLZ17, Def. 7.2.5]; in particular, $M(\mathbf{f})^*$ is a finite and projective module over a local $\mathbf{Z}_p[[\mathbf{Z}_p^\times]]$-algebra $\Lambda_{\mathbf{f}}$, and there is a short exact sequence of $\Lambda_{\mathbf{f}}[\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)]$-modules

$$0 \longrightarrow \mathscr{F}^+M(\mathbf{f})^* \longrightarrow M(\mathbf{f})^* \longrightarrow \mathscr{F}^-M(\mathbf{f})^* \longrightarrow 0$$

with the Galois action on $\mathscr{F}^-M(\mathbf{f})^*$ being unramified. If $\mathbf{f}$ is the Hida family associated with a p-ordinary newform f as above, then there is a height one prime $\mathfrak{P} \subseteq \Lambda_{\mathbf{f}}$ with residue field $\mathbf{Q}_p$ for which we have an isomorphism

$$T \simeq M(\mathbf{f})^* \otimes_{\Lambda_{\mathbf{f}}} \Lambda_{\mathbf{f}}/\mathfrak{P}$$

compatible with the filtrations $\mathscr{F}^\pm$. Set $\Lambda_{\mathrm{cyc}} = \mathbf{Z}_p[[\mathrm{Gal}(K(\mu_{p^\infty})/K)]]$.

Theorem 2.4 (Kings–Loeffler–Zerbes)**.** *There exists an element $\mathcal{BF} \in \mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, \mathbf{T})$, a fractional ideal $I_{\mathbf{g}} \subseteq \mathrm{Frac}(\mathbf{Z}_p[[H_{\mathfrak{p}^\infty}]])$, and Λ-linear injections with pseudo-null cokernel:*

$$\begin{aligned}
\mathrm{Col}^{(1)} : H^1(K_{\overline{\mathfrak{p}}}, \mathscr{F}^-\mathbf{T}) &\longrightarrow \Lambda \otimes_{\mathbf{Z}_p} \mathbf{Q}_p, \\
\mathrm{Col}^{(2)} : H^1(K_{\mathfrak{p}}, \mathscr{F}^+\mathbf{T}) &\longrightarrow I_{\mathbf{g}} \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{\mathrm{cyc}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p,
\end{aligned}$$

such that

$$\mathrm{Col}^{(1)}(\mathrm{loc}_{\overline{\mathfrak{p}}}(\mathcal{BF})) = L_p(f/K, \Sigma^{(1)}), \qquad \mathrm{Col}^{(2)}(\mathrm{loc}_{\mathfrak{p}}(\mathcal{BF})) = L_p(f/K, \Sigma^{(2)}),$$

where $L_p(f/K, \Sigma^{(i)})$ are the p-adic L-functions of Theorem 1.2.

Proof. The fields $K(\mathfrak{p}^\infty)$ and $K(\mu_{p^\infty})$ are linearly disjoint over K and their compositum is $K(p^\infty)$, and so

$$\mathbf{Z}_p[[H_{p^\infty}]] \simeq \mathbf{Z}_p[[H_{\mathfrak{p}^\infty}]] \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{\mathrm{cyc}} \tag{2.5}$$

as Galois modules, where $\mathbf{Z}_p[[H_{p^\infty}]]$ is equipped with the G_K-action induced by the inverse of the character $G_K \twoheadrightarrow H_{p^\infty} \hookrightarrow \mathbf{Z}_p[[H_{p^\infty}]]^\times$, and similarly for the terms on the right-hand side. Let $\mathbf{f}$ be the Hida family attached to f, and consider the formal q-expansion

$$\mathbf{g} = \sum_{(\mathfrak{a},\mathfrak{p})=1} [\mathfrak{a}] q^{N(\mathfrak{a})} \in \Lambda_{\mathfrak{p}}[[q]], \tag{2.6}$$

where $\Lambda_{\mathfrak{p}} = \mathbf{Z}_p[[H_{\mathfrak{p}^\infty}]]$. Taking $m = 1$, $c > 1$ an integer coprime to $6N_fD_Kp$, and N a positive integer divisible by N_f and D_K and with the same prime factors as N_fD_K, the Beilinson–Flach class ${}_c\mathcal{BF}_m^{\mathbf{f},\mathbf{g}}$ constructed in [KLZ17, Def. 8.1.1] gives rise to an element

$${}_c\mathcal{BF}^{\mathbf{f},\mathbf{g}} \in H^1(\mathfrak{G}_{\mathbf{Q},S}, M(\mathbf{f})^* \hat{\otimes}_{\mathbf{Z}_p} M(\mathbf{g})^* \hat{\otimes}_{\mathbf{Z}_p} \Lambda_\Gamma), \tag{2.7}$$

where $\mathfrak{G}_{\mathbf{Q},S}$ is the Galois group of the maximal extension of $\mathbf{Q}$ unramified outside the primes dividing N_fD_Kp, $M(\mathbf{f})^*$ and $M(\mathbf{g})^*$ are the Galois modules associated to the corresponding Hida families as in [*loc.cit.*, Def. 7.2.5], and $\Lambda_\Gamma := \mathbf{Z}_p[[\mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})]]$ is equipped with the inverse of the natural $G_{\mathbf{Q}}$-action. With a slight abuse of notation, we continue to denote by ${}_c\mathcal{BF}^{\mathbf{f},\mathbf{g}}$ the image of the class (2.7) in $H^1(\mathbf{Q}, M(\mathbf{f})^* \hat{\otimes} M(\mathbf{g})^* \hat{\otimes} \Lambda_\Gamma)$ under inflation.

The CM Hida family $\mathbf{g}$ satisfies $M(\mathbf{g})^* \simeq \mathrm{Ind}_K^{\mathbf{Q}} \Lambda_{\mathfrak{p}}$, where $\Lambda_{\mathfrak{p}}$ is equipped with the G_K-action described in (2.5) above (see [LLZ15, Thm. 5.2.4]), and we let ${}_c\mathcal{BF}$ denote the image of ${}_c\mathcal{BF}^{\mathbf{f},\mathbf{g}}$ under the composite map

$$\begin{aligned}
H^1(\mathbf{Q}, M(\mathbf{f})^* \hat{\otimes}_{\mathbf{Z}_p} M(\mathbf{g})^* \hat{\otimes}_{\mathbf{Z}_p} \Lambda_\Gamma) &\longrightarrow H^1(\mathbf{Q}, T \hat{\otimes}_{\mathbf{Z}_p} M(\mathbf{g})^* \hat{\otimes}_{\mathbf{Z}_p} \Lambda_\Gamma) \\
&\xrightarrow{\simeq} H^1(K, T \hat{\otimes}_{\mathbf{Z}_p} \mathbf{Z}_p[[H_{p^\infty}]]) \\
&\longrightarrow H^1(K, \mathbf{T}),
\end{aligned}$$

where the first arrow is the natural map induced by the specialization $M(\mathbf{f})^* \to T$, the second isomorphism is given by Shapiro's lemma, and the third arrow is induced by the projection $H_{p^\infty} \twoheadrightarrow \Gamma_K$. By [LLZ14, Lem. 6.8.9], we may dispense with c by noting that the tame character of $\mathbf{g}$ is non-trivial at some prime dividing D_K; thus we have defined a class $\mathcal{BF} \in H^1(K, \mathbf{T})$.

By its geometric construction, the resulting class $\mathcal{BF}$ is unramified outside p, and the vanishing of its natural image in $H^1(K_{\mathfrak{p}}, \mathscr{F}^-\mathbf{T})$ follows from [KLZ17, Prop. 8.1.7]; therefore, we have $\mathcal{BF} \in \mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, \mathbf{T})$.

Now let $\omega_{\mathbf{f}}$, $\eta_{\mathbf{f}}$, $\omega_{\mathbf{g}}$ and $\eta_{\mathbf{g}}$ be as constructed in [KLZ17, Prop. 10.1.1], and define $\mathrm{Col}^{(1)}$ and $\mathrm{Col}^{(2)}$ to be the specializations at f of the maps

$$\langle \mathcal{L}(-), \eta_{\mathbf{f}} \otimes \omega_{\mathbf{g}} \rangle : H^1(K_{\overline{\mathfrak{p}}}, \mathscr{F}^{-+}M(\mathbf{f} \otimes \mathbf{g})^* \hat{\otimes}_{\mathbf{Z}_p} \Lambda_\Gamma) \longrightarrow \big(I_{\mathbf{f}} \hat{\otimes}_{\mathbf{Z}_p} \Lambda_{\mathfrak{p}} \hat{\otimes}_{\mathbf{Z}_p} \Lambda_\Gamma\big) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\mu_N],$$
$$\langle \mathcal{L}(-), \eta_{\mathbf{g}} \otimes \omega_{\mathbf{f}} \rangle : H^1(K_{\mathfrak{p}}, \mathscr{F}^{+-}M(\mathbf{f} \otimes \mathbf{g})^* \hat{\otimes}_{\mathbf{Z}_p} \Lambda_\Gamma) \longrightarrow \big(I_{\mathbf{g}} \hat{\otimes} \Lambda_{\mathbf{f}} \hat{\otimes}_{\mathbf{Z}_p} \Lambda_\Gamma\big) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p[\mu_N],$$

obtained by composing the "big logarithm map" $\mathcal{L}$ of [KLZ17, Thm. 8.2.8] with the pairing against the classes $\eta_{\mathbf{f}} \otimes \omega_{\mathbf{g}}$ and $\eta_{\mathbf{g}} \otimes \omega_{\mathbf{f}}$, respectively. (Here $\mathscr{F}^{-+}M(\mathbf{f}\otimes\mathbf{g})^*$ and $\mathscr{F}^{+-}M(\mathbf{f}\otimes\mathbf{g})^*$ are the subquotients of $M(\mathbf{f})^* \hat{\otimes} M(\mathbf{g})^*$ defined in [*loc.cit.*, p. 75], and $I_{\mathbf{g}} \subseteq \mathrm{Frac}(\Lambda_{\mathfrak{p}})$ and $I_{\mathbf{f}} \subseteq \mathrm{Frac}(\Lambda_{\mathbf{f}})$ are the *congruence ideal* of the corresponding Hida family; see [*loc.cit.*, Rem. 9.6.3].)

Thus defined, the claim about the kernel and cokernel of the maps $\mathrm{Col}^{(i)}$ follows from the last two claims in [KLZ17, Thm. 8.2.3], and their claimed relation with the p-adic L-functions $L_p(f/K, \Sigma^{(i)})$ is a consequence of the explicit reciprocity law of [KLZ17, Thm. 10.2.2]. □

Denote by $\mathcal{BF}^{\mathrm{ac}}$ the image of $\mathcal{BF}$ under the natural map $H^1(K, \mathbf{T}) \to H^1(K, \mathbf{T}^{\mathrm{ac}})$.

Corollary 2.5. *For all primes $v \mid p$, the class $\mathrm{loc}_v(\mathcal{BF})$ is non-torsion over Λ. Moreover, if hypothesis* (Heeg) *holds, then $\mathrm{loc}_{\mathfrak{p}}(\mathcal{BF}^{\mathrm{ac}})$ is non-torsion over Λ^{ac}, and $\mathrm{loc}_{\overline{\mathfrak{p}}}(\mathcal{BF}^{\mathrm{ac}})$ lies in the kernel of the natural map*

$$H^1(K_{\overline{\mathfrak{p}}}, \mathbf{T}^{\mathrm{ac}}) \longrightarrow H^1(K_{\overline{\mathfrak{p}}}, \mathscr{F}^-\mathbf{T}^{\mathrm{ac}}).$$

In particular, $\mathcal{BF}^{\mathrm{ac}} \in \mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}})$.

Proof. In light of Theorem 2.4, the first claim follows from the nonvanishing of the p-adic L-functions $L_p(f/K, \Sigma^{(i)})$ of Theorem 1.2 (see Remark 1.3); the second claim from the nonvanishing of the projection $L_p^{\mathrm{ac}}(f/K, \Sigma^{(2)})$ in Theorem 1.7; and the last claim from the vanishing of the projection $L_p^{\mathrm{ac}}(f/K, \Sigma^{(1)})$ in Proposition 1.4 and the injectivity of $\mathrm{Col}^{(1)}$. □

2.3. **Two-variable main conjectures.** Recall that the generalized Selmer groups $\mathrm{Sel}_{\mathscr{L}}(K, \mathbf{T})$ and $\mathrm{Sel}_{\mathscr{L}}(K, \mathbf{T}^{\mathrm{ac}})$ are submodules of $H^1(\mathfrak{G}_{K,\Sigma}, \mathbf{T})$ and $H^1(\mathfrak{G}_{K,\Sigma}, \mathbf{T}^{\mathrm{ac}})$, respectively.

Lemma 2.6. *Assume that the Galois representation $G_K \to \mathrm{Aut}_{\mathbf{Z}_p}(T)$ is surjective. Then the modules $H^1(\mathfrak{G}_{K,\Sigma}, \mathbf{T})$ and $H^1(\mathfrak{G}_{K,\Sigma}, \mathbf{T}^{\mathrm{ac}})$ are torsion-free over Λ and Λ^{ac}, respectively.*

Proof. As shown in [How04a, Lem. 2.2.9], this follows immediately from [PR00, §1.3.3] (whose argument readily extends to the two-variable setting). □

For any $\mathbf{Z}_p$-module M, define $M_{R_0} := M \hat{\otimes}_{\mathbf{Z}_p} R_0$. Let $\gamma^{\mathrm{ac}} \in \Gamma^{\mathrm{ac}}$ be a topological generator, and let $P \subseteq \Lambda$ be the pullback of the augmentation ideal of $P_{\mathrm{ac}} := (\gamma^{\mathrm{ac}} - 1) \subseteq \Lambda^{\mathrm{ac}}$.

Theorem 2.7. *Assume that $G_K \to \mathrm{Aut}_{\mathbf{Z}_p}(T)$ is surjective. Then the following two statements are equivalent:*

(1) *$X_{\mathrm{Gr},0}(K, \mathbf{A})$ is Λ-torsion, $\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, \mathbf{T})$ has Λ-rank 1, and*

$$Ch_\Lambda(X_{\mathrm{Gr},0}(K, \mathbf{A})) = Ch_\Lambda\left(\frac{\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, \mathbf{T})}{\Lambda \cdot \mathcal{BF}}\right)$$

in $\Lambda[1/P]$.

(2) *Both* $X_{\emptyset,0}(K,\mathbf{A})$ *and* $\mathrm{Sel}_{0,\emptyset}(K,\mathbf{T})$ *are* Λ*-torsion, and*

$$Ch_{\Lambda_{R_0}}(X_{\emptyset,0}(K,\mathbf{A})_{R_0}) = (\mathscr{L}_{\mathfrak{p}}(f/K))$$

in $\Lambda_{R_0}[1/P]$.

Proof. We only prove the implication (2) $\Rightarrow$ (1), which is the only direction we will use in the following, but the proof of the converse implication follows from the same ideas. Assume that $\mathrm{Sel}_{0,\emptyset}(K,\mathbf{T})$ is Λ-torsion, and consider the tautological exact sequence

$$0 \longrightarrow \mathrm{Sel}_{0,\emptyset}(K,\mathbf{T}) \longrightarrow \mathrm{Sel}_{\mathrm{Gr},\emptyset}(K,\mathbf{T}) \xrightarrow{\mathrm{loc}_{\mathfrak{p}}} H^1_{\mathrm{Gr}}(K_{\mathfrak{p}},\mathbf{T}).$$

By Corollary 2.5 the image of $\mathrm{loc}_{\mathfrak{p}}$ is nontorsion over Λ, and so (since $\mathrm{rank}_{\Lambda}(H^1_{\mathrm{Gr}}(K_{\mathfrak{p}},\mathbf{T}))=1$) the cokernel of $\mathrm{loc}_{\mathfrak{p}}$ is Λ-torsion and $\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K,\mathbf{T})$ has Λ-rank 1. Since $\mathrm{Sel}_{0,\emptyset}(K,\mathbf{T})$ is has no Λ-torsion by Lemma 2.6, our assumption implies that $\mathrm{Sel}_{0,\emptyset}(K,\mathbf{T})$ is trivial, and so Poitou–Tate duality gives rise to the exact sequence

$$0 \longrightarrow \mathrm{Sel}_{\mathrm{Gr},\emptyset}(K,\mathbf{T}) \longrightarrow H^1_{\mathrm{Gr}}(K_{\mathfrak{p}},\mathbf{T}) \longrightarrow X_{\emptyset,0}(K,\mathbf{A}) \longrightarrow X_{\mathrm{Gr},0}(K,\mathbf{A}) \longrightarrow 0,$$

and moding out by the images of $\mathcal{BF}$ to the exact sequence

$$0 \longrightarrow \frac{\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K,\mathbf{T})}{\Lambda\cdot\mathcal{BF}} \longrightarrow \frac{H^1_{\mathrm{Gr}}(K_{\mathfrak{p}},\mathbf{T})}{\Lambda\cdot\mathrm{loc}_{\mathfrak{p}}(\mathcal{BF})} \longrightarrow X_{\emptyset,0}(K,\mathbf{A}) \longrightarrow X_{\mathrm{Gr},0}(K,\mathbf{A}) \longrightarrow 0. \tag{2.8}$$

Now recall the congruence ideal $I_{\mathbf{g}}$ of CM forms $\mathbf{g}$ introduced in the proof of Theorem 2.4, which we shall view as an element in $\mathrm{Frac}(\Lambda^{\mathrm{ac}})$ in the following. By [HT91, Thm. A.4] (see also [HT93, Thm. I] and [HT94, Thm. 1.4.7]) we then have the divisibilities

$$(h_K\cdot\mathscr{L}^{\mathrm{ac}}_{\mathfrak{p}}(K)) \supseteq I_{\mathbf{g}}^{-1} \supseteq (h_K\cdot\mathscr{F}^{\mathrm{ac}}_{\mathfrak{p}}(K)) \tag{2.9}$$

in $\Lambda^{\mathrm{ac}}_{R_0}[1/P_{\mathrm{ac}}]$, where h_K is the class number of K, $\mathscr{L}^{\mathrm{ac}}_{\mathfrak{p}}(K)$ is an anticyclotomic projection of the Katz p-adic L-function of K as in Theorem 1.9, and $\mathscr{F}^{\mathrm{ac}}_{\mathfrak{p}}(K)$ generates the characteristic ideal of the Pontrjagin dual of the maximal abelian pro-p-extension of K^{ac}_{∞} unramified outside $\mathfrak{p}$. On the other hand, Rubin's proof [Rub91] of the Iwasawa main conjecture for K yields the equality

$$(\mathscr{F}^{\mathrm{ac}}_{\mathfrak{p}}(K)) = (\mathscr{L}^{\mathrm{ac}}_{\mathfrak{p}}(K)) \tag{2.10}$$

as ideals in $\Lambda^{\mathrm{ac}}_{R_0}$. Combining (2.9) and (2.10), it follows that $I_{\mathbf{g}}^{-1}$ is generated by $h_K\cdot\mathscr{L}^{\mathrm{ac}}_{\mathfrak{p}}(K)$ up to powers of P, and hence by Theorem 2.4 and the factorization in Theorem 1.9, the map $(h_K\cdot\mathscr{L}^{\mathrm{ac}}_{\mathfrak{p}}(K))\times\mathrm{Col}^{(2)}$ yields an injection

$$\frac{H^1_{\mathrm{Gr}}(K_{\mathfrak{p}},\mathbf{T})_{R_0}}{\Lambda_{R_0}\cdot\mathrm{loc}_{\mathfrak{p}}(\mathcal{BF})} \hookrightarrow \frac{\Lambda_{R_0}}{\Lambda_{R_0}\cdot\mathscr{L}_{\mathfrak{p}}(f/K)} \tag{2.11}$$

after extending scalars to $\Lambda_{R_0}[1/P]$ with pseudo-null cokernel. By multiplicativity of characteristic ideals in exact sequences, taking characteristic ideals in (2.8) and (2.11), the result follows. ☐

We record the following result for our later use.

Proposition 2.8. *Assume that equality in part (1) of Theorem 2.7 holds as ideals in* Λ*. Then*

$$Ch_{\Lambda^{\mathrm{ac}}}(X_{\mathrm{Gr},0}(K,\mathbf{A}^{\mathrm{ac}})) = Ch_{\Lambda^{\mathrm{ac}}}\left(\frac{\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K,\mathbf{T}^{\mathrm{ac}})}{\Lambda^{\mathrm{ac}}\cdot\mathcal{BF}^{\mathrm{ac}}}\right)$$

as ideals in Λ^{ac}.

Proof. Of course, this follows from descending from K_∞ to K_∞^{ac}. Let $\gamma^{\mathrm{cyc}} \in \Gamma^{\mathrm{cyc}}$ be a topological generator, and let I^{cyc} be the principal ideal $(\gamma^{\mathrm{cyc}}-1)\Lambda \subseteq \Lambda$. Then by [SU14, Prop. 3.9] (with the roles of the cyclotomic and anticyclotomic $\mathbf{Z}_p$-extensions reversed) we have

$$X_{\mathrm{Gr},0}(K,\mathbf{A})/I^{\mathrm{cyc}}X_{\mathrm{Gr},0}(K,\mathbf{A}) \simeq X_{\mathrm{Gr},0}(K,\mathbf{A}^{\mathrm{ac}}).$$

In particular, $X_{\mathrm{Gr},0}(K,\mathbf{A}^{\mathrm{ac}})$ is Λ^{ac}-torsion and by [Rub91, Lemma 6.2(ii)] it follows that

$$Ch_{\Lambda^{\mathrm{ac}}}(X_{\mathrm{Gr},0}(K,\mathbf{A}^{\mathrm{ac}})) = Ch_{\Lambda}(X_{\mathrm{Gr},0}(K,\mathbf{A}))\cdot\mathfrak{D}, \tag{2.12}$$

where $\mathfrak{D} := Ch_{\Lambda^{\mathrm{ac}}}(X_{\mathrm{Gr},0}(K,\mathbf{A})[I^{\mathrm{cyc}}])$. On the other hand, set

$$Z(K_\infty) := \mathrm{Sel}_{\mathrm{Gr},\emptyset}(K,\mathbf{T})/(\mathcal{BF}), \quad Z(K_\infty^{\mathrm{ac}}) := \mathrm{Sel}_{\mathrm{Gr},\emptyset}(K,\mathbf{T}^{\mathrm{ac}})/(\mathcal{BF}^{\mathrm{ac}}).$$

Using the fact that I^{cyc} is principal, a straightforward application of Snake's Lemma yields the exactness of

$$\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K,\mathbf{T})[I^{\mathrm{cyc}}] \longrightarrow Z(K_\infty)[I^{\mathrm{cyc}}] \longrightarrow (\mathcal{BF})/I^{\mathrm{cyc}}(\mathcal{BF}). \tag{2.13}$$

Arguing as in the proof of [AH06, Prop. 2.4.15] we see that the natural Λ^{ac}-module map

$$Z(K_\infty)/I^{\mathrm{cyc}}Z(K_\infty) \longrightarrow Z(K_\infty^{\mathrm{ac}})$$

is injective with cokernel having characteristic ideal $\mathfrak{D}$, and hence

$$Ch_{\Lambda^{\mathrm{ac}}}(Z(K_\infty^{\mathrm{ac}})) = Ch_{\Lambda^{\mathrm{ac}}}(Z(K_\infty)/I^{\mathrm{cyc}}Z(K_\infty))\cdot\mathfrak{D}. \tag{2.14}$$

Since $\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K,\mathbf{T})$ has no Λ-torsion by Lemma 2.6, the leftmost term in (2.13) vanishes; on the other hand the rightmost term is clearly torsion-free, and hence $Z(K_\infty)[I^{\mathrm{cyc}}]$ is torsion-free. Since [Rub91, Lem. 6.2(i)] and equality (2.14) imply that $Z(K_\infty)[I^{\mathrm{cyc}}]$ is also a torsion Λ^{ac}-module (using the nonvanishing of the terms in that equality), we conclude that $Z(K_\infty)[I^{\mathrm{cyc}}] = 0$, and by [Rub91, Lem. 6.2(ii)] it follows that

$$Ch_{\Lambda}(Z(K_\infty))\cdot\Lambda^{\mathrm{ac}} = Ch_{\Lambda^{\mathrm{ac}}}(Z(K_\infty)/I^{\mathrm{cyc}}Z(K_\infty)). \tag{2.15}$$

Combined with (2.12), we thus arrive at

$$\begin{aligned} Ch_{\Lambda^{\mathrm{ac}}}(X_{\mathrm{Gr},0}(K,\mathbf{A}^{\mathrm{ac}})) &= Ch_{\Lambda}(X_{\mathrm{Gr},0}(K,\mathbf{A}))\cdot\mathfrak{D} \\ &= Ch_{\Lambda}(Z(K_\infty))\cdot\mathfrak{D} \\ &= Ch_{\Lambda^{\mathrm{ac}}}(Z(K_\infty^{\mathrm{ac}})), \end{aligned}$$

using (2.14) and (2.15) for the last equality. This completes the proof. □

3. Λ-adic Gross–Zagier formula

Throughout this section, we let $E/\mathbf{Q}$ be an elliptic curve of conductor N, $f = \sum_{n=1}^{\infty} a_n(f)q^n$ be the normalized newform of weight 2 associated with E, and $p \nmid 6N$ be a prime of ordinary reduction for E. We also let $K/\mathbf{Q}$ be an imaginary quadratic field of discriminant prime to Np satisfying the generalized Heegner hypothesis (Heeg) in the Introduction and such that $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K.

3.1. Heegner point main conjecture.

Recall that for every integer $m \geqslant 1$ we let $K[m]$ be the ring class field of K of conductor m. In particular, $K[1]$ is the Hilbert class field of K.

Proposition 3.1. *There exists a collection of Heegner points $z_{f,p^n} \in E(K[p^n])\otimes_{\mathbf{Z}}\mathbf{Z}_p$ satisfying the norm-compatibility relations*

$$a_p(f)\cdot z_{f,p^{n-1}} = \begin{cases} z_{f,p^{n-2}} + \mathrm{Norm}_{K[p^{n-1}]}^{K[p^n]}(z_{f,p^n}) & \text{if } n>1; \\ \sigma_{\mathfrak{p}}z_{f,1} + \sigma_{\overline{\mathfrak{p}}}z_{f,1} + w_K\cdot\mathrm{Norm}_{K[1]}^{K[p]}(z_{f,p}) & \text{if } n=1, \end{cases} \tag{3.1}$$

where $w_K = |\mathcal{O}_K^\times|$, and $\sigma_{\mathfrak{p}}, \sigma_{\overline{\mathfrak{p}}} \in \mathrm{Gal}(K[1]/K)$ are the Frobenius elements of $\mathfrak{p}, \overline{\mathfrak{p}}$, respectively.

Proof. This is standard, but for our later reference we briefly recall the construction, referring the reader to [How04b, §1.2]) and the references therein for further details. Let $X := X_{N^+,N^-}$ be the Shimura curve attached to a quaternion algebra $B/\mathbf{Q}$ of discriminant N^- equipped with the $\Gamma_0(N^+)$-level structure defined be an Eichler order of level N^+. By [How04b, Prop. 1.2.1], there exists a collection of CM points $h_{p^n} \in X(K[p^n])$ satisfying the norm relations (3.1) with the Hecke correspondence T_p in place of $a_p(f)$.

By a construction due to S.-W. Zhang [Zha01b] extending a classical result of Shimura, after possibly replacing E by a $\mathbf{Q}$-isogenous elliptic curve we may fix a parametrization

$$\Phi_E : \mathrm{Jac}(X) \longrightarrow E$$

defined over $\mathbf{Q}$. If $N^- \neq 1$, the curve X has no cusps, and an auxiliary choice is necessary in order to embed X into $\mathrm{Jac}(X)$. To that end, we fix a prime $\ell \nmid N$ such that $\ell + 1 - a_\ell(f)$ is a p-adic unit,[4] and define $z_{f,p^n} \in E(K[p^n]) \otimes_{\mathbf{Z}} \mathbf{Z}_p$ be the image of the degree zero divisor $(\ell+1-a_\ell(f))^{-1}(\ell+1-T_\ell)h_{p^n}$ under the map $\mathrm{Jac}(X)(K[p^n]) \to E(K[p^n])$ induced by Φ_E. $\square$

Now let α be the unit root of $x^2 - a_p(f)x + p$, and define the *regularized Heegner points* $z_{f,p^n,\alpha} \in E(K[p^n]) \otimes_{\mathbf{Z}} \mathbf{Z}_p$ by

$$z_{f,p^n,\alpha} := \begin{cases} z_{f,p^n} - \frac{1}{\alpha} z_{f,p^{n-1}} & \text{if } n > 0; \\ \frac{1}{w_K}\left(1 - \frac{\sigma_{\mathfrak{p}}}{\alpha}\right)\left(1 - \frac{\sigma_{\overline{\mathfrak{p}}}}{\alpha}\right) z_{f,1} & \text{if } n = 0. \end{cases}$$

By Proposition 3.1, we then have

$$\mathrm{Norm}^{K[p^n]}_{K[p^{n-1}]}(z_{f,p^n,\alpha}) = \alpha \cdot z_{f,p^{n-1},\alpha}$$

for all $n \geqslant 1$. Letting $\mathbf{z}_{f,p^n,\alpha}$ be the image of the point $z_{f,p^n,\alpha}$ under the Kummer map

$$E(K[p^n]) \otimes_{\mathbf{Z}} \mathbf{Z}_p \longrightarrow H^1(K[p^n], T_pE),$$

the classes $\alpha^{-n} \cdot \mathbf{z}_{f,p^n,\alpha}$ are then compatible under corestriction, thus defining a class

$$\mathbf{z}_f \in H^1(K, \mathbf{T}^{\mathrm{ac}})$$

which lands in the Selmer group $\mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}})$.

Conjecture 3.2. *Both* $\mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}})$ *and* $X(K, \mathbf{A}^{\mathrm{ac}})$ *have* Λ^{ac}*-rank* 1*, and*

$$Ch_{\Lambda^{\mathrm{ac}}}(X(K, \mathbf{A}^{\mathrm{ac}})_{\mathrm{tors}}) = Ch_{\Lambda^{\mathrm{ac}}}\left(\frac{\mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}})}{\Lambda^{\mathrm{ac}} \cdot \mathbf{z}_f}\right)^2$$

as ideals in $\Lambda^{\mathrm{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$*.*

Remark 3.3. When $N^- = 1$ (so that $X = X_{N^+,N^-}$ is just the classical modular curve $X_0(N^+)$), an integral form of Conjecture 3.2 incorporating the Manin constant associated to the modular parametrization Φ_E was originally formulated by Perrin-Riou [PR87a, Conj. B]. In the above form, Conjecture 3.2 is the case $F = \mathbf{Q}$ of a conjecture formulated by Howard [How04b, p.3].

Thanks to the work of several authors, Conjecture 3.2 is now known under mild hypotheses.

Theorem 3.4. *Let* $E/\mathbf{Q}$ *be an elliptic curve of conductor* N *with good ordinary reduction at* $p \geqslant 5$*, and let* K *be an imaginary quadratic field of discriminant prime to* N *satisfying hypothesis* (Heeg)*. Assume in addition that:*

- N *is square-free,*
- $N^- \neq 1$*,*
- $E[p]$ *is ramified at every prime* $\ell \mid N^-$*,*
- $\mathrm{Gal}(\overline{\mathbf{Q}}/K) \to \mathrm{Aut}_{\mathbf{Z}_p}(T_pE)$ *is surjective.*

[4]In our application, the existence of such a prime ℓ will be guaranteed by a certain big image hypothesis; without this assumption, one may use a certain *Hodge class* to exhibit an embedding $X \to \mathrm{Jac}(X)$.

Then:

(1) *Conjecture 3.2 holds.*
(2) $X_{\emptyset,0}(K,\mathbf{A}^{\mathrm{ac}})$ *is* Λ^{ac}*-torsion, and*

$$Ch_{\Lambda^{\mathrm{ac}}_{R_0}}(X_{\emptyset,0}(K,\mathbf{A}^{\mathrm{ac}})_{R_0}) = (\mathscr{L}_{\mathfrak{p}}^{\mathtt{BDP}}(f/K)^2)$$

as ideals in $\Lambda^{\mathrm{ac}}_{R_0}$.

Proof. This follows by combining the works of Howard [How04a, How04b] and Wan [Wan14b], and the link between the tow provided by (the weight 2 case of) the explicit reciprocity law of [CH17] and Poitou–Tate duality. The details are given in the Appendix to this paper. ☐

Corollary 3.5. *Under the hypotheses of Theorem 3.4, the following hold:*

(1) $X_{\mathrm{Gr},0}(K,\mathbf{A})$ *is* Λ*-torsion,* $\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K,\mathbf{T})$ *has* Λ*-rank* 1*, and*

$$Ch_{\Lambda}(X_{\mathrm{Gr},0}(K,\mathbf{A})) = Ch_{\Lambda}\left(\frac{\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K,\mathbf{T})}{\Lambda\cdot\mathcal{BF}}\right).$$

(2) $X_{\emptyset,0}(K,\mathbf{A})$ *is* Λ*-torsion, and*

$$Ch_{\Lambda_{R_0}}(X_{\emptyset,0}(K,\mathbf{A})_{R_0}) = (\mathscr{L}_{\mathfrak{p}}(f/K)).$$

Proof. Part (2) is deduced from Theorem 3.4 by the same argument as in [CW16, Thm. 5.1]; part (1) then follows from Theorem 2.7. The details, which involve many of the ingredients that go into the proof of Theorem 3.4 is relegated to the Appendix of this paper. ☐

3.2. **Rubin's height formula.** We maintain the notations from §1.2, and continue to denote by $L_p(f/K,\Sigma^{(1)})$ the image of the p-adic L-function $L_p(f/K,\Sigma^{(1)})$ of Theorem 1.2 under the natural projection

$$\mathbf{Q}_p\otimes_{\mathbf{Z}_p}\mathbf{Z}_p[[H_{p^\infty}]] \longrightarrow \mathbf{Q}_p\otimes_{\mathbf{Z}_p}\Lambda.$$

Let $\gamma^{\mathrm{cyc}}\in\Gamma^{\mathrm{cyc}}$ be a topological generator, and using the identification $\Lambda\simeq\Lambda^{\mathrm{ac}}[[\Gamma^{\mathrm{cyc}}]]$ expand

$$L_p(f/K,\Sigma^{(1)}) = L_{p,0}(f/K) + L_{p,1}^{\mathrm{cyc}}(f/K)(\gamma^{\mathrm{cyc}}-1)+\cdots \tag{3.2}$$

as a power series in $(\gamma^{\mathrm{cyc}}-1)$. The 'constant term' in this expansion is just the anticyclotomic projection $L_p^{\mathrm{ac}}(f/K,\Sigma^{(1)})$, which vanishes by Proposition 1.4.

In light of the isomorphism (2.1), the Beilinson–Flach class $\mathcal{BF}$ from Theorem 2.4 can be seen a compatible system of classes

$$\mathcal{BF} = \varprojlim_{K\subseteq F\subseteq K_\infty}\mathcal{BF}_F$$

with $\mathcal{BF}_F\in\mathrm{Sel}_{\mathrm{Gr},\emptyset}(F,T)$, and where F runs over the finite extensions of K contained in K_∞. Let $L_n = K_n^{\mathrm{ac}}K_\infty^{\mathrm{cyc}}$, and set $\mathcal{BF}(L_n) := \varprojlim_{K\subseteq F\subseteq L_n}\mathcal{BF}_F$.

Recall that for any free $\mathbf{Z}_p$-module M equipped with a linear action of $G_{\mathbf{Q}_p}$, and a p-adic Lie extension $E_\infty/\mathbf{Q}_p$, one defines

$$H^1_{\mathrm{Iw}}(E_\infty,M) := \varprojlim_{\mathbf{Q}_p\subseteq E\subseteq E_\infty} H^1(E,M),$$

where similarly as before, E runs over the finite extensions of $\mathbf{Q}_p$ contained in E_∞.

Lemma 3.6. *For every* $n\geqslant 0$ *there is an element*

$$\beta_n\in H^1_{\mathrm{Iw}}(L_{n,\overline{\mathfrak{p}}},T)\otimes_{\mathbf{Z}_p}\mathbf{Q}_p,$$

unique modulo the natural image of $H^1_{\mathrm{Iw}}(L_{n,\overline{\mathfrak{p}}},\mathscr{F}^+T)\otimes\mathbf{Q}_p$*, such that*

$$(\gamma^{\mathrm{cyc}}-1)\beta_n = \mathrm{loc}_{\overline{\mathfrak{p}}}(\mathcal{BF}(L_n)).$$

Furthermore, the natural images of β_n in $H^1(K^{\rm ac}_{n,\overline{\mathfrak{p}}},T)\otimes\mathbf{Q}_p/H^1_{\rm Gr}(K^{\rm ac}_{n,\overline{\mathfrak{p}}},T)\otimes\mathbf{Q}_p$ define a class $\beta_\infty(\mathbb{1})\in H^1(K_{\overline{\mathfrak{p}}},\mathbf{T}^{\rm ac})\otimes\mathbf{Q}_p/H^1_{\rm Gr}(K_{\overline{\mathfrak{p}}},\mathbf{T}^{\rm ac})\otimes\mathbf{Q}_p$, and the Λ-linear map $\mathrm{Col}^{(1)}$ of Theorem 2.4 yields an identification

$$\frac{H^1(K_{\overline{\mathfrak{p}}},\mathbf{T}^{\rm ac})}{H^1_{\rm Gr}(K_{\overline{\mathfrak{p}}},\mathbf{T}^{\rm ac})}\otimes_{\mathbf{Z}_p}\mathbf{Q}_p\simeq\Lambda^{\rm ac}\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$$

sending $\beta_\infty(\mathbb{1})$ to the 'linear term' $L_{p,1}^{\rm cyc}(f/K)$ in (3.2).

Proof. By the injectivity of $\mathrm{Col}^{(1)}$, the first claim follows easily from the explicit reciprocity law of Theorem 2.4 and the vanishing of $L_{p,0}(f/K)$. The second claim is a direct consequence of the definitions of $\beta_\infty(\mathbb{1})$ and $L_{p,1}^{\rm cyc}(f/K)$ (and again Theorem 2.4). □

Recall that by Corollary 2.5 we have the inclusion $\mathcal{BF}_n^{\rm ac}\in\mathrm{Sel}(K_n^{\rm ac},T)$ for every $n\geqslant 0$. Let $\mathcal{I}$ be the augmentation ideal of $\mathbf{Z}_p[[\Gamma^{\rm cyc}]]$, and set $\mathcal{J}=\mathcal{I}/\mathcal{I}^2$.

Theorem 3.7. *For every $n\geqslant 0$ there is a p-adic height pairing*

$$\langle\ ,\ \rangle^{\rm cyc}_{K_n^{\rm ac}}:\mathrm{Sel}(K_n^{\rm ac},T)\times\mathrm{Sel}(K_n^{\rm ac},T)\longrightarrow p^{-k}\mathbf{Z}_p\otimes_{\mathbf{Z}_p}\mathcal{J}$$

for some $k\in\mathbf{Z}_{\geqslant 0}$ independent of n, such that for every $b\in\mathrm{Sel}(K_n^{\rm ac},T)$, we have

$$\langle\mathcal{BF}_n^{\rm ac},b\rangle^{\rm cyc}_{K_n^{\rm ac}}=(\beta_n(\mathbb{1}),\mathrm{loc}_{\overline{\mathfrak{p}}}(b))_n\otimes(\gamma^{\rm cyc}-1),\tag{3.3}$$

where $(\ ,\ )_n$ is the $\mathbf{Q}_p$-linear extension of the local Tate pairing

$$\frac{H^1(K^{\rm ac}_{n,\overline{\mathfrak{p}}},T)}{H^1_{\rm Gr}(K^{\rm ac}_{n,\overline{\mathfrak{p}}},T)}\times H^1_{\rm Gr}(K^{\rm ac}_{n,\overline{\mathfrak{p}}},T)\longrightarrow\mathbf{Z}_p.$$

Proof. The construction of the p-adic height pairing follows work of Perrin-Riou [PR92], and the proof of the height formula (3.3) follows from a straightforward extension of a well-known result due to Rubin [Rub94] (cf. [AH06, §3.2] and [Nek06, (13.3.14)]). □

3.3. **Main results.** Define the $\Lambda^{\rm ac}$-adic height pairing

$$\langle\ ,\ \rangle^{\rm cyc}_{K_\infty^{\rm ac}}:\mathrm{Sel}(K,\mathbf{T}^{\rm ac})\otimes_{\Lambda^{\rm ac}}\mathrm{Sel}(K,\mathbf{T}^{\rm ac})^\iota\longrightarrow\mathbf{Q}_p\otimes_{\mathbf{Z}_p}\Lambda^{\rm ac}\otimes_{\mathbf{Z}_p}\mathcal{J}\tag{3.4}$$

by the formula

$$\langle a_\infty,b_\infty\rangle^{\rm cyc}_{K_\infty^{\rm ac}}=\varprojlim_n\sum_{\sigma\in\mathrm{Gal}(K_n^{\rm ac}/K)}\langle a_n,b_n^\sigma\rangle^{\rm cyc}_{K_n^{\rm ac}}\cdot\sigma,$$

and define the cyclotomic regulator $\mathcal{R}_{\rm cyc}\subseteq\Lambda^{\rm ac}$ to be the characteristic ideal of the cokernel of (3.4). The following $\Lambda^{\rm ac}$-adic Birch and Swinnerton-Dyer formula corresponds to Theorem B in the Introduction.

Theorem 3.8. *Let the hypotheses be as in Theorem 3.4, and denote by $\mathcal{X}_{\rm tors}$ the characteristic ideal of $X(K,\mathbf{A}^{\rm ac})_{\rm tors}$. Then*

$$\mathcal{R}_{\rm cyc}\cdot\mathcal{X}_{\rm tors}=(L_{p,1}^{\rm cyc}(f/K))$$

as ideals in $\Lambda^{\rm ac}\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$.

Proof. The height formula of Theorem 3.7 and Lemma 3.6 immediately yield the equality

$$\mathcal{R}_{\rm cyc}\cdot Ch_{\Lambda^{\rm ac}}\left(\frac{\mathrm{Sel}(K,\mathbf{T}^{\rm ac})}{\Lambda^{\rm ac}\cdot\mathcal{BF}^{\rm ac}}\right)=(L_{p,1}^{\rm cyc}(f/K))\cdot\eta^\iota,\tag{3.5}$$

where $\eta\subseteq\Lambda^{\rm ac}$ is the characteristic ideal of $H^1_{\rm Gr}(K_{\overline{\mathfrak{p}}},\mathbf{T}^{\rm ac})/\mathrm{loc}_{\overline{\mathfrak{p}}}(\mathrm{Sel}(K,\mathbf{T}^{\rm ac}))$. By Theorem 3.4 we have $\mathrm{rank}_{\Lambda^{\rm ac}}(\mathrm{Sel}(K,\mathbf{T}^{\rm ac}))=1$, and so $\mathrm{Sel}_{0,\rm Gr}(L,\mathbf{T}^{\rm ac})=\{0\}$ by Lemma A.3, from where the nonvanishing of η follows by the exactness of (A.4) in the proof of that lemma.

Now, from global duality we have the exact sequence

$$0 \longrightarrow \frac{H^1_{\mathrm{Gr}}(K_{\overline{\mathfrak{p}}}, \mathbf{T}^{\mathrm{ac}})}{\mathrm{loc}_{\overline{\mathfrak{p}}}(\mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}}))} \longrightarrow X_{\mathrm{Gr},\emptyset}(K, \mathbf{A}^{\mathrm{ac}}) \longrightarrow X(K, \mathbf{A}^{\mathrm{ac}}) \longrightarrow 0. \tag{3.6}$$

Since $\mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}})$ has Λ^{ac}-rank 1, by Lemma A.2 we know that $X_{\mathrm{Gr},0}(K, \mathbf{A}^{\mathrm{ac}})$ is Λ^{ac}-torsion and that

$$\mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}}) = \mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, \mathbf{T}^{\mathrm{ac}}). \tag{3.7}$$

The discussion above also shows that the second and third terms in (3.6) both have Λ^{ac}-rank 1. Taking Λ^{ac}-torsion in (3.6) and applying Lemma 2.3, we thus obtain the equality

$$Ch_{\Lambda^{\mathrm{ac}}}(X_{\mathrm{Gr},0}(K, \mathbf{A}^{\mathrm{ac}})) = \mathcal{X}_{\mathrm{tors}} \cdot \eta^{\iota} \tag{3.8}$$

as ideals in $\Lambda^{\mathrm{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$. By Proposition 2.8 (whose conclusion can be invoked thanks to Corollary 3.5), equality (3.8) amounts to

$$Ch_{\Lambda^{\mathrm{ac}}}\left(\frac{\mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}})}{\Lambda^{\mathrm{ac}} \cdot \mathcal{BF}^{\mathrm{ac}}}\right) = \mathcal{X}_{\mathrm{tors}} \cdot \eta^{\iota}, \tag{3.9}$$

using equality (3.7) in the left-hand side. Substituting (3.9) into (3.5), the result follows from the nonvanishing of η. □

The Λ^{ac}-adic Gross–Zagier formula of Theorem A in the Introduction is now a direct consequence of Theorem 3.8 and the Heegner point main conjecture of Theorem 3.4.

Theorem 3.9. *Under the hypotheses of Theorem 3.4 we have the equality*

$$(L^{\mathrm{cyc}}_{p,1}(f/K)) = (\langle \mathbf{z}_f, \mathbf{z}_f \rangle^{\mathrm{cyc}}_{K^{\mathrm{ac}}_\infty})$$

as ideals of $\Lambda^{\mathrm{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

Proof. By Theorem 3.4 we have the equality

$$Ch_{\Lambda^{\mathrm{ac}}}(X(K, \mathbf{A}^{\mathrm{ac}})_{\mathrm{tors}}) = Ch_{\Lambda^{\mathrm{ac}}}\left(\frac{\mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}})}{\Lambda^{\mathrm{ac}} \cdot \mathbf{z}_f}\right)^2$$

up to powers of $p\Lambda^{\mathrm{ac}}$. Combined with Theorem 3.8, we conclude that

$$(L^{\mathrm{cyc}}_{p,1}(f/K)) = \mathcal{R}_{\mathrm{cyc}} \cdot Ch_{\Lambda^{\mathrm{ac}}}\left(\frac{\mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}})}{\Lambda^{\mathrm{ac}} \cdot \mathbf{z}_f}\right)^2 = (\langle \mathbf{z}_f, \mathbf{z}_f \rangle^{\mathrm{cyc}}_{K^{\mathrm{ac}}_\infty})$$

as ideals in $\Lambda^{\mathrm{ac}} \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$, as was to be shown. □

Appendix A. Proofs of Theorem 3.4 and Corollary 3.5

The purpose of this Appendix is to develop some of the details that go into the proofs of Theorem 3.4 and Corollary 3.5. As already mentioned, the first result follows easily from the combination of three ingredients: Howard's divisibility in the Heegner point main conjecture, Wan's converse divisibility in the main conjecture for $\mathscr{L}_{\mathfrak{p}}(f/K)$, and an explicit reciprocity law for Heegner points building a bridge between the two. The arguments that follow closely parallel those in [CW16, §4.3], and the interested reader may also wish to consult [Wan14a].

Throughout, we let $E/\mathbf{Q}$ be an elliptic curve of conductor N, $f \in S_2(\Gamma_0(N))$ the normalized newform associated with E, $p \geqslant 5$ a prime of good reduction for E, and $K/\mathbf{Q}$ an imaginary quadratic field of discriminant prime to Np satisfying hypothesis (Heeg) in the Introduction. We begin by recalling the third of the aforementioned results.

Theorem A.1. *Assume in addition that $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K and that f is p-ordinary. Then there exists an injective Λ^{ac}-linear map*

$$\mathcal{L}_+ : H^1(K_\mathfrak{p}, \mathscr{F}^+\mathbf{T}^{\mathrm{ac}})_{R_0} \longrightarrow \Lambda_{R_0}$$

with finite cokernel such that

$$\mathcal{L}_+(\mathrm{res}_\mathfrak{p}(\mathbf{z}_f)) = -\mathscr{L}_\mathfrak{p}^{\mathsf{BDP}}(f/K) \cdot \sigma_{-1,\mathfrak{p}},$$

where $\sigma_{-1,\mathfrak{p}} \in \Gamma^{\mathrm{ac}}$ has order two.

Proof. This follows from the weight 2 case of [CH17, Thm. 5.7]. (We note that the injectivity of the map $\mathcal{L}_+$ is not explicitly stated in *loc.cit.*, but it follows from [LZ14, Prop. 4.11] and the construction in [CH17, Thm. 5.1]. Also, since $\mathbf{z}_f \in \mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}})$, the restriction $\mathrm{res}_\mathfrak{p}(\mathbf{z}_f)$ lands in $H^1_{\mathrm{Gr}}(K_\mathfrak{p}, \mathbf{T}^{\mathrm{ac}})$, which is naturally identified with $H^1(K_\mathfrak{p}, \mathscr{F}^+\mathbf{T}^{\mathrm{ac}})$ by the vanishing of $H^0(K_\mathfrak{p}, \mathscr{F}^-\mathbf{T}^{\mathrm{ac}})$.) $\square$

The next three lemmas are applications of Poitou–Tate duality, combined with the explicit reciprocity law of Theorem A.1, and in all of them the assumption that $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits in K is in order.

Lemma A.2. *Assume that* $\mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}})$ *has* Λ^{ac}*-rank* 1*. Then*

$$\mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}}) = \mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, \mathbf{T}^{\mathrm{ac}}) \tag{A.1}$$

and $X_{\mathrm{Gr},0}(K, \mathbf{A}^{\mathrm{ac}})$ is Λ^{ac}-torsion.

Proof. Consider the exact sequence

$$\mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}}) \xrightarrow{\mathrm{loc}_\mathfrak{p}} H^1_{\mathrm{Gr}}(K_\mathfrak{p}, \mathbf{T}^{\mathrm{ac}}) \longrightarrow X_{\emptyset,\mathrm{Gr}}(K, \mathbf{A}^{\mathrm{ac}}) \longrightarrow X(K, \mathbf{A}^{\mathrm{ac}}) \longrightarrow 0. \tag{A.2}$$

Since $\mathbf{z}_f$ lands in $\mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}})$, by Theorem A.1 the nonvanishing of $\mathscr{L}_\mathfrak{p}^{\mathsf{BDP}}(f/K)$ implies that the image of the map $\mathrm{loc}_\mathfrak{p}$ is not Λ^{ac}-torsion. Since $H^1_{\mathrm{Gr}}(K_\mathfrak{p}, \mathbf{T}^{\mathrm{ac}})$ has Λ^{ac}-rank 1, it follows that $\mathrm{coker}(\mathrm{loc}_\mathfrak{p})$ is Λ^{ac}-torsion, and since part (1) of Lemma 2.3 combined with our assumption implies that $X(K, \mathbf{A}^{\mathrm{ac}})$ has Λ^{ac}-rank 1, we conclude from (A.2) that

$$\mathrm{rank}_{\Lambda^{\mathrm{ac}}}(X_{\emptyset,\mathrm{Gr}}(K, \mathbf{A}^{\mathrm{ac}})) = 1. \tag{A.3}$$

Since $X_{\emptyset,\mathrm{Gr}}(K, \mathbf{A}^{\mathrm{ac}}) \simeq X_{\mathrm{Gr},\emptyset}(K, \mathbf{A}^{\mathrm{ac}})$ by the action of complex conjugation, we deduce from (A.3) and part (2) of Lemma 2.3 that $X_{\mathrm{Gr},0}(K, \mathbf{A}^{\mathrm{ac}})$ is Λ^{ac}-torsion. Finally, since the quotient $H^1(K_{\overline{\mathfrak{p}}}, \mathbf{T}^{\mathrm{ac}})/H^1_{\mathrm{Gr}}(K_{\overline{\mathfrak{p}}}, \mathbf{T}^{\mathrm{ac}})$ has Λ^{ac}-rank 1, counting ranks in the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}}) \longrightarrow \mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, \mathbf{T}^{\mathrm{ac}}) &\xrightarrow{\mathrm{loc}_{\overline{\mathfrak{p}}}} \frac{H^1(K_{\overline{\mathfrak{p}}}, \mathbf{T}^{\mathrm{ac}})}{H^1_{\mathrm{Gr}}(K_{\overline{\mathfrak{p}}}, \mathbf{T}^{\mathrm{ac}})} \\ &\longrightarrow X(K, \mathbf{A}^{\mathrm{ac}}) \longrightarrow X_{\mathrm{Gr},0}(K, \mathbf{A}^{\mathrm{ac}}) \longrightarrow 0, \end{aligned}$$

we conclude that $\mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}})$ and $\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K, \mathbf{T}^{\mathrm{ac}})$ have both Λ^{ac}-rank 1, and since the quotient $H^1(K_{\overline{\mathfrak{p}}}, \mathbf{T}^{\mathrm{ac}})/H^1_{\mathrm{Gr}}(K_{\overline{\mathfrak{p}}}, \mathbf{T}^{\mathrm{ac}})$ is also Λ^{ac}-torsion-free (in fact, it injects into $H^1(K_{\overline{\mathfrak{p}}}, \mathscr{F}^-\mathbf{T}^{\mathrm{ac}})$ by the vanishing of $H^0(K_{\overline{\mathfrak{p}}}, \mathscr{F}^-\mathbf{T}^{\mathrm{ac}})$), equality (A.1) follows. $\square$

Lemma A.3. *Assume that* $\mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}})$ *has* Λ^{ac}*-rank* 1*. Then* $\mathrm{Sel}_{0,\mathrm{Gr}}(K, \mathbf{T}^{\mathrm{ac}}) = \{0\}$*, and for any height one prime $\mathfrak{P}$ of $\Lambda^{\mathrm{ac}}_{R_0}$ we have*

$$\mathrm{ord}_\mathfrak{P}(\mathscr{L}_\mathfrak{p}^{\mathsf{BDP}}(f/K)) = \mathrm{length}_\mathfrak{P}(\mathrm{coker}(\mathrm{loc}_\mathfrak{p})_{R_0}) + \mathrm{length}_\mathfrak{P}\left(\frac{\mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}})_{R_0}}{\Lambda^{\mathrm{ac}}_{R_0} \cdot \mathbf{z}_f}\right),$$

where $\mathrm{loc}_\mathfrak{p} : \mathrm{Sel}(K, \mathbf{T}^{\mathrm{ac}}) \to H^1_{\mathrm{Gr}}(K_\mathfrak{p}, \mathbf{T}^{\mathrm{ac}})$ is the natural restriction map.

Proof. Denote by $\Lambda^{\rm ac}_{\mathfrak{P}}$ the localization of $\Lambda^{\rm ac}_{R_0}$ at $\mathfrak{P}$, and consider the tautological exact sequence

$$0 \longrightarrow \mathrm{Sel}_{0,\mathrm{Gr}}(K,\mathbf{T}^{\rm ac}) \longrightarrow \mathrm{Sel}(K,\mathbf{T}^{\rm ac}) \xrightarrow{\mathrm{loc}_{\mathfrak{p}}} H^1_{\mathrm{Gr}}(K_{\mathfrak{p}},\mathbf{T}^{\rm ac}) \longrightarrow \mathrm{coker}(\mathrm{loc}_{\mathfrak{p}}) \longrightarrow 0. \tag{A.4}$$

By Theorem A.1, the nonvanishing of $\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K)$ implies that the image of $\mathbf{z}_f \in \mathrm{Sel}(K,\mathbf{T}^{\rm ac})$ under the map $\mathrm{loc}_{\mathfrak{p}}$ is not $\Lambda^{\rm ac}$-torsion. Since we assume that $\mathrm{Sel}(K,\mathbf{T}^{\rm ac})$ has $\Lambda^{\rm ac}$-rank 1, this shows that $\mathrm{Sel}_{0,\mathrm{Gr}}(K,\mathbf{T}^{\rm ac})$ is $\Lambda^{\rm ac}$-torsion, and so $\mathrm{Sel}_{0,\mathrm{Gr}}(K,\mathbf{T}^{\rm ac})=\{0\}$ by Lemma 2.6.

From (A.4) we thus deduce the exact sequence

$$0 \longrightarrow \frac{\mathrm{Sel}(K,\mathbf{T}^{\rm ac})}{\Lambda^{\rm ac}\cdot\mathbf{z}_f} \xrightarrow{\mathrm{loc}_{\mathfrak{p}}} \frac{H^1_{\mathrm{Gr}}(K_{\mathfrak{p}},\mathbf{T}^{\rm ac})}{\Lambda^{\rm ac}\cdot\mathrm{loc}_{\mathfrak{p}}(\mathbf{z}_f)} \longrightarrow \mathrm{coker}(\mathrm{loc}_{\mathfrak{p}}) \longrightarrow 0,$$

and since by Theorem A.1 we have a $\Lambda^{\rm ac}_{\mathfrak{P}}$-module pseudo-isomorphism

$$\left(\frac{H^1_{\mathrm{Gr}}(K_{\mathfrak{p}},\mathbf{T}^{\rm ac})}{\Lambda^{\rm ac}_{R_0}\cdot\mathrm{loc}_{\mathfrak{p}}(\mathbf{z}_f)}\right)\otimes_{\Lambda^{\rm ac}_{R_0}}\Lambda^{\rm ac}_{\mathfrak{P}} \xrightarrow{\sim} \frac{\Lambda^{\rm ac}_{\mathfrak{P}}}{\Lambda^{\rm ac}_{\mathfrak{P}}\cdot\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K)},$$

the result follows. ☐

Lemma A.4. *Assume that* $\mathrm{Sel}(K,\mathbf{T}^{\rm ac})$ *has* $\Lambda^{\rm ac}$*-rank* 1. *Then* $X_{\emptyset,0}(K,\mathbf{A}^{\rm ac})$ *is* $\Lambda^{\rm ac}$*-torsion, and for any height one prime* $\mathfrak{P}$ *of* $\Lambda^{\rm ac}$ *we have*

$$\mathrm{length}_{\mathfrak{P}}(X_{\emptyset,0}(K,\mathbf{A}^{\rm ac})) = \mathrm{length}_{\mathfrak{P}}(X(K,\mathbf{A}^{\rm ac})_{\rm tors}) + 2\,\mathrm{length}_{\mathfrak{P}}(\mathrm{coker}(\mathrm{loc}_{\mathfrak{p}})),$$

where $\mathrm{loc}_{\mathfrak{p}}:\mathrm{Sel}(K,\mathbf{T}^{\rm ac})\to H^1_{\mathrm{Gr}}(K_{\mathfrak{p}},\mathbf{T}^{\rm ac})$ *is the natural restriction map.*

Proof. Global duality yields the exact sequence

$$0 \longrightarrow \mathrm{coker}(\mathrm{loc}_{\mathfrak{p}}) \longrightarrow X_{\emptyset,\mathrm{Gr}}(K,\mathbf{A}^{\rm ac}) \longrightarrow X(K,\mathbf{A}^{\rm ac}) \longrightarrow 0. \tag{A.5}$$

As shown in the proof of Lemma A.3, the first term in this sequence is $\Lambda^{\rm ac}$-torsion; since by Lemma 2.3(1) the assumption implies that $X(K,\mathbf{A}^{\rm ac})$ has $\Lambda^{\rm ac}$-rank 1, this shows that the same is true for $X_{\emptyset,\mathrm{Gr}}(K,\mathbf{A}^{\rm ac})$, and by Lemma 2.3(2) it follows that $X_{0,\mathrm{Gr}}(K,\mathbf{A}^{\rm ac})$ is $\Lambda^{\rm ac}$-torsion. Thus taking $\Lambda^{\rm ac}$-torsion in (A.5) and using Lemma 2.3(3), it follows that

$$\mathrm{length}_{\mathfrak{P}}(X_{0,\mathrm{Gr}}(K,\mathbf{A}^{\rm ac})) = \mathrm{length}_{\mathfrak{P}}(X(K,\mathbf{A}^{\rm ac})_{\rm tors}) + \mathrm{length}_{\mathfrak{P}}(\mathrm{coker}(\mathrm{loc}_{\mathfrak{p}})) \tag{A.6}$$

for any height one prime $\mathfrak{P}$ of $\Lambda^{\rm ac}$.

Another application of global duality yields the exact sequence

$$0 \longrightarrow \mathrm{coker}(\mathrm{loc}_{\mathfrak{p}}^{\emptyset}) \longrightarrow X_{\emptyset,0}(K,\mathbf{A}^{\rm ac}) \longrightarrow X_{\mathrm{Gr},0}(K,\mathbf{A}^{\rm ac}) \longrightarrow 0, \tag{A.7}$$

where $\mathrm{loc}_{\mathfrak{p}}^{\emptyset}:\mathrm{Sel}_{\mathrm{Gr},\emptyset}(K,\mathbf{T}^{\rm ac})\to H^1_{\mathrm{Gr}}(K_{\mathfrak{p}},\mathbf{T}^{\rm ac})$ is the natural restriction map. By Lemma A.2, this is the same as the map $\mathrm{loc}_{\mathfrak{p}}$ in the statement, and hence $\mathrm{coker}(\mathrm{loc}_{\mathfrak{p}}^{\emptyset})$is $\Lambda^{\rm ac}$-torsion. Since $X_{\mathrm{Gr},0}(K,\mathbf{A}^{\rm ac})$ is $\Lambda^{\rm ac}$-torsion by Lemma A.2, we conclude from (A.7) that $X_{\emptyset,0}(K,\mathbf{A}^{\rm ac})$ is $\Lambda^{\rm ac}$-torsion. Combining (A.7) and (A.6), we thus have

$$\begin{aligned}\mathrm{length}_{\mathfrak{P}}(X_{\emptyset,0}(K,\mathbf{A}^{\rm ac})) &= \mathrm{length}_{\mathfrak{P}}(X_{\mathrm{Gr},0}(K,\mathbf{A}^{\rm ac})) + \mathrm{length}_{\mathfrak{P}}(\mathrm{coker}(\mathrm{loc}_{\mathfrak{p}}))\\ &= \mathrm{length}_{\mathfrak{P}}(X(K,\mathbf{A}^{\rm ac})_{\rm tors}) + 2\,\mathrm{length}_{\mathfrak{P}}(\mathrm{coker}(\mathrm{loc}_{\mathfrak{p}}))\end{aligned}$$

for any height one prime $\mathfrak{P}$ of $\Lambda^{\rm ac}$, as was to be shown. ☐

The next two theorems recall the results by Howard and Wan that we need.

Theorem A.5 (Howard)**.** *Assume that* f *is* p*-ordinary and that* $\mathrm{Gal}(\overline{\mathbf{Q}}/K)\to\mathrm{Aut}_{\mathbf{Z}_p}(T_pE)$ *is surjective. Then both* $\mathrm{Sel}(K,\mathbf{T}^{\rm ac})$ *and* $X(K,\mathbf{A}^{\rm ac})$ *have* $\Lambda^{\rm ac}$*-rank* 1 *and we have the divisibility*

$$Ch_{\Lambda^{\rm ac}}(X(K,\mathbf{A}^{\rm ac})_{\rm tors}) \supseteq Ch_{\Lambda^{\rm ac}}\left(\frac{\mathrm{Sel}(K,\mathbf{T}^{\rm ac})}{\Lambda^{\rm ac}\cdot\mathbf{z}_f}\right)^2.$$

Proof. This follows from [How04a, Thm. B], as extended by [How04b, Thm. 3.4.2, Thm. 3.4.3] to the context of Shimura curves attached to possibly non-split indefinite quaternion algebras, *cf.* [Wan14a, Thm. 2.1]. (Note also that by the work of Cornut–Vatsal [CV07, Thm. 1.10] the assumption in [How04b, Thm. 3.4.3] is now known to hold.) □

Remark A.6. In [How04a] and [How04b] it is assumed that p does not divide the class number of K, but similarly as in [Kim07, §4] (see esp. [*loc.cit.*, Prop. 4.19]), this assumption may be easily relaxed.

Theorem A.7 (Wan)**.** *Assume that:*

- N *is square-free,*
- $N^- \neq 1$,
- $E[p]|_{\mathrm{Gal}(\overline{\mathbf{Q}}/K)}$ *is absolutely irreducible.*

Then we have the divisibility

$$Ch_{\Lambda_{R_0}}(X_{\emptyset,0}(K,\mathbf{T})_{R_0}) \subseteq (\mathscr{L}_{\mathfrak{p}}(f/K))$$

in $\Lambda_{R_0}\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$.

Proof. The follows from the main result of [Wan14b]. □

The last ingredient that we need is the following result on the vanishing of the μ-invariant for the p-adic L-function $\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K)$ of Theorem 1.5.

Theorem A.8 (Burungale, Hsieh)**.** *Assume that:*

- N *is square-free,*
- $E[p]|_{\mathrm{Gal}(\overline{\mathbf{Q}}/K)}$ *is absolutely irreducible,*
- $E[p]$ *is ramified at every prime* $\ell \mid N^-$.

Then $\mu(\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K))=0$.

Proof. This follows from [Hsi14, Thm. B]; or alternatively [Bur17, Thm. B], noting that by the discussion in [Pra06, p. 912] our last assumption guarantees that the ratio of Petersson norms $\alpha(f,f_B)$ in [Bur17, Thm. 5.6] is a p-adic unit. □

Now we finally assemble all the pieces together.

Proof of Theorem 3.4. By Theorem A.5, $\mathrm{Sel}(K,\mathbf{T}^{\mathrm{ac}})$ has Λ^{ac}-rank 1, and so $X_{\emptyset,0}(K,\mathbf{A}^{\mathrm{ac}})$ is Λ^{ac}-torsion by Lemma A.4. Let $\mathfrak{P}$ be a height one prime of $\Lambda_{R_0}^{\mathrm{ac}}$, and set $\mathfrak{P}_0=\mathfrak{P}\cap\Lambda^{\mathrm{ac}}$. By the divisibility in Theorem A.5 we have

$$\mathrm{length}_{\mathfrak{P}_0}(X(K,\mathbf{A}^{\mathrm{ac}})_{\mathrm{tors}}) \leqslant 2\,\mathrm{length}_{\mathfrak{P}_0}\left(\frac{\mathrm{Sel}(K,\mathbf{A}^{\mathrm{ac}})}{\Lambda^{\mathrm{ac}}\cdot\mathbf{z}_f}\right). \tag{A.8}$$

Combined with Lemma A.4 and Lemma A.3, respectively, this implies that

$$\begin{aligned}\mathrm{length}_{\mathfrak{P}}(X_{\mathfrak{p}}(K,\mathbf{A}^{\mathrm{ac}})_{R_0}) &\leqslant 2\,\mathrm{length}_{\mathfrak{P}}\left(\frac{\mathrm{Sel}(K,\mathbf{T}^{\mathrm{ac}})_{R_0}}{\Lambda_{R_0}^{\mathrm{ac}}\cdot\mathbf{z}_f}\right)+2\,\mathrm{length}_{\mathfrak{P}}(\mathrm{coker}(\mathrm{loc}_{\mathfrak{p}})_{R_0})\\ &= 2\,\mathrm{ord}_{\mathfrak{P}}(\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K)),\end{aligned}$$

and hence

$$Ch_{\Lambda_{R_0}^{\mathrm{ac}}}(X_{\emptyset,0}(K,\mathbf{A}^{\mathrm{ac}}))\ \supseteq\ (\mathscr{L}_{\mathfrak{p}}^{\mathsf{BDP}}(f/K)^2) \tag{A.9}$$

and ideals in $\Lambda_{R_0}^{\mathrm{ac}}$. Now let I^{cyc} be the principal ideal of Λ generated by $\gamma^{\mathrm{cyc}}-1$. By a standard control theorem (*cf.* [SU14, Prop. 3.9], or see [JSW15, §3.3] for a proof in our same context), the natural restriction map $H^1(K_\infty^{\mathrm{ac}},E[p^\infty])\to H^1(K_\infty,E[p^\infty])$ induces an isomorphism

$$X_{\emptyset,0}(K,\mathbf{A})/I^{\mathrm{cyc}}X_{\emptyset,0}(K,\mathbf{A})\simeq X_{\emptyset,0}(K,\mathbf{A}^{\mathrm{ac}})$$

as $\Lambda^{\rm ac}$-modules. Combined with Corollary 1.10, the divisibility in Theorem A.7 thus yields the divisibility

$$Ch_{\Lambda^{\rm ac}_{R_0}}(X_{\emptyset,0}(K,\mathbf{A}^{\rm ac})_{R_0}) \subseteq (\mathscr{L}_{\mathfrak{p}}^{\tt BDP}(f/K)^2)$$

in $\Lambda^{\rm ac}_{R_0}\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$. In particular, equality holds in (A.9) up to powers of $p\Lambda^{\rm ac}_{R_0}$, and by Theorem A.8 the equality holds integrally. Combined with Lemmas A.3 and A.4, this implies that for any height one prime $\mathfrak{P}_0$ of $\Lambda^{\rm ac}$ equality in (A.8) holds, thus concluding the proof. ☐

Proof of Corollary 3.5. As before, let $I^{\rm cyc}\subseteq\Lambda$ be the ideal generated by $\gamma^{\rm cyc}-1$, and set

$$X := Ch_{\Lambda_{R_0}}(X_{\emptyset,0}(K,\mathbf{A})_{R_0}), \quad Y := (\mathscr{L}_{\mathfrak{p}}(f/K)) \subseteq \Lambda_{R_0}.$$

As in the proof of Theorem 3.4, the divisibility in Theorem A.7 combined with Theorem A.8 yields the divisibility $X\subseteq Y$ as ideals in Λ_{R_0}. On the other hand, from the combination of (A.10) and Corollary 1.10, Theorem 3.4 implies that $X_{\emptyset,0}(K,\mathbf{A})$ is Λ-torsion and that $X=Y$ (mod $I^{\rm cyc}$). The equality $X=Y$ in Λ_{R_0} thus follows from [SU14, Lem. 3.2]. This completes the proof of part (2) of Corollary 3.5. By Theorem 2.7, it follows that the equality in part (1) holds in $\Lambda[1/P]$. Since by Theorem 2.4 the Beilinson–Flach class $\mathcal{BF}$ restricts nontrivially to the cyclotomic line (see Remark 1.3), the aforementioned equality holds in Λ, concluding the proof of the result. ☐

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