CM ELLIPTIC CURVES OF RANK 2 AND NONVANISHING OF GENERALISED KATO CLASSES

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Abstract. Let $E_{/\mathbf{Q}}$ be a CM elliptic curve and p an odd prime of good ordinary reduction for E. Suppose $L(E,s)$ has sign $w=+1$ and vanishes at $s=1$, so in particular $\mathrm{ord}_{s=1}L(E,s)\geq 2$. In this paper we modify a construction of Darmon–Rotger [DR17] to define a new generalised Kato class $\kappa_p\in\mathrm{Sel}(\mathbf{Q},V_pE)$ conjectured to be nonzero precisely when E has rank 2. Our main results show that, under some hypothesis, $\kappa_p\neq 0$ precisely when $\mathrm{Sel}(\mathbf{Q},V_pE)$ is 2-dimensional. Moreover, combined with work of Rubin [Rub92], we can exhibit explicit bases for $\mathrm{Sel}(\mathbf{Q},V_pE)$ in certain cases.

1. Introduction

Fix an odd prime p. Let $E_{/\mathbf{Q}}$ be an elliptic curve with complex multiplication by the ring of integers of an imaginary quadratic field K in which $p=\mathfrak{p}\overline{\mathfrak{p}}$ splits. Let $L(E,s)$ be the Hasse–Weil L-function of $E_{/\mathbf{Q}}$, and denote by $w\in\{\pm1\}$ the sign in its functional equation:

$$L(E,s)=wL(E,2-s).$$

1.1. **Rubin's p-converse theorem.** We begin by recalling the following result by Rubin [Rub94], giving the first "p-converse" to the theorem of Gross–Zagier and Kolyvagin.

Theorem (Rubin)**.** *Assume that $w=-1$. Then there exists a Heegner point $P\in E(\mathbf{Q})$ such that*

$$\left.\begin{array}{r}\mathrm{rank}_{\mathbf{Z}}E(\mathbf{Q})=1\\ \#\text{Ш}(E_{/\mathbf{Q}})[p^\infty]<\infty\end{array}\right\}\implies P\notin E(\mathbf{Q})_{\mathrm{tors}},$$

and hence $\mathrm{ord}_{s=1}L(E,s)=1$*.*

The proof of this result is deduced in [Rub94, Thm. 4] as a consequence of the Iwasawa main conjecture for K [Rub91], Perrin-Riou's calculation of the Euler characteristic of a Selmer group attached to $E_{/K}$ over the $\mathbf{Z}_p$-extension of K unramified outside $\mathfrak{p}$ [PR84], and Bertrand's transcendence result [Ber84]; the claim that $\mathrm{ord}_{s=1}L(E,1)$ is a consequence of the Gross–Zagier formula [GZ86].

1.2. **Statement of the main result.** In this paper we prove an analogue of Rubin's p-converse theorem in rank 2, in which the Heegner point P is replaced by a certain Selmer class

$$\kappa_p\in\mathrm{Sel}(\mathbf{Q},V_pE).$$

Here $\mathrm{Sel}(\mathbf{Q},V_pE)$ denotes the p-adic Selmer group of $E_{/\mathbf{Q}}$ fitting into the exact sequence

$$0\to E(\mathbf{Q})\otimes_{\mathbf{Z}}\mathbf{Q}_p\to\mathrm{Sel}(\mathbf{Q},V_pE)\to\mathrm{Ta}_p\text{Ш}(E_{/\mathbf{Q}})\otimes_{\mathbf{Z}_p}\mathbf{Q}_p\to 0,$$

where $\mathrm{Ta}_p\text{Ш}(E_{/\mathbf{Q}})$ is the p-adic Tate module of the Tate–Shafarevich group of $E_{/\mathbf{Q}}$.

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The class κ_p may be viewed as a variant of the *generalised Kato classes* introduced by Darmon–Rotger [DR16] and a natural extension of [*op. cit.*, Conj. 3.2] predicts that $\kappa_p \neq 0$ precisely when E has rank 2.

Recall that the *fine Selmer group* $\mathrm{Sel}_0(\mathbf{Q}, V_pE)$ is defined as the kernel of the restriction map

$$\mathrm{loc}_p : \mathrm{Sel}(\mathbf{Q}, V_pE) \to \mathrm{H}^1(\mathbf{Q}_p, V_pE).$$

Our first main result in this paper is the following.

Theorem A (Theorem 6.3.2)**.** *Suppose* $w = +1$ *and* $L(E, 1) = 0$*, so in particular* $\mathrm{ord}_{s=1}L(E, s) \geq 2$*. If* $\mathrm{Sel}(\mathbf{Q}, V_pE) \neq \mathrm{Sel}_0(\mathbf{Q}, V_pE)$*, then the following implication holds:*

$$\dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2 \implies \kappa_p \neq 0;$$

more precisely, κ_p *is a nonzero class in the fine Selmer group* $\mathrm{Sel}_0(\mathbf{Q}, V_pE)$*.*

It is worth-noting that the vanishing of $L(E, 1)$ implies that the p-adic Selmer group $\mathrm{Sel}(\mathbf{Q}, V_pE)$ is non-trivial [Rub91], so the above condition $\mathrm{Sel}(\mathbf{Q}, V_pE) \neq \mathrm{Sel}_0(\mathbf{Q}, V_pE)$ holds unless $E(\mathbf{Q})$ is finite and $Ш(E_{/\mathbf{Q}})[p^\infty]$ is infinite.

1.3. **Application to rank 2 Selmer basis.** The construction of κ_p depends on certain auxiliary choices, but it follows from our results that different choices give rise to the same Selmer class up to scaling. Hence it is natural to ask for a class in the 2-dimensional $\mathrm{Sel}(\mathbf{Q}, V_pE)$ complementary to the line spanned by κ_p.

Let λ be the Hecke character of K attached to E, so that

$$L(E, s) = L(\lambda, s),$$

and denote by $\lambda^{\mathbf{c}}$ the composition of λ with the action of the nontrivial element $\mathbf{c} \in \mathrm{Gal}(K/\mathbf{Q})$. Denote by $\mathcal{W}$ the completion of the ring of integers of the maximal unramified extension of $\mathbf{Q}_p$, and let $\mathcal{L}_{\mathfrak{p}} = \mathcal{L}_{\mathfrak{p},\mathfrak{c}} \in \mathcal{W}[\![Z(\mathfrak{c})]\!]$ be the Katz p-adic L-function recalled in Theorem 2.1.1 below, where $Z(\mathfrak{c})$ is the Galois group of the extension $K(E[p^\infty])/K$. Following [Rub94], for $s \in \mathbf{Z}_p$ define

$$L_{\mathfrak{p}}(s) = \mathcal{L}_{\mathfrak{p}}(\lambda\langle\lambda\rangle^{s-1}), \quad L_{\mathfrak{p}}^*(s) = \mathcal{L}_{\mathfrak{p}}(\lambda^{\mathbf{c}}\langle\lambda^{\mathbf{c}}\rangle^{s-1}),$$

where $\langle - \rangle : \mathbf{Z}_p^\times \to 1 + p\mathbf{Z}_p$ is the projection onto the 1-units. Assuming that $E(\mathbf{Q})$ has a point of infinite order, the p-adic Birch and Swinnerton-Dyer conjectures formulated in *op. cit.* predict that $\mathrm{ord}_{s=1}L_{\mathfrak{p}}(s) = r$ and $\mathrm{ord}_{s=1}L_{\mathfrak{p}}^*(s) = r - 1$, where $r = \mathrm{rank}_{\mathbf{Z}}E(\mathbf{Q}) \geq 1$.

Theorem B. *Assume that* $\#Ш(E_{/\mathbf{Q}})[p^\infty] < \infty$ *and the following conditions hold:*

$$\mathrm{ord}_{s=1}L_{\mathfrak{p}}(s) = 2, \quad \mathrm{ord}_{s=1}L_{\mathfrak{p}}^*(s) = 1.$$

Then $\mathrm{Sel}(\mathbf{Q}, V_pE)$ *is* 2*-dimensional, with*

$$\mathrm{Sel}(\mathbf{Q}, V_pE) = \mathbf{Q}_p\kappa_p \oplus \mathbf{Q}_px_{\mathfrak{p}},$$

where κ_p *is a generalised Kato class and* $x_{\mathfrak{p}}$ *is a derived elliptic unit.*

Proof. The assumption that $\mathrm{ord}_{s=1}L_{\mathfrak{p}}(s) = 2$ implies, on the one hand, that

$$r := \dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) \leq 2$$

by [Rub91, Thm. 4.1] and [PR84, Ch. IV, Thm. 22], and on the other hand that $r \geq 2$ by the theorem of Coates–Wiles [CW77] and the p-parity conjecture [Guo93]. Therefore $r = 2$, and by [Rub92, Prop. 4.4] the construction of derived elliptic units in [*op. cit.*, §6] yields a class

$$x_{\mathfrak{p}} \in \mathrm{Sel}(K, T_{\mathfrak{p}}E) \simeq \mathrm{Sel}(\mathbf{Q}, T_pE)$$

denoted $x_{\mathfrak{p}}^{(2)}$ in *loc. cit.*, where $T_{\mathfrak{p}}E$ is the $\mathfrak{p}$-adic Tate module of E. By [Rub92, Thm. 9.5(ii)], the assumptions $\#Ш(E_{/\mathbf{Q}})[p^\infty] < \infty$ and $\mathrm{ord}_{s=1}L_{\mathfrak{p}}^*(s) = 1$ imply that $x_{\mathfrak{p}} \notin \mathrm{Sel}_0(\mathbf{Q}, V_pE)$, so the result follows from Theorem A. □

1.4. **A rank 2 Kolyvagin theorem.** We can also prove a converse to Theorem A, going in the direction of Kolyvagin's theorem [Kol88].

Theorem C (Theorem 7.0.1)**.** *Suppose $w = +1$ and $L(E, 1) = 0$. If* $\mathrm{Sel}(\mathbf{Q}, V_pE) \neq \mathrm{Sel}_0(\mathbf{Q}, V_pE)$, *then the following implication holds:*

$$\kappa_p \neq 0 \quad \Longrightarrow \quad \dim_{\mathbf{Q}_p}\mathrm{Sel}(\mathbf{Q}, V_pE) = 2.$$

The proof of Theorem C builds on a "degeneration" to weights $(1, 1)$ of the diagonal cycle Euler system for Rankin–Selberg convolutions constructed in a joint work with Alonso and Rivero [ACR21], and the general theory of anticyclotomic Euler systems developed by Jetchev–Nekovář–Skinner [JNS]. The key new observation here is that such a result in rank 2 can be deduced from a rank 1 Kolyvagin theorem for a non-classical Selmer group.

1.5. **Relation to other works.** In [CH22], the author and M.-L. Hsieh obtained results analogous to Theorem A and Theorem C for certain generalised Kato classes κ_p' introduced by Darmon–Rotger [DR16], but the CM case eluded their approach. Indeed, the generalised Kato classes κ_p' studied in [CH22] are attached to the choice of:

(i) a quadratic imaginary field M in which p splits such that $L(E^M, 1) \neq 0$;
(ii) a ring class character $\chi = \psi/\psi^{\mathbf{c}}$ of M such that $L(E_{/M}, \chi, 1) \neq 0$,

so that the twisted L-value $L(E, \mathrm{ad}^0(\theta_\psi), 1)$ is nonzero. It is under these nonvanishing hypotheses that [DR16, Conj. 3.2] predicts that $\kappa_p' \neq 0$ precisely when E has rank 2.

For elliptic curves $E_{/\mathbf{Q}}$ with CM by K, condition (i) can be arranged by taking $M \neq K$ [BFH90], but condition (ii) is problematic, as Vatsal's nonvanishing results for $L(E_{/M}, \chi, 1)$ [Vat03] require E to have some prime of multiplicative reduction, but unfortunately (having integral j-invariant) CM elliptic curves have no such primes. Thus, lacking stronger nonvanishing results, in the CM setting the construction of κ_p' from [DR16] appears to be ill-suited.

The proof of Theorem A and Theorem C is therefore completely different, and our results for the class κ_p constructed here hint at a new variant of the conjectures in [DR16] for CM elliptic curves. The approach in this paper reveals a close link between [DR16, Conj. 3.2] (a nonvanishing criterion for generalised Kato classes) in the *rank (2,0) adjoint CM setting* and a main conjecture in anticyclotomic Iwasawa theory. Moreover, the approach should generalise to other contexts, starting with the case studied in [CH22] and its analogue for supersingular primes p [CH].

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2. p-adic L-functions

In this section we recall the two p-adic L-functions that will appear in our arguments, one due to Katz [Kat78] attached to Hecke characters of an imaginary quadratic fields, and another due to Hsieh [Hsi21] attached to triple products of modular forms in Hida families.

From now on we fix a prime $p > 2$ and an imaginary quadratic field K with ring of integers $\mathcal{O}_K$ in which

$$(p) = \mathfrak{p}\overline{\mathfrak{p}} \text{ splits,} \tag{spl}$$

with $\mathfrak{p}$ the prime of K above p determined by a fixed embedding $\iota_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$.

2.1. **Katz p-adic L-function.** Denote by $D_K < 0$ the discriminant of K, and fix an integral ideal $\mathfrak{c} \subset \mathcal{O}_K$ coprime to p. Let $\mathcal{W}$ be a finite extension of the Witt ring $W(\overline{\mathbf{F}}_p)$, and denote by $Z(\mathfrak{c})$ the Galois group of the ray class field extension $K(\mathfrak{c}p^\infty)/K$.

We say that a Hecke character $\chi : K^\times\backslash\mathbb{A}_K^\times \to \mathbf{C}^\times$ has infinity type (a, b) if $\chi_\infty(z) = z^a\overline{z}^b$.

Theorem 2.1.1. *There exists an element $\mathcal{L}_{\mathfrak{p},\mathfrak{c}} \in \mathcal{W}[\![Z(\mathfrak{c})]\!]$ such that for all Hecke characters χ of conductor dividing $\mathfrak{c}p^\infty$ and infinity type $(k + j, -j)$ with $k \geq 1$, $j \geq 0$, we have*

$$\mathcal{L}_{\mathfrak{p},\mathfrak{c}}(\chi) = \left(\frac{\Omega_p}{\Omega_\infty}\right)^{k+2j} \cdot \Gamma(k + j) \cdot \left(\frac{2\pi}{\sqrt{D_K}}\right)^j \cdot \mathcal{E}_\mathfrak{p}(\chi) \cdot L^{(p\mathfrak{c})}(\chi, 0),$$

where $\mathcal{E}_\mathfrak{p}(\chi)$ is the modified Euler factor

$$\mathcal{E}_\mathfrak{p}(\chi) = \frac{L(0, \chi_\mathfrak{p})}{\varepsilon(0, \chi_\mathfrak{p}) \cdot L(1, \chi_\mathfrak{p}^{-1})}$$

and $L^{(p\mathfrak{c})}(\chi, s)$ is the L-function $L^{(p\mathfrak{c})}(\chi, 0)$ deprived from the Euler factors at the primes $\mathfrak{l}|p\mathfrak{c}$. Moreover, we have the functional equation

$$\mathcal{L}_{\mathfrak{p},\mathfrak{c}}(\chi) = \mathcal{L}_{\mathfrak{p},\overline{\mathfrak{c}}}((\chi^\mathbf{c})^{-1}\mathbf{N}^{-1}),$$

where $\chi^\mathbf{c}$ is the composition of χ with the action of the non-trivial automorphism of $K/\mathbf{Q}$, and the equality is up to a p-adic unit.

Proof. See [Kat78], [HT93]; our formulation follows of the interpolation property follows [Hsi14, Prop. 4.19] most closely. The functional equation is shown in [dS87, Thm. II.6.4]. □

2.2. **Triple product p-adic L-function.** Let $\mathbb{I}$ be a normal domain finite flat over

$$\Lambda := \mathcal{O}[\![1 + p\mathbf{Z}_p]\!],$$

where $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbf{Q}_p$. For a positive integer N with $p \nmid N$ and a Dirichlet character $\chi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathcal{O}^\times$, we denote by $S^o(N, \chi, \mathbb{I}) \subset \mathbb{I}[\![q]\!]$ the space of ordinary $\mathbb{I}$-adic cusp forms of tame level N and branch character χ as defined in [Hsi21, §3.1].

Denote by $\mathfrak{X}_\mathbb{I}^+ \subset \operatorname{Spec} \mathbb{I}(\overline{\mathbf{Q}}_p)$ the set of *arithmetic points* of $\mathbb{I}$, consisting of the ring homomorphisms $Q : \mathbb{I} \to \overline{\mathbf{Q}}_p$ such that $Q|_{1+p\mathbf{Z}_p}$ is given by $z \mapsto z^{k_Q}\epsilon_Q(z)$ for some $k_Q \in \mathbf{Z}_{\geq 2}$ called the *weight of* Q and $\epsilon_Q(z) \in \mu_{p^\infty}$. As in [Hsi21, §3.1], we say that $\boldsymbol{f} = \sum_{n=1}^\infty a_n(\boldsymbol{f})q^n \in S^o(N, \chi, \mathbb{I})$ is a *primitive Hida family* if for every $Q \in \mathfrak{X}_\mathbb{I}^+$ the specialisation $\boldsymbol{f}_Q$ gives the q-expansion of an ordinary p-stabilised newform of weight of k_Q and tame conductor N. Attached to such $\boldsymbol{f}$ we let $\mathfrak{X}_\mathbb{I}^{\rm cls} \subset \mathfrak{X}_\mathbb{I}^+$ be the set of ring homomorphisms Q as above with $k_Q \in \mathbf{Z}$ such that $\boldsymbol{f}_Q$ is the q-expansion of a classical modular form.

For $\boldsymbol{f}$ a primitive Hida family, we let

$$\rho_{\boldsymbol{f}} : G_\mathbf{Q} \to \operatorname{Aut}_\mathbb{I}(V_{\boldsymbol{f}}) \simeq \mathrm{GL}_2(\mathbb{I})$$

denote the associated Galois representation, which here we take to be the *dual* of that in [Hsi21, §3.2]; in particular, the determinant of $\rho_{\boldsymbol{f}}$ is $\chi_\mathbb{I} \cdot \varepsilon_{\rm cyc}$ in the notations of *loc. cit.*, where $\varepsilon_{\rm cyc}$ is the

p-adic cyclotomic character. By [Wil88, Thm. 2.2.2], restricted to $G_{\mathbf{Q}_p}$ the Galois representation $V_{\boldsymbol{f}}$ fits into a short exact sequence

$$0 \to V_{\boldsymbol{f}}^{+} \to V_{\boldsymbol{f}} \to V_{\boldsymbol{f}}^{-} \to 0,$$

where the quotient $V_{\boldsymbol{f}}^{-}$ is free of rank one over $\mathbb{I}$, with the $G_{\mathbf{Q}_p}$-action given by the unramified character sending an arithmetic Frobenius Fr_p to $a_p(\boldsymbol{f})$.

Let

$$(\boldsymbol{\varphi}, \boldsymbol{g}, \boldsymbol{h}) \in S^o(N_\varphi, \chi_\varphi, \mathbb{I}_\varphi) \times S^o(N_g, \chi_g, \mathbb{I}_g) \times S^o(N_h, \chi_h, \mathbb{I}_h)$$

be a triple of primitive Hida families with

$$\chi_\varphi \chi_g \chi_h = \omega^{2a} \text{ for some } a \in \mathbf{Z}, \tag{2.1}$$

where ω is the Teichmüller character. Put

$$\mathcal{R} = \mathbb{I}_\varphi \hat{\otimes}_{\mathcal{O}} \mathbb{I}_g \hat{\otimes}_{\mathcal{O}} \mathbb{I}_h,$$

which is a finite extension of the three-variable Iwasawa algebra $\Lambda \hat{\otimes}_{\mathcal{O}} \Lambda \hat{\otimes}_{\mathcal{O}} \Lambda$, and let

$$\begin{aligned}\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{\varphi}} := \{(Q_0, Q_1, Q_2) \in \mathfrak{X}_{\mathbb{I}_\varphi}^{+} \times \mathfrak{X}_{\mathbb{I}_g}^{\mathrm{cls}} \times \mathfrak{X}_{\mathbb{I}_h}^{\mathrm{cls}} \\ : k_{Q_0} \geq k_{Q_1} + k_{Q_2} \text{ and } k_{Q_0} \equiv k_{Q_1} + k_{Q_2} \pmod 2\}\end{aligned}$$

be the weight space for $\mathcal{R}$ in the so-called $\boldsymbol{\varphi}$*-unbalanced range.*

Let $\mathbf{V} = V_{\boldsymbol{\varphi}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}$ be the triple tensor product Galois representation attached to $(\boldsymbol{\varphi}, \boldsymbol{g}, \boldsymbol{h})$, and writing $\det \mathbf{V} = \mathcal{X}^2 \varepsilon_{\mathrm{cyc}}$ define

$$\mathbf{V}^{\dagger} := \mathbf{V} \otimes \mathcal{X}^{-1}, \tag{2.2}$$

which is a self-dual twist of $\mathbf{V}$. Define the rank four $G_{\mathbf{Q}_p}$-invariant subspace $\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}^{\dagger}) \subset \mathbf{V}^{\dagger}$ by

$$\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}^{\dagger}) := V_{\boldsymbol{\varphi}}^{+} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}} \otimes \mathcal{X}^{-1}. \tag{2.3}$$

For every $\underline{Q} = (Q_0, Q_1, Q_2) \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{\varphi}}$ we denote by $\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}_{\underline{Q}}^{\dagger}) \subset \mathbf{V}_{\underline{Q}}^{\dagger}$ the corresponding specialisations. Finally, for every rational prime ℓ denote by $\varepsilon_\ell(\mathbf{V}_{\underline{Q}}^{\dagger})$ the epsilon factor attached to the restriction of $\mathbf{V}_{\underline{Q}}^{\dagger}$ to $G_{\mathbf{Q}_\ell}$ as in [Tat79, p. 21], and assume that

$$\text{for some } \underline{Q} \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{\varphi}}, \text{ we have } \varepsilon_\ell(\mathbf{V}_{\underline{Q}}^{\dagger}) = +1 \text{ for all prime factors } \ell \text{ of } N_{\boldsymbol{\varphi}} N_{\boldsymbol{g}} N_{\boldsymbol{h}}. \tag{2.4}$$

As explained in [Hsi21, §1.2], it is known that condition (2.4) is independent of $\underline{Q}$, and it implies that the sign in the functional equation for the triple product L-function

$$L(\mathbf{V}_{\underline{Q}}^{\dagger}, s)$$

(relating its values at s and $-s$) is $+1$ for all $\underline{Q} \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{\varphi}}$.

For the next statement, we refer the reader to §2.3 for a review of the congruence ideal associated with a primitive Hida family.

Theorem 2.2.1. *Let* $(\boldsymbol{\varphi}, \boldsymbol{g}, \boldsymbol{h})$ *be a triple of primitive Hida families as above satisfying conditions (2.1) and (2.4). Assume in addition that:*

- $\gcd(N_\varphi, N_g, N_h)$ *is square-free,*
- *the residual representation* $\bar{\rho}_{\boldsymbol{\varphi}}$ *is irreducible and* p*-distinguished,*

and fix a generator $\eta_{\boldsymbol{\varphi}}$ *of the congruence ideal of* $\boldsymbol{\varphi}$*. Then there exists a unique element*

$$\mathcal{L}_p^{\boldsymbol{\varphi}}(\boldsymbol{\varphi},\boldsymbol{g},\boldsymbol{h})\in\mathcal{R}$$

such that for all $\underline{Q}=(Q_0,Q_1,Q_2)\in\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{\varphi}}$ *of weight* (k_0,k_1,k_2) *with* $\epsilon_{Q_0}=1$ *we have*

$$(\mathcal{L}_p^{\boldsymbol{\varphi}}(\boldsymbol{\varphi},\boldsymbol{g},\boldsymbol{h})(\underline{Q}))^2=\Gamma_{\mathbf{V}_{\underline{Q}}^\dagger}(0)\cdot\frac{L(\mathbf{V}_{\underline{Q}}^\dagger,0)}{(\sqrt{-1})^{2k_0}\cdot\Omega_{\boldsymbol{\varphi}_{Q_0}}^2}\cdot\mathcal{E}_p(\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}_{\underline{Q}}^\dagger))\cdot\prod_{\ell\in\Sigma_{\rm exc}}(1+\ell^{-1})^2,$$

where:

- $\Gamma_{\mathbf{V}_{\underline{Q}}^\dagger}(0)=\Gamma_{\mathbf{C}}(c_{\underline{Q}})\Gamma_{\mathbf{C}}(c_{\underline{Q}}+2-k_1-k_2)\Gamma_{\mathbf{C}}(c_{\underline{Q}}+1-k_1)\Gamma_{\mathbf{C}}(c_{\underline{Q}}+1-k_2)$*, with*
$$c_{\underline{Q}}=(k_0+k_1+k_2-2)/2$$
and $\Gamma_{\mathbf{C}}(s)=2(2\pi)^{-s}\Gamma(s)$*;*
- $\Omega_{\boldsymbol{\varphi}_{Q_0}}$ *is the canonical period*
$$\Omega_{\boldsymbol{\varphi}_{Q_0}}:=(-2\sqrt{-1})^{k_0+1}\cdot\frac{\|\boldsymbol{\varphi}_{Q_0}^\circ\|^2_{\Gamma_0(N_\varphi)}}{\imath_p(\eta_{\boldsymbol{\varphi}_{Q_0}})}\cdot\Big(1-\frac{\chi_\varphi'(p)p^{k_0-1}}{\alpha_{Q_0}^2}\Big)\Big(1-\frac{\chi_\varphi'(p)p^{k_0-2}}{\alpha_{Q_0}^2}\Big),$$
with $\boldsymbol{\varphi}_{Q_0}^\circ\in S_{k_0}(N_\varphi)$ *the newform of conductor* N_φ *associated with* $\boldsymbol{\varphi}_{Q_0}$*,* χ_φ' *the prime-to-p part of* χ_φ*, and* α_{Q_0} *the specialisation of* $a_p(\boldsymbol{\varphi})\in\mathbb{I}_\varphi^\times$ *at* Q_0*;*
- $\mathcal{E}_p(\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}_{\underline{Q}}^\dagger))$ *is the modified p-Euler factor*
$$\mathcal{E}_p(\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}_{\underline{Q}}^\dagger)):=\frac{L_p(\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}_{\underline{Q}}^\dagger),0)}{\varepsilon_p(\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}_{\underline{Q}}^\dagger))\cdot L_p(\mathbf{V}_{\underline{Q}}^\dagger/\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}_{\underline{Q}}^\dagger),0)}\cdot\frac{1}{L_p(\mathbf{V}_{\underline{Q}}^\dagger,0)},$$

and $\Sigma_{\rm exc}$ *is an explicitly defined subset of the prime factors of* $N_\varphi N_g N_h$*,* [Hsi21, p. 416].

Proof. This is [Hsi21, Thm. A]. □

Remark 2.2.2. For simplicity, we have stated the interpolation property of $\mathcal{L}_p^{\boldsymbol{\varphi}}(\boldsymbol{\varphi},\boldsymbol{g},\boldsymbol{h})$ restricted to $\underline{Q}$ with $\epsilon_{Q_0}=1$, as this will suffice for our purposes; see [Hsi21, Thm. A] for the interpolation property for all $\underline{Q}\in\mathfrak{X}_{\mathcal{R}}^{\boldsymbol{\varphi}}$.

2.3. **Congruence ideal.** Let $\boldsymbol{f}\in S^o(N_f,\chi_f,\mathbb{I})$ be a primitive Hida family defined over $\mathbb{I}$. Associated with $\boldsymbol{f}$ there is a $\mathbb{I}$-algebra homomorphism

$$\lambda_{\boldsymbol{f}}:\mathbb{T}(N_f,\mathbb{I})\to\mathbb{I}$$

where $\mathbb{T}(N_f,\mathbb{I})$ is the Hecke algebra acting on $\oplus_\chi S^o(N_f,\chi,\mathbb{I})$, where χ runs over the characters of $(\mathbf{Z}/pN_f\mathbf{Z})^\times$. Let $\mathbb{T}_\mathfrak{m}$ be the local component of $\mathbb{T}(N_f,\mathbb{I})$ through which $\lambda_{\boldsymbol{f}}$ factors, and following [Hid88] define the *congruence ideal* $C(\boldsymbol{f})$ of $\boldsymbol{f}$ by

$$C(\boldsymbol{f}):=\lambda_{\boldsymbol{f}}(\mathrm{Ann}_{\mathbb{T}_\mathfrak{m}}(\ker\lambda_{\boldsymbol{f}}))\subset\mathbb{I}.$$

When the residual representation $\bar\rho_{\boldsymbol{f}}$ is irreducible and p-distinguished, it follows from the results of [Wil95] and [Hid88] that $C(\boldsymbol{f})$ is generated by a nonzero element $\eta_{\boldsymbol{f}}\in\mathbb{I}$.

3. Factorisation of p-adic triple product L-function

In this section we relate the triple product p-adic L-function attached to triples of forms with CM by K to a product of anticyclotomic Katz p-adic L-functions.

3.1. **Hida families with CM.** We review the construction of CM Hida families following the exposition in [Hsi21, §8.1]. Since it will suffice for our purposes, we assume that the class number $h_K = |\mathrm{Pic}(\mathcal{O}_K)|$ of K is coprime to p. Let K_∞ be the unique $\mathbf{Z}_p^2$-extension of K, and denote by $K_{\mathfrak{p}^\infty}$ the maximal subfield of K_∞ unramified outside $\mathfrak{p}$. Put

$$\Gamma_\infty := \mathrm{Gal}(K_\infty/K) \simeq \mathbf{Z}_p^2, \qquad \Gamma_{\mathfrak{p}^\infty} := \mathrm{Gal}(K_{\mathfrak{p}^\infty}/K) \simeq \mathbf{Z}_p.$$

For every integral ideal $\mathfrak{c} \subset \mathcal{O}_K$ we denote by $K(\mathfrak{c})$ the ray class field of K of conductor $\mathfrak{c}$. Thus $K_{\mathfrak{p}^\infty}$ is also the maximal $\mathbf{Z}_p$-extension inside $K(\mathfrak{p}^\infty)$. By our assumption on h_K, the restriction of the Artin map to $K_{\mathfrak{p}}^\times$ induces an isomorphism $1+p\mathbf{Z}_p \simeq \Gamma_{\mathfrak{p}^\infty}$, where we identified $\mathbf{Z}_p^\times$ and $\mathcal{O}_{K_{\mathfrak{p}}}^\times$ by the embedding ι_p. Denote by $\gamma_{\mathfrak{p}}$ the topological generator of $\Gamma_{\mathfrak{p}^\infty}$ corresponding to $1+p$ under this isomorphism, and for each variable S let $\Psi_S : \Gamma_\infty \to \mathbf{Z}_p[\![S]\!]^\times$ be the universal character given by

$$\Psi_S(\sigma) = (1+S)^{l(\sigma)},$$

where $l(\sigma) \in \mathbf{Z}_p$ is such that $\sigma|_{K_{\mathfrak{p}^\infty}} = \gamma_{\mathfrak{p}}^{l(\sigma)}$. Fix $\mathfrak{c}$ prime to p, and for any finite order character $\psi : G_K \to \mathcal{O}^\times$ of tame conductor $\mathfrak{c}$ put

$$\boldsymbol{\theta}_\psi(S)(q) = \sum_{(\mathfrak{a},p\mathfrak{c})=1} \psi(\sigma_{\mathfrak{a}})\Psi_S^{-1}(\sigma_{\mathfrak{a}})q^{\mathbf{N}(\mathfrak{a})} \in \mathcal{O}[\![S]\!][\![q]\!],$$

where $\sigma_{\mathfrak{a}} \in \mathrm{Gal}(K(\mathfrak{c}\mathfrak{p}^\infty)/K)$ is the Artin symbol of $\mathfrak{a}$. Then $\boldsymbol{\theta}_\psi(S)$ is a primitive Hida family defined over $\mathcal{O}[\![S]\!]$ of level $D_K\mathbf{N}(\mathfrak{c})$ and tame character $(\psi \circ \mathscr{V})\eta_{K/\mathbf{Q}}\omega^{-1}$, where $\mathscr{V} : G_{\mathbf{Q}}^{\mathrm{ab}} \to G_K^{\mathrm{ab}}$ is the transfer map and $\eta_{K/\mathbf{Q}}$ is the quadratic character associated to $K/\mathbf{Q}$.

3.2. **Setting.** Let $E_{/\mathbf{Q}}$ be an elliptic curve of conductor N with CM by the maximal order $\mathcal{O}_K$. Assume that the prime $p > 2$ is a prime of good *ordinary* reduction for E, so in particular (spl) holds, and note that $h_K = 1$. Let λ be the Grossencharacter of K associated to E by the theory of complex multiplication, so we have

$$L(E,s) = L(\lambda,s).$$

We now introduce a triple

$$(\varphi, \boldsymbol{g}, \boldsymbol{h})$$

attached to certain auxiliary choices. Let ℓ, ℓ' be *distinct* primes split in K with $(\ell\ell', Np) = 1$. Take Φ (resp. Ξ) a ring class character of conductor $\ell^m\mathcal{O}_K$ (resp. $(\ell')^n\mathcal{O}_K$) for some $m, n > 0$, and write

$$\Phi = \phi^{1-\mathbf{c}}, \qquad \Xi = \xi^{1-\mathbf{c}}$$

with ϕ (resp. ξ) a ray class character modulo $\ell^m\mathcal{O}_K$ (resp. $q^n\mathcal{O}_K$), where $\mathbf{c}$ is the non-trivial automorphism of $K/\mathbf{Q}$ (see [Hid06b, Lemma 5.31]). As usual, here we write $\phi^{\mathbf{c}}$ for the composition of ϕ with $\mathbf{c}$, and write $\phi^{1-\mathbf{c}}$ (and similarly $\xi^{1-\mathbf{c}}$) to denote the product $\phi(\phi^{\mathbf{c}})^{-1}$. Consider the primitive CM Hida families

$$\boldsymbol{g} = \boldsymbol{\theta}_\phi(S_1), \qquad \boldsymbol{h} = \boldsymbol{\theta}_\xi(S_2)$$

of level $D_K\ell^{2m}$ and $D_K(\ell')^{2n}$, respectively. On the other hand, put $\psi = (\phi^{\mathbf{c}}\xi^{\mathbf{c}})^{-1}$ and let

$$\varphi := \theta(\lambda\psi) \tag{3.1}$$

be the theta series of weight 2 associated to $\lambda\psi$.

Note that

$$L(E,s) = L(\overline{\lambda},s) = L(\lambda^{-1},s-1),$$

since E is defined over $\mathbf{Q}$. The following immediate consequence of the nonvanishing results due to Greenberg and Rohrlich will be a key ingredient in the proof of our main result.

Theorem 3.2.1. *There exists ring class characters Φ and Ξ as above such that:*

(1) *Both Φ and Ξ have order prime to p,*
(2) *The restrictions $\phi|_{G_{\mathbf{Q}_p}}$ and $\xi|_{G_{\mathbf{Q}_p}}$ are both non-trivial,*
(3) *Both ϕ and ξ have order at 3,*

and the following nonvanishing condition holds:

$$L(\lambda^{-1}\Phi^{-1},0)\cdot L(\lambda^{-1}\Xi^{-1},0)\cdot L(\lambda^{-1},\Phi^{-1}\Xi^{-1},0)\neq 0.$$

Proof. Since conditions (1)–(3) exclude only finitely many ring class characters, this follows from the nonvanishing results of [Gre85] and [Roh84]. □

Choice 3.2.2. From now on, we fix a pair of ring class characters Φ and Ξ satisfying the conditions of Theorem 3.2.1, and let $(\varphi,\boldsymbol{g},\boldsymbol{h})$ be the resulting triple as defined above.

For the construction of nontrivial Selmer classes in the Selmer group of E, we will be interested in the Iwasawa main conjecture and the p-adic L-function for the triple $(\varphi,\boldsymbol{g},\boldsymbol{h})$.

3.3. **Congruence ideal of CM Hida families.** Denote by Γ^- the maximal $\mathbf{Z}_p$-free quotient of $Z^-(\mathfrak{c}):=Z(\mathfrak{c})/Z(\mathfrak{c})^{1+\mathbf{c}}$, and let

$$\pi_-:Z(\mathfrak{c})\to\Gamma^-$$

be the natural projection. For a Hecke character η of conductor dividing $\mathfrak{c}p^\infty$, we denote by $\mathcal{L}^-_{\mathfrak{p},\eta}$ the image of $\mathcal{L}_{\mathfrak{p},\mathfrak{c}}$ under the map $\mathcal{W}[\![Z(\mathfrak{c})]\!]\to\mathcal{W}[\![\Gamma^-]\!]$ given by $\gamma\mapsto\eta(\gamma)\pi_-(\gamma)$ for $\gamma\in Z(\mathfrak{c})$. Here, with a slight abuse of notation, we enlarge $\mathcal{W}$ if necessary so that is contains the values of η, but we continue to denote it by $\mathcal{W}$.

Lemma 3.3.1. *Let $\boldsymbol{\varphi}$ be the primitive CM Hida passing through the ordinary p-stabilisation of φ in (3.1). Then the congruence ideal $C(\boldsymbol{\varphi})$ is generated by*

$$\frac{h_K}{w_K}\cdot\mathcal{L}^-_{\mathfrak{p},\lambda^{\mathbf{c}-1}\psi^{\mathbf{c}-1}},$$

where $h_K=|\mathrm{Pic}(\mathcal{O}_K)|$ and $w_K=|\mathcal{O}_K^\times|$.

Proof. A generator of $C(\boldsymbol{\varphi})$ is given by a congruence power series $H(\boldsymbol{\varphi})$ attached to $\boldsymbol{\varphi}$ as in [Hid06a]. By our Choice 3.2.2 of Φ and Ξ, the $H(\boldsymbol{\varphi})$ corresponds to a branch character satisfying the hypotheses (1)–(4) in [Hid06a, p. 466], so as noted in p. 469 of *op. cit*, the result follows from the proof of the anticyclotomic Iwasawa main conjecture by Hida–Tilouine [HT93, HT94] and Hida [Hid06a]. □

3.4. **Proof of the factorisation.** Let $\boldsymbol{\varphi}\in\mathcal{O}[\![S_0]\!][\![q]\!]$ be the primitive CM Hida family associated to the character φ in (3.1), and let $Q_0\in\mathfrak{X}^+_{\mathcal{O}[\![S_0]\!]}$ be such that $\boldsymbol{\varphi}_{Q_0}$ is the ordinary p-stabilisation of φ. Letting

$$\mathcal{R}\simeq\mathcal{O}[\![S_0]\!]\hat{\otimes}_{\mathcal{O}}\mathcal{O}[\![S_1,S_2]\!]\to\mathcal{O}[\![S_1,S_2]\!]\tag{3.2}$$

be the specialisation map at Q_0, in the followiing we denote by

$$\mathscr{L}_p^{\varphi}(\varphi,\boldsymbol{g},\boldsymbol{h})(S_1,S_2)\in\mathcal{O}[\![S_1,S_2]\!]$$

the image of the triple product p-adic L-function $\mathscr{L}_p^{\boldsymbol{\varphi}}(\boldsymbol{\varphi},\boldsymbol{g},\boldsymbol{h})\in\mathcal{R}$ of Theorem 2.2.1 under this map. Upon the choice of a topological generator $\gamma^-\in\Gamma^-$, as usual we shall identify $\mathcal{W}[\![\Gamma^-]\!]$ with the power series ring $\mathcal{W}[\![W]\!]$ via $\gamma^-\mapsto 1+W$. Denote by $\lambda\mapsto\lambda^\iota$ the involution of $\mathcal{W}[\![\Gamma^-]\!]$ given by $\gamma^-\mapsto(\gamma^-)^{-1}$.

Proposition 3.4.1. *Put*

$$W_1 = (1+S_1)^{1/2}(1+S_2)^{1/2} - 1, \quad W_2 = (1+S_1)^{1/2}(1+S_2)^{-1/2} - 1.$$

Then we have the factorisation

$$\begin{aligned}\mathscr{L}_p^{\varphi}(\varphi, \boldsymbol{g}, \boldsymbol{h})^2(S_1, S_2) = u \cdot \mathcal{L}^-_{\mathfrak{p},\lambda^{-1}\psi^{\mathbf{c}-1}}(W_1)^{\iota} \cdot \mathcal{L}^-_{\mathfrak{p},\lambda^{-1}}(W_1) \\ \times \mathcal{L}^-_{\mathfrak{p},\lambda^{-1}\phi^{\mathbf{c}-1}}(W_2)^{\iota} \cdot \mathcal{L}^-_{\mathfrak{p},\lambda^{-1}\xi^{\mathbf{c}-1}}(W_2),\end{aligned}$$

where $u \in \mathcal{W}^\times$ is a p-adic unit.

Proof. Denote by $\mathbf{V}^\dagger_{Q_0}$ the image of $\mathbf{V}^\dagger$ under the specialisation map (3.2). Then from (2.2) and the definition of φ we see that

$$\mathbf{V}^\dagger_{Q_0} = \mathrm{Ind}_K^{\mathbf{Q}}(\lambda^{-1}\psi^{-1}) \otimes \mathrm{Ind}_K^{\mathbf{Q}}(\phi^{-1}\Psi_{S_1}) \otimes \mathrm{Ind}_K^{\mathbf{Q}}(\xi^{-1}\Psi_{S_2}) \otimes \mathcal{X}^{-1},$$

where $\mathcal{X} = (\Psi_{S_1}^{1/2}\Psi_{S_2}^{1/2} \circ \mathscr{V}) : G_{\mathbf{Q}} \to \mathbf{Z}_p[\![S_1, S_2]\!]^\times$, and so an immediate computation shows that

$$\begin{aligned}\mathbf{V}^\dagger_{Q_0} \simeq \mathrm{Ind}_K^{\mathbf{Q}}(\lambda^{-1}\psi^{\mathbf{c}-1}\Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{Ind}_K^{\mathbf{Q}}(\lambda^{-1}\Psi_{W_1}^{\mathbf{c}-1}) \\ \oplus \mathrm{Ind}_K^{\mathbf{Q}}(\lambda^{-1}\phi^{\mathbf{c}-1}\Psi_{W_2}^{1-\mathbf{c}}) \oplus \mathrm{Ind}_K^{\mathbf{Q}}(\lambda^{-1}\xi^{\mathbf{c}-1}\Psi_{W_2}^{\mathbf{c}-1}).\end{aligned} \tag{3.3}$$

For $i = 1, 2$, let ζ_i be a primitive p^{n_i}-th root of unity with $n_i > 0$, and put $Q_1 = \zeta_1\zeta_2 - 1$, $Q_2 = \zeta_1\zeta_2^{-1} - 1$, so the specialisations $\boldsymbol{g}_{Q_1}, \boldsymbol{h}_{Q_2}$ are both CM forms of weight 1. Let $\epsilon_i : \Gamma^- \to \mu_{p^\infty}$ be the finite order character given by $\epsilon_i(\gamma^-) = \zeta_i$. Letting $\underline{Q} = (Q_0, Q_1, Q_2)$, we thus see that

$$\begin{aligned}\mathbf{V}^\dagger_{\underline{Q}} &= \mathrm{Ind}_K^{\mathbf{Q}}(\lambda^{-1}\psi^{\mathbf{c}-1}\epsilon_1^{-1}) \oplus \mathrm{Ind}_K^{\mathbf{Q}}(\lambda^{-1}\epsilon_1) \\ &\quad \oplus \mathrm{Ind}_K^{\mathbf{Q}}(\lambda^{-1}\phi^{\mathbf{c}-1}\epsilon_2^{-1}) \oplus \mathrm{Ind}_K^{\mathbf{Q}}(\lambda^{-1}\xi^{\mathbf{c}-1}\epsilon_2), \\ \mathscr{F}_p^{\varphi}(\mathbf{V}^\dagger_{\underline{Q}}) &= \lambda_{\mathfrak{p}}^{-1}\psi_{\mathfrak{p}}^{\mathbf{c}-1}\epsilon_{1,\mathfrak{p}}^{-1} \oplus \lambda_{\mathfrak{p}}^{-1}\epsilon_{1,\mathfrak{p}} \oplus \lambda_{\mathfrak{p}}^{-1}\phi_{\mathfrak{p}}^{\mathbf{c}-1}\epsilon_{2,\mathfrak{p}}^{-1} \oplus \lambda_{\mathfrak{p}}^{-1}\xi_{\mathfrak{p}}^{\mathbf{c}-1}\epsilon_{2,\mathfrak{p}}.\end{aligned}$$

Thus the terms appearing in the interpolation formula of Theorem 2.2.1 become:

$$\begin{aligned}\Gamma_{\mathbf{V}^\dagger_{\underline{Q}}}(0) \cdot L(\mathbf{V}^\dagger_{\underline{Q}}, 0) = \pi^{-4} \cdot L(\lambda^{-1}\psi^{\mathbf{c}-1}\epsilon_1^{-1}, 0) \cdot L(\lambda^{-1}\epsilon_1, 0) \\ \times L(\lambda^{-1}\phi^{\mathbf{c}-1}\epsilon_2^{-1}, 0) \cdot L(\lambda^{-1}\xi^{\mathbf{c}-1}\epsilon_2, 0);\end{aligned} \tag{3.4}$$

$$\mathcal{E}_p(\mathscr{F}_p^{\varphi}(\mathbf{V}^\dagger_{\underline{Q}})) = \mathcal{E}_{\mathfrak{p}}(\lambda^{-1}\psi^{\mathbf{c}-1}\epsilon_1^{-1}) \cdot \mathcal{E}_{\mathfrak{p}}(\lambda^{-1}\epsilon_1) \cdot \mathcal{E}_{\mathfrak{p}}(\lambda^{-1}\phi^{\mathbf{c}-1}\epsilon_2^{-1}) \cdot \mathcal{E}_{\mathfrak{p}}(\lambda^{-1}\xi^{\mathbf{c}-1}\epsilon_2); \tag{3.5}$$

$$\Omega_{\boldsymbol{\varphi}_{Q_0}} = (-2\sqrt{-1})^3 \cdot \frac{\|\varphi\|^2_{\Gamma_0(N_\varphi)}}{\imath_p(\eta_{\boldsymbol{\varphi}_{Q_0}})} \cdot \left(1 - \frac{\lambda\psi(\mathfrak{p})}{\lambda\psi(\bar{\mathfrak{p}})}\right)\left(1 - \frac{\lambda\psi(\mathfrak{p})}{p\lambda\psi(\bar{\mathfrak{p}})}\right); \tag{3.6}$$

and we note that $\Sigma_{\mathrm{exc}} = \emptyset$ in this case.

On the other hand, from Hida's formula for the adjoint L-value [HT93, Thm. 7.1] and Dirichlet's class number formula we obtain

$$\|\varphi\|^2_{\Gamma_0(N_\varphi)} = \frac{D_K^2}{2^4\pi^3} \cdot \frac{2\pi h_K}{w_K\sqrt{D_K}} \cdot L(\lambda^{1-\mathbf{c}}\psi^{1-\mathbf{c}}, 1).$$

Noting that $L(\lambda^{1-\mathbf{c}}\psi^{1-\mathbf{c}},0)=L(\lambda^{\mathbf{c}-1}\psi^{\mathbf{c}-1}\mathbf{N}^{-1},0)$ and $\lambda^{\mathbf{c}-1}\psi^{\mathbf{c}-1}\mathbf{N}^{-1}$ has infinity type $(2,0)$, the interpolation property in Theorem 2.1.1 can thus be rewritten as

$$\begin{aligned}\mathcal{L}_{\mathfrak{p},\mathfrak{c}\ell\ell'}(\lambda^{\mathbf{c}-1}\psi^{\mathbf{c}-1}\mathbf{N}^{-1}) &= \left(\frac{\Omega_p}{\Omega_\infty}\right)^2\cdot\frac{2^3\pi^2}{\sqrt{D_K}^3}\\ &\quad\times\left(1-\frac{\lambda\psi(\mathfrak{p})}{\lambda\psi(\overline{\mathfrak{p}})}\right)\left(1-\frac{\lambda\psi(\mathfrak{p})}{p\lambda\psi(\overline{\mathfrak{p}})}\right)\cdot\frac{w_K}{h_K}\cdot\|\varphi\|^2_{\Gamma_0(N_\varphi)}\\ &= -\left(\frac{\Omega_p}{\Omega_\infty}\right)^2\cdot\frac{\pi^2\sqrt{-1}}{\sqrt{D_K}^3}\cdot\Omega_{\boldsymbol{\varphi}_{Q_0}}\cdot\eta_{\boldsymbol{\varphi}_{Q_0}}\cdot\frac{w_K}{h_K},\end{aligned}$$

using (3.6) for the second equality, and where $\mathfrak{c}$ denotes the conductor of $\lambda^{\mathbf{c}-1}$. By the functional equation for Katz's p-adic L-function (see Theorem 2.1.1) together with Lemma 3.3.1 this shows that

$$\frac{1}{\Omega^2_{\boldsymbol{\varphi}_{Q_0}}}=\left(\frac{\Omega_p}{\Omega_\infty}\right)^4\cdot\frac{\pi^4}{D_K^3},\tag{3.7}$$

where the equality is up to a p-adic unit independent of Q_1,Q_2. Thus substituting (3.4), (3.5), (3.6) and (3.7) into the interpolation formula for $\mathscr{L}_p^{\varphi}(\varphi,\boldsymbol{g},\boldsymbol{h})$ we thus arrive at

$$\begin{aligned}\mathscr{L}_p^{\varphi}(\varphi,\boldsymbol{g},\boldsymbol{h})^2(\zeta_1\zeta_2-1,\zeta_1\zeta_2^{-1}-1) &= D_K^{-3}\cdot\mathcal{L}^-_{\mathfrak{p},\lambda^{-1}\psi^{\mathbf{c}-1}}(\zeta_1^{-1}-1)\cdot\mathcal{L}^-_{\mathfrak{p},\lambda^{-1}}(\zeta_1-1)\\ &\quad\times\mathcal{L}^-_{\mathfrak{p},\lambda^{-1}\phi^{\mathbf{c}-1}}(\zeta_2^{-1}-1)\cdot\mathcal{L}^-_{\mathfrak{p},\lambda^{-1}\xi^{\mathbf{c}-1}}(\zeta_2-1),\end{aligned}$$

for all non-trivial p-power roots of unity ζ_1,ζ_2, and the result follows. □

4. Selmer group decompositions

In this section we define different Selmer groups attached to Hecke characters and triple products of modular forms. Then, for the so-called unbalanced Selmer groups attached to the triple product, we prove a decomposition mirroring the factorisation in Proposition 4.3.1.

4.1. **Selmer groups for Hecke characters.** Let ν be a Hecke character of K with values in the ring of integers $\mathcal{O}$ of a finite extension Φ of $\mathbf{Q}_p$. Denote by $\mathcal{O}_\nu$ the free $\mathcal{O}$-module of rank 1 on which G_K acts via ν, and put

$$T_\nu=\mathcal{O}_\nu,\quad V_\nu=T_\nu\otimes_{\mathcal{O}}\Phi,\quad A_\nu=V_\nu/T_\nu=T_\nu\otimes_{\mathcal{O}}(\Phi/\mathcal{O}).$$

Let Σ be any finite set of places of K containing ∞ and the primes dividing p or the conductor of ν, and for any finite extension F/K denote by $G_{F,\Sigma}$ the Galois group of the maximal extension of F unramified outside the places above Σ.

Definition 4.1.1. Let F/K be a finite extension, and for $v|p$ a prime of F above p put

$$\mathrm{H}^1_\emptyset(F_v,V_\nu)=\mathrm{H}^1(F_v,V_\nu),\quad \mathrm{H}^1_0(F_v,V_\nu)=\{0\}.$$

For $(\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}})\in\{\emptyset,0\}^{\oplus 2}$ define the Selmer group $\mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(F,V_\nu)$ by

$$\begin{aligned}\mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(F,V_\nu)=\ker\biggl(\mathrm{H}^1(G_{F,\Sigma},V_\nu)\rightarrow\prod_{v|\mathfrak{p}}\frac{\mathrm{H}^1(F_v,V_\nu)}{\mathrm{H}^1_{\mathcal{L}_\mathfrak{p}}(F_v,V_\nu)}\times\prod_{v|\overline{\mathfrak{p}}}\frac{\mathrm{H}^1(F_v,V_\nu)}{\mathrm{H}^1_{\mathcal{L}_{\overline{\mathfrak{p}}}}(F_v,V_\nu)}\\ \times\prod_{v\in\Sigma,v\nmid p\infty}\mathrm{H}^1(F_v,V_\nu)\biggr).\end{aligned}$$

Remark 4.1.2. In particular, if ν has infinity type $(-1,0)$, the Selmer group

$$\mathrm{Sel}_{\emptyset,0}(F,V_\nu)=\ker\Big(\mathrm{H}^1(G_{F,\Sigma},V_\nu)\to\prod_{v\mid\overline{\mathfrak{p}}}\mathrm{H}^1(F_v,V_\nu)\times\prod_{v\in\Sigma,v\nmid p\infty}\mathrm{H}^1(F_v,V_\nu)\Big)$$

agrees with the Bloch–Kato Selmer group of V_ν (see e.g. [AH06, §1.1] or [Arn07, §1.2]). Similarly, if ν has infinity type $(0,-1)$ then $\mathrm{Sel}_{0,\emptyset}(F,V_\nu)$ agrees with the Bloch–Kato Selmer group of V_ν.

For $?\in\{\emptyset,0\}$, the local condition $\mathrm{H}^1_?(F_v,T_\nu)$ and $\mathrm{H}^1_?(F_v,A_\nu)$ are defined from the above by propagation, and using these we define $\mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(F,T_\nu)$ and $\mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(F,A_\nu)$ by the same recipe as before. Then, letting K_∞/K denote the anticyclotomic $\mathbf{Z}_p$-extension of K, we set

$$\begin{aligned}\mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(K_\infty,T_\nu)&=\varprojlim_n\mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(K_n,T_\nu),\\ \mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(K_\infty,A_\nu)&=\varinjlim_n\mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(K_n,A_\nu).\end{aligned}\tag{4.1}$$

Let $\Lambda=\mathcal{O}[\![\Gamma^-]\!]\simeq\mathcal{O}[\![W]\!]$ be the anticyclotomic Iwasawa algebra, and denote by $\Psi_W:G_K\to\Lambda^\times$ the universal character given $g\mapsto[g|_{K_\infty}]$, where $\gamma\mapsto[\gamma]$ is the inclusion of Γ^- intro Λ^- as a group-like element. Then we have isomorphisms

$$\mathrm{H}^1(K,T_\nu\otimes\Psi_W^{-1})\simeq\varprojlim_n\mathrm{H}^1(K_n,T_\nu),\quad \mathrm{H}^1(K,A_\nu\otimes\Psi_W)\simeq\varinjlim_n\mathrm{H}^1(K_n,A_\nu)$$

by Shapiro's lemma. In the following we let $\mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(K,T_\nu\otimes\Psi_W^{-1})$ and $\mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(K,A_\nu\otimes\Psi_W)$ be the Selmer groups corresponding to (4.1) under these isomorphism, and let

$$X_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(K,A_\nu\otimes\Psi_W)=\mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(K,A_\nu\otimes\Psi_W),\mathbf{Q}_p/\mathbf{Z}_p)$$

denote the Pontryagin dual of $\mathrm{Sel}_{\mathcal{L}_\mathfrak{p},\mathcal{L}_{\overline{\mathfrak{p}}}}(K,A_\nu\otimes\Psi_W)$.

4.2. **Selmer groups for triple products.** Let $(\varphi,\boldsymbol{g},\boldsymbol{h})$ be a triple as in Choice 3.2.2. As in the proof of Proposition 3.4.1, suppose φ arising from the specialisation of $\boldsymbol{\varphi}$ at an arithmetic point Q_0, and denote by $\mathbf{V}^\dagger$ the triple product Galois representation associated to $(\boldsymbol{\varphi},\boldsymbol{g},\boldsymbol{h})$ as in (2.2).

Definition 4.2.1. Put

$$\mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger)=\mathscr{F}_p^2(\mathbf{V}^\dagger):=\big(V_{\boldsymbol{\varphi}}^+\otimes V_{\boldsymbol{g}}^+\otimes V_{\boldsymbol{h}}+V_{\boldsymbol{\varphi}}^+\otimes V_{\boldsymbol{g}}\otimes V_{\boldsymbol{h}}^++V_{\boldsymbol{\varphi}}\otimes V_{\boldsymbol{g}}^+\otimes V_{\boldsymbol{h}}^+\big)\otimes\mathcal{X}^{-1}$$

and define the *balanced local condition* $\mathrm{H}^1_{\mathrm{bal}}(\mathbf{Q}_p,\mathbf{V}^\dagger)$ by

$$\mathrm{H}^1_{\mathrm{bal}}(\mathbf{Q}_p,\mathbf{V}^\dagger):=\mathrm{im}\big(\mathrm{H}^1(\mathbf{Q}_p,\mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger))\to\mathrm{H}^1(\mathbf{Q}_p,\mathbf{V}^\dagger)\big).$$

Similarly, put $\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}^\dagger):=\big(V_{\boldsymbol{\varphi}}^+\otimes V_{\boldsymbol{g}}\otimes V_{\boldsymbol{h}}\big)\otimes\mathcal{X}^{-1}$ and define the *$\boldsymbol{\varphi}$-unbalanced local condition* $\mathrm{H}^1_{\varphi}(\mathbf{Q}_p,\mathbf{V}^\dagger)$ by

$$\mathrm{H}^1_{\boldsymbol{\varphi}}(\mathbf{Q}_p,\mathbf{V}^\dagger):=\mathrm{im}\big(\mathrm{H}^1(\mathbf{Q}_p,\mathscr{F}_p^{\boldsymbol{\varphi}}(\mathbf{V}^\dagger))\to\mathrm{H}^1(\mathbf{Q}_p,\mathbf{V}^\dagger)\big).$$

It is easy to see that the maps appearing in these definitions are injective, and in the following we shall use this to identify $\mathrm{H}^1_?(\mathbf{Q}_p,\mathbf{V}^\dagger)$ with $\mathrm{H}^1(\mathbf{Q}_p,\mathscr{F}_p^?(\mathbf{V}^\dagger))$ for $?\in\{\mathrm{bal},\boldsymbol{\varphi}\}$.

Definition 4.2.2. Let $?\in\{\mathrm{bal},\boldsymbol{\varphi}\}$, and define the Selmer group $\mathrm{Sel}^?(\mathbf{Q},\mathbf{V}^\dagger)$ by

$$\mathrm{Sel}^?(\mathbf{Q},\mathbf{V}^\dagger):=\ker\Big\{\mathrm{H}^1(\mathbf{Q},\mathbf{V}^\dagger)\to\frac{\mathrm{H}^1(\mathbf{Q}_p,\mathbf{V}^\dagger)}{\mathrm{H}^1_?(\mathbf{Q}_p,\mathbf{V}^\dagger)}\times\prod_{v\neq p}\mathrm{H}^1(\mathbf{Q}_v^{\mathrm{nr}},\mathbf{V}^\dagger)\Big\}.$$

We call $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbf{V}^\dagger)$ (resp. $\mathrm{Sel}^{\boldsymbol{\varphi}}(\mathbf{Q},\mathbf{V}^\dagger)$) the *balanced* (resp. *$\boldsymbol{\varphi}$-unbalanced*) Selmer group.

Let $\mathbf{A}^\dagger = \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{V}^\dagger, \mu_{p^\infty})$ and for $? \in \{\mathrm{bal}, \boldsymbol{\varphi}\}$ define $\mathrm{H}^1_?(\mathbf{Q}_p, \mathbf{A}^\dagger) \subset \mathrm{H}^1(\mathbf{Q}_p, \mathbf{A}^\dagger)$ to be the orthogonal complement of $\mathrm{H}^1_?(\mathbf{Q}_p, \mathbf{V}^\dagger)$ under the local Tate duality

$$\mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}^\dagger) \times \mathrm{H}^1(\mathbf{Q}_p, \mathbf{A}^\dagger) \to \mathbf{Q}_p/\mathbf{Z}_p.$$

Similarly as above, we then define the balanced and $\boldsymbol{\varphi}$-unbalanced Selmer groups with coefficients in $\mathbf{A}^\dagger$ by

$$\mathrm{Sel}^?(\mathbf{Q}, \mathbf{A}^\dagger) := \ker\biggl\{\mathrm{H}^1(\mathbf{Q}, \mathbf{A}^\dagger) \to \frac{\mathrm{H}^1(\mathbf{Q}_p, \mathbf{A}^\dagger)}{\mathrm{H}^1_?(\mathbf{Q}_p, \mathbf{A}^\dagger)} \times \prod_{v \neq p} \mathrm{H}^1(\mathbf{Q}_v^{\mathrm{nr}}, \mathbf{A}^\dagger)\biggr\},$$

and let $X^?(\mathbf{Q}, \mathbf{A}^\dagger) = \mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Sel}^?(\mathbf{Q}, \mathbf{A}^\dagger), \mathbf{Q}_p/\mathbf{Z}_p)$ denote the Pontryagin dual of $\mathrm{Sel}^?(\mathbf{Q}, \mathbf{A}^\dagger)$.

4.3. **Proof of the decompositions.** Denoting by $\mathbf{V}^\dagger_{Q_0}$ the image of $\mathbf{V}^\dagger$ under the specialisation map (3.2) at Q_0, we have an isomorphism

$$\begin{aligned} \mathrm{H}^1(\mathbf{Q}, \mathbf{V}^\dagger_{Q_0}) \simeq{} & \mathrm{H}^1(K, T_{\lambda^{-1}\psi^{\mathbf{c}-1}} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{H}^1(K, T_{\lambda^{-1}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) \\ & \oplus \mathrm{H}^1(K, T_{\lambda^{-1}\phi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus \mathrm{H}^1(K, T_{\lambda^{-1}\xi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{\mathbf{c}-1}) \end{aligned} \tag{4.2}$$

arising from (3.3) and Shapiro's lemma.

Proposition 4.3.1. *Under (4.2), the balanced Selmer group* $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}^\dagger_{Q_0})$ *decomposes as*

$$\begin{aligned} \mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}^\dagger_{Q_0}) \simeq{} & \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda^{-1}\psi^{\mathbf{c}-1}} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{Sel}_{0,\emptyset}(K, T_{\lambda^{-1}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) \\ & \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda^{-1}\phi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda^{-1}\xi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{\mathbf{c}-1}), \end{aligned}$$

and the φ-unbalanced Selmer group $\mathrm{Sel}^{\varphi}(K, \mathbf{V}^\dagger_{Q_0})$ *decomposes as*

$$\begin{aligned} \mathrm{Sel}^{\varphi}(\mathbf{Q}, \mathbf{V}^\dagger_{Q_0}) \simeq{} & \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda^{-1}\psi^{\mathbf{c}-1}} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda^{-1}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) \\ & \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda^{-1}\phi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda^{-1}\xi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{\mathbf{c}-1}). \end{aligned}$$

The proof will follow easily from the following lemma.

Lemma 4.3.2. *Under (4.2), the Selmer group* $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}^\dagger_{Q_0})$ *corresponds to the submodule of*

$$\mathrm{H}^1(K, (T_{\lambda^{-1}\psi^{\mathbf{c}-1}} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus (T_{\lambda^{-1}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) \oplus (T_{\lambda^{-1}\phi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus (T_{\lambda^{-1}\xi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{\mathbf{c}-1}))$$

consisting of unramified-outside-p classes x with $\mathrm{res}_v(x)$ belonging to

$$\begin{cases} \mathrm{H}^1(K_{\mathfrak{p}}, (T_{\lambda^{-1}\psi^{\mathbf{c}-1}} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus (T_{\lambda^{-1}\phi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus (T_{\lambda^{-1}\xi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{\mathbf{c}-1})) & \text{if } v = \mathfrak{p}, \\ \mathrm{H}^1(K_{\overline{\mathfrak{p}}}, T_{\lambda^{-1}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) & \text{if } v = \overline{\mathfrak{p}}, \end{cases}$$

and the Selmer group $\mathrm{Sel}^{\varphi}(\mathbf{Q}, \mathbf{V}^\dagger_{Q_0})$ *similarly corresponds to the submodule consisting of unramified-outside-p classes x with $\mathrm{res}_{\overline{\mathfrak{p}}}(x) = 0$ (and no condition at $\mathfrak{p}$).*

Proof. Using (3.3) we see that the balanced local condition is given by

$$\begin{aligned} \mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger_{Q_0}) ={} & (T_{\lambda^{-1}\psi^{\mathbf{c}-1}} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus (T_{\lambda^{-\mathbf{c}}} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \\ & \oplus (T_{\lambda^{-1}\phi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus (T_{\lambda^{-1}\xi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{\mathbf{c}-1}), \end{aligned}$$

from where we obtain

$$\begin{aligned} & \mathscr{F}_{\mathfrak{p}}^{\mathrm{bal}}(\mathbf{V}^\dagger_{Q_0}) = (T_{\lambda^{-1}\psi^{\mathbf{c}-1}} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus (T_{\lambda^{-1}\phi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus (T_{\lambda^{-1}\xi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{\mathbf{c}-1}), \\ & \mathscr{F}_{\overline{\mathfrak{p}}}^{\mathrm{bal}}(\mathbf{V}^\dagger_{Q_0}) = T_{\lambda^{-1}} \otimes \Psi_{W_1}^{\mathbf{c}-1}, \end{aligned} \tag{4.3}$$

yielding the stated descriptions of $\mathrm{Sel}^{\mathrm{bal}}(K, \mathbf{V}^\dagger_{Q_0})$. Similarly, we see that the φ-unbalanced local condition is given by

$$\begin{aligned}\mathscr{F}_p^\varphi(\mathbf{V}^\dagger_{Q_0}) = (T_{\lambda^{-1}\psi^{\mathbf{c}-1}} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus (T_{\lambda^{-1}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) \\ \oplus (T_{\lambda^{-1}\phi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus (T_{\lambda^{-1}\xi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{\mathbf{c}-1}),\end{aligned}$$

and this immediately yields the stated description of $\mathrm{Sel}^\varphi(\mathbf{Q}, \mathbf{V}^\dagger)$. □

Proof of Proposition 4.3.1. Put

$$\begin{aligned}\widetilde{\mathbf{V}}^\dagger_{Q_0} := (T_{\lambda^{-1}\psi^{\mathbf{c}-1}} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus (T_{\lambda^{-1}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) \\ \oplus (T_{\lambda^{-1}\phi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus (T_{\lambda^{-1}\xi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{\mathbf{c}-1}),\end{aligned} \tag{4.4}$$

so we have

$$\mathrm{H}^1(\mathbf{Q}, \mathbf{V}^\dagger_{Q_0}) \simeq \mathrm{H}^1(K, \widetilde{\mathbf{V}}^\dagger_{Q_0}).$$

Denoting by $\mathrm{Sel}^{\{p\}}(K, \widetilde{\mathbf{V}}^\dagger_{Q_0})$ the submodule of $\mathrm{H}^1(K, \widetilde{\mathbf{V}}^\dagger_{Q_0})$ consisting of unramified-outside-p classes, it follows from Lemma 4.3.2 that under the above isomorphism the balanced Selmer group $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}^\dagger_{Q_0})$ corresponds to the kernel of the restriction map from $\mathrm{Sel}^{\{p\}}(K, \widetilde{\mathbf{V}}^\dagger_{Q_0})$ to

$$\begin{aligned}&\frac{\mathrm{H}^1(K_\mathfrak{p}, \widetilde{\mathbf{V}}^\dagger_{Q_0})}{\mathrm{H}^1(K_\mathfrak{p}, (T_{\lambda^{-1}\psi^{\mathbf{c}-1}} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus (T_{\lambda^{-1}\phi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \oplus (T_{\lambda^{-1}\xi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{\mathbf{c}-1}))} \\ &\times \frac{\mathrm{H}^1(K_{\overline{\mathfrak{p}}}, \widetilde{\mathbf{V}}^\dagger_{Q_0})}{\mathrm{H}^1(K_{\overline{\mathfrak{p}}}, T_{\lambda^{-1}} \otimes \Psi_{W_1}^{\mathbf{c}-1})},\end{aligned}$$

and this kernel is isomorphic to

$$\begin{aligned}\mathrm{Sel}_{\emptyset,0}(K, T_{\lambda^{-1}\psi^{\mathbf{c}-1}} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda^{-1}\phi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{1-\mathbf{c}}) \\ \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda^{-1}\xi^{\mathbf{c}-1}} \otimes \Psi_{W_2}^{\mathbf{c}-1}) \oplus \mathrm{Sel}_{0,\emptyset}(K, T_{\lambda^{-1}} \otimes \Psi_{W_1}^{\mathbf{c}-1}).\end{aligned}$$

This shows the result for $\mathrm{Sel}^{\mathrm{bal}}(K, \mathbf{V}^\dagger_{Q_0})$, and the case of $\mathrm{Sel}^\varphi(K, \mathbf{V}^\dagger_{Q_0})$ follows from Lemma 4.3.2 in the same manner. □

As a consequence we also obtain the following decomposition for the Selmer groups with coefficients in $\mathbf{A}^\dagger_{Q_0} = \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{V}^\dagger_{Q_0}, \mu_{p^\infty})$, mirroring in the case of $\mathrm{Sel}^\varphi(K, \mathbf{A}^\dagger_{Q_0})$ the factorisation of p-adic L-functions in Proposition 3.4.1.

Corollary 4.3.3. *The balanced Selmer group* $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{A}^\dagger_{Q_0})$ *decomposes as*

$$\begin{aligned}\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{A}^\dagger_{Q_0}) \simeq \mathrm{Sel}_{0,\emptyset}(K, A_{\lambda^{-\mathbf{c}}\psi^{1-\mathbf{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) \oplus \mathrm{Sel}_{\emptyset,0}(K, A_{\lambda^{-\mathbf{c}}} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \\ \oplus \mathrm{Sel}_{0,\emptyset}(K, A_{\lambda^{-\mathbf{c}}\phi^{1-\mathbf{c}}} \otimes \Psi_{W_2}^{\mathbf{c}-1}) \oplus \mathrm{Sel}_{0,\emptyset}(K, A_{\lambda^{-\mathbf{c}}\xi^{1-\mathbf{c}}} \otimes \Psi_{W_2}^{1-\mathbf{c}}),\end{aligned}$$

and the φ-unbalanced Selmer group $\mathrm{Sel}^\varphi(\mathbf{Q}, \mathbf{A}^\dagger_{Q_0})$ *decomposes as*

$$\begin{aligned}\mathrm{Sel}^\varphi(\mathbf{Q}, \mathbf{A}^\dagger_{Q_0}) \simeq \mathrm{Sel}_{0,\emptyset}(K, A_{\lambda^{-\mathbf{c}}\psi^{1-\mathbf{c}}} \otimes \Psi_{W_1}^{\mathbf{c}-1}) \oplus \mathrm{Sel}_{0,\emptyset}(K, A_{\lambda^{-\mathbf{c}}} \otimes \Psi_{W_1}^{1-\mathbf{c}}) \\ \oplus \mathrm{Sel}_{0,\emptyset}(K, A_{\lambda^{-\mathbf{c}}\phi^{1-\mathbf{c}}} \otimes \Psi_{W_2}^{\mathbf{c}-1}) \oplus \mathrm{Sel}_{0,\emptyset}(K, A_{\lambda^{-\mathbf{c}}\xi^{1-\mathbf{c}}} \otimes \Psi_{W_2}^{1-\mathbf{c}}).\end{aligned}$$

Proof. This is immediate from Proposition 4.3.1 and local Tate duality. □

Note that the only difference between $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{A}^{\dagger}_{Q_0})$ and $\mathrm{Sel}^{\varphi}(\mathbf{Q}, \mathbf{A}^{\dagger}_{Q_0})$ as described in Corollary 4.3.3 is their second direct summand: The direct summand $\mathrm{Sel}_{\emptyset,0}(K, A_{\lambda^{-\mathbf{c}}} \otimes \Psi_{W_1}^{1-\mathbf{c}})$ in the former is replaced by $\mathrm{Sel}_{0,\emptyset}(K, A_{\lambda^{-\mathbf{c}}} \otimes \Psi_{W_1}^{1-\mathbf{c}})$ in the latter (i.e., their local conditions at the primes above p are *reversed*).

Under some hypotheses, the Pontryagin dual of $\mathrm{Sel}_{0,\emptyset}(K, A_{\lambda^{-\mathbf{c}}} \otimes \Psi_{W_1}^{1-\mathbf{c}})$ will be Λ-torsion (see Theorem 5.0.5), while $\mathrm{Sel}_{\emptyset,0}(K, A_{\lambda^{-\mathbf{c}}} \otimes \Psi_{W_1}^{1-\mathbf{c}})$ will have Λ-corank one.

5. Iwasawa main conjectures

In this section we explain a variant of the Iwasawa main conjecture for triple products formulated in [ACR21].

In the notations of §3.2 we now set $S_1 = S_2$, and call it S (i.e., the weights of $\boldsymbol{g}$ and $\boldsymbol{h}$ move in tandem). Then the change of variables of Proposition 3.4.1 becomes $(W_1, W_2) = (S, 0)$, and we denote by

$$\mathscr{L}_p^{\varphi}(\varphi, \boldsymbol{gh}) \in \Lambda := \mathcal{O}[\![S]\!]$$

the resulting restriction of $\mathscr{L}_p^{\varphi}(\varphi, \boldsymbol{g}, \boldsymbol{h})$. Similarly, we put $\mathbb{V}^{\dagger} = \mathbf{V}^{\dagger}|_{S_1=S_2}$ and denote by

$$\kappa(\varphi, \boldsymbol{gh}) \in \mathrm{H}^1(\mathbf{Q}, \mathbb{V}^{\dagger}) \tag{5.1}$$

the resulting restriction of the three-variable big diagonal class $\kappa(\boldsymbol{\varphi}, \boldsymbol{g}, \boldsymbol{h})$ constructed in [BSV21, §8.1].

Remark 5.0.1. Directly from [BSV21, §8.1] one obtains a class in the cohomology of a representation non-canonically isomorphic to finitely many copies of $\mathbb{V}^{\dagger}$; to obtain a class as in (5.1) one further needs to pick a choice of level-$N = \mathrm{lcm}(N_{\varphi}, N_g, N_h)$ test vectors for the triple $(\varphi, \boldsymbol{g}, \boldsymbol{h})$, which we implictly take to be the one furnished by [Hsi21, Thm. A].

It follows from [BSV21, Cor. 8.2] that the class $\kappa(\varphi, \boldsymbol{gh})$ lands in $\mathrm{Sel}^{\mathrm{bal}}(K, \mathbb{V}^{\dagger})$. Put

$$\mathscr{F}_p^3(\mathbf{V}^{\dagger}) = V_{\boldsymbol{\varphi}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}^+ \otimes \mathcal{X}^{-1} \subset \mathbf{V}^{\dagger}$$

and denote by $\mathscr{F}_p^3(\mathbb{V}^{\dagger}) \subset \mathbb{V}^{\dagger}$ the corresponding restriction to the line $S_1 = S_2$. Then we have

$$\mathscr{F}_p^{\mathrm{bal}}(\mathbb{V}^{\dagger}) / \mathscr{F}_p^3(\mathbb{V}^{\dagger}) \simeq \mathbb{V}_{\boldsymbol{\varphi}}^{\boldsymbol{gh}} \oplus \mathbb{V}_{\boldsymbol{g}}^{\boldsymbol{\varphi h}} \oplus \mathbb{V}_{\boldsymbol{h}}^{\boldsymbol{\varphi g}},$$

where $\mathbb{V}_{\boldsymbol{\varphi}}^{\boldsymbol{gh}}$ is the restriction of $\mathbf{V}_{\boldsymbol{\varphi}}^{\boldsymbol{gh}} = V_{\varphi}^- \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}^+ \otimes \mathcal{X}^{-1}$ restricted to the line $S_1 = S_2$, and similarly for the other two direct summands. In terms of the description given in the proof of Lemma 4.3.2, we find that

$$\mathscr{F}_{\mathfrak{p}}^3(\mathbb{V}^{\dagger}) = T_{\lambda^{-1}\psi^{\mathbf{c}-1}} \otimes \Psi_S^{1-\mathbf{c}}, \quad \mathscr{F}_{\overline{\mathfrak{p}}}^3(\mathbb{V}^{\dagger}) = \{0\},$$

and so together with (4.3) we obtain

$$\mathrm{H}^1(\mathbf{Q}_p, \mathbb{V}_{\varphi}^{\boldsymbol{gh}}) \simeq \mathrm{H}^1(K_{\overline{\mathfrak{p}}}, T_{\lambda^{-1}} \otimes \Psi_S^{\mathbf{c}-1}), \tag{5.2}$$

and under this isomorphism the inclusion $\kappa(\varphi, \boldsymbol{gh}) \in \mathrm{Sel}_{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^{\dagger})$ implies that

$$\mathrm{res}_{\overline{\mathfrak{p}}}(\kappa(\varphi, \boldsymbol{gh})) \in \mathrm{H}^1(K_{\overline{\mathfrak{p}}}, \lambda^{-1}\Psi_S^{\mathbf{c}-1}).$$

Put $u := 1 + p$ and for any Λ-module M and integer k denote by M_k the specialisation of M at $S = u^{k-2} - 1$. Then in particular we see that there are isomorphisms

$$\begin{aligned} \log_{\overline{\mathfrak{p}}} &: \mathrm{H}^1(K_{\overline{\mathfrak{p}}}, T_{\lambda^{-1}} \otimes \Psi_S^{\mathbf{c}-1})_k \to \mathbf{Q}_p, \quad k \geq 3, \\ \exp_{\overline{\mathfrak{p}}}^* &: \mathrm{H}^1(K_{\overline{\mathfrak{p}}}, T_{\lambda^{-1}} \otimes \Psi_S^{\mathbf{c}-1})_k \to \mathbf{Q}_p, \quad k = 2, \end{aligned} \tag{5.3}$$

given by the Bloch–Kato logarithm and dual exponential maps.

Theorem 5.0.2 (Explicit reciprocity Law)**.** *There is an injective Λ-module homomorphism*

$$\mathrm{Log}^{\varphi} : \mathrm{H}^1(K_{\bar{\mathfrak{p}}}, T_{\lambda^{-1}} \otimes \Psi_S^{\mathbf{c}-1}) \to \Lambda$$

with pseudo-null cokernel satisfying for any $\mathbf{z} \in \mathrm{H}^1(K_{\bar{\mathfrak{p}}}, T_{\lambda^{-1}} \otimes \Psi_S^{\mathbf{c}-1})$ the interpolation property

$$\mathrm{Log}^{\varphi}(\mathbf{z})_k = \begin{cases} c_k \cdot \log_{\bar{\mathfrak{p}}}(\mathbf{z}_k) & \text{if } k \geq 3, \\ c_k \cdot \exp_{\bar{\mathfrak{p}}}(\mathbf{z}_k) & \text{if } k = 2, \end{cases}$$

where c_k is an explicit nonzero constant, and such that

$$\mathrm{Log}^{\varphi}\big(\mathrm{res}_{\bar{\mathfrak{p}}}(\kappa(\varphi, \boldsymbol{gh}))\big) = \mathscr{L}_p^{\varphi}(\varphi, \boldsymbol{gh}).$$

Proof. In light of (5.2), the construction of Log^{φ} follows from specialising the three-variable p-adic regulator map of [BSV21, §7.1] similarly as in the proof of [ACR21, Prop. 7.3]; the reciprocity law then follows from [BSV21, Thm. A] (see also [DR21, Thm. 10]). ☐

As explained in more detail in Remark 5.0.4 below (see also [ACR21, §7.3]), the following can be seen as the equivalence between two different formulations of the Iwasawa–Greenberg main conjecture [Gre94] for $\mathbb{V}^{\dagger}$.

Proposition 5.0.3. *The following statements (1)-(2) are equivalent:*

(1) *$\kappa(\varphi, \boldsymbol{gh})$ is not Λ-torsion, the modules $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^{\dagger})$ and $X^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}^{\dagger})$ have both Λ-rank one, and*

$$\mathrm{char}_{\Lambda}\big(X^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}^{\dagger})_{\mathrm{tors}}\big) = \mathrm{char}_{\Lambda}\left(\frac{\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^{\dagger})}{\Lambda \cdot \kappa(\varphi, \boldsymbol{gh})}\right)^2$$

in $\Lambda \otimes \mathbf{Q}_p$, where the subscript tors *denotes the Λ-torsion submodule.*

(2) *$\mathscr{L}_p^{\varphi}(\varphi, \boldsymbol{gh})$ is nonzero, the modules $\mathrm{Sel}^{\varphi}(\mathbf{Q}, \mathbb{V}^{\dagger})$ and $X^{\varphi}(\mathbf{Q}, \mathbb{A}^{\dagger})$ are both Λ-torsion, and*

$$\mathrm{char}_{\Lambda}\big(X^{\varphi}(\mathbf{Q}, \mathbb{A}^{\dagger})\big) = \big(\mathscr{L}_p^{\varphi}(\varphi, \boldsymbol{gh})\big)^2$$

in $\Lambda \otimes \mathbf{Q}_p$.

Proof. This follows from Theorem 5.0.2 and global duality in the same way as [ACR21, Thm. 7.15]. Since the required arguments are virtually the same, we omit the details. ☐

Remark 5.0.4. For any $k \in \mathbf{Z}$ and $\zeta \in \mu_{p^\infty}$ denote by $\mathfrak{P}_{k,\zeta}$ the height one prime of Λ given by $\mathfrak{P}_{k,\zeta} = (S - \gamma^{k-2}\zeta)$, and put

$$C^{\mathrm{bal}} = \{\mathfrak{P}_{k,\zeta} \colon k \geq 2,\ \zeta \in \mu_{p^\infty}\}, \qquad C^{\varphi} = \{\mathfrak{P}_{1,\zeta} \colon \zeta \in \mu_{p^\infty}\}.$$

Then the pairs $(\mathscr{F}_p^{\mathrm{bal}}(\mathbb{V}^{\dagger}), C^{\mathrm{bal}})$ and $(\mathscr{F}_p^{\varphi}(\mathbb{V}^{\dagger}), C^{\varphi})$ both satisfy the *Panchishkin condition for* $\mathbb{V}^{\dagger}$ introduced in [Gre94, p. 216], and statements (1) and (2) of Proposition 5.0.3 can be viewed as instances of the corresponding Iwasawa main conjectures formulated in *op. cit.*.

The results of Proposition 3.4.1, Proposition 4.3.1, and Corollary 4.3.3 reduce the study of the Selmer groups $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{A}^{\dagger})$, $\mathrm{Sel}^{\varphi}(\mathbf{Q}, \mathbb{A}^{\dagger})$ and their associated main conjectures to a corresponding study for Selmer groups of twists of E.

The next result due to Agboola–Howard and Arnold (building on work of Greenberg, Yager, and Rubin, among others) will therefore be useful.

Theorem 5.0.5. *Let $E_{/\mathbf{Q}}$ be an elliptic curve with CM by the ring of integers of K having root number $w=+1$. Let ν be a ring class character of K of conductor divisible only by primes $\ell\nmid p$ split in K. Then* $\mathrm{Sel}_{\emptyset,0}(K,T_{\lambda^{-1}\nu}\otimes\Psi_S^{1-\mathbf{c}})=0$*,* $X_{0,\emptyset}(K,A_{\lambda^{-\mathbf{c}}\nu^{-1}}\otimes\Psi_S^{\mathbf{c}-1})$ *is Λ-torsion, and*

$$\mathrm{char}_\Lambda\big(X_{0,\emptyset}(K,A_{\lambda^{-\mathbf{c}}\nu^{-1}}\otimes\Psi_S^{\mathbf{c}-1})\big)=\big(\mathcal{L}^-_{\mathfrak{p},\lambda^{-1}\nu}(S)\big)$$

as ideals in $\Lambda_{\mathcal{W}}[1/p]$.

Proof. For $\nu=\mathbb{1}$ this is [AH06, Thm. 2.4.17]; the general case follows from [Arn07, Thm. 2.1]. □

6. Proof of main result

In this section we assemble all the pieces to conclude the proof of Theorem A. The definition of the generalised Kato class κ_p appearing in the statement is given in Definition 6.2.1 below.

6.1. Preliminaries.

The link between one of the Selmer groups for characters appeared in §4.3 and the Selmer group of E arises from the following lemma, whose proof we learnt from [Agb07, §6]. Denote by

$$\mathrm{loc}_p:\mathrm{Sel}(\mathbf{Q},V_pE)\to\mathrm{H}^1(\mathbf{Q}_p,V_pE)$$

the restriction map at p.

Lemma 6.1.1. *Assume that* $\mathrm{Sel}(\mathbf{Q},V_pE)\neq\ker(\mathrm{loc}_p)$*. Then*

$$\mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}}\mathrm{Sel}_{0,\emptyset}(K,T_{\lambda^{-1}})=\mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}}\mathrm{Sel}(K,T_{\lambda^{-1}})-1.$$

Proof. Put $\mathcal{D}_{\overline{\mathfrak{p}}}=K_{\overline{\mathfrak{p}}}/\mathcal{O}_{K,\overline{\mathfrak{p}}}$. By global duality we have the exact sequence

$$0\to\mathrm{Sel}(K,T_{\lambda^{-1}})\to\mathrm{Sel}_{\mathrm{rel}}(K,T_{\lambda^{-1}})\xrightarrow{\alpha}\prod_{v\mid p}\frac{\mathrm{H}^1(K_v,T_{\lambda^{-1}})}{E(K_v)\otimes\mathcal{O}_{K,\mathfrak{p}}}\xrightarrow{\beta^\vee}\mathrm{Sel}(K,A_{\lambda^{-\mathbf{c}}})^\vee,\tag{6.1}$$

where the last arrow is identified with Pontryagin dual of the restriction map

$$\beta:\mathrm{Sel}(K,A_{\lambda^{-\mathbf{c}}})\to\prod_{v\mid p}E(K_v)\otimes\mathcal{D}_{\overline{\mathfrak{p}}}$$

via local duality. By assumption, the image of β has $\mathcal{O}_{K,\overline{\mathfrak{p}}}$-rank one. By (6.1) this implies that the map α has finite image (noting that its target has $\mathcal{O}_{K,\mathfrak{p}}$-rank one), and so $\mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}}\mathrm{Sel}_{\mathrm{rel}}(K,T_{\lambda^{-1}})=\mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}}\mathrm{Sel}(K,T_{\lambda^{-1}})$. In particular, this shows that

$$\mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}}\mathrm{Sel}_{0,\emptyset}(K,T_{\lambda^{-1}})=\mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}}\ker\big(\mathrm{loc}_{\mathfrak{p}}:\mathrm{Sel}(K,T_{\lambda^{-1}})\to E(K_{\mathfrak{p}})\otimes\mathcal{O}_{K,\mathfrak{p}}\big).$$

Since the image of the restriction map $\mathrm{loc}_{\mathfrak{p}}$ has $\mathcal{O}_{K,\mathfrak{p}}$-rank one by our hypotheses, this implies the result. □

For any $\mathcal{O}[\![S]\!]$-module M we denote by $M_S=M/SM$ the denote the cokernel of multiplication by S. We shall also need the following variant of Mazur's control theorem. (Note however that the Selmer groups in the statement have reversed local conditions at the primes above p with respect to the usual Selmer groups.)

Proposition 6.1.2. *Multiplication by S induces natural maps*

$$r^*:\mathrm{Sel}_{\emptyset,0}(K,A_{\lambda^{-\mathbf{c}}})\to\mathrm{Sel}_{\emptyset,0}(K,A_{\lambda^{-\mathbf{c}}}\otimes\Psi_S^{1-\mathbf{c}})|_{S=0},$$
$$r:\mathrm{Sel}_{0,\emptyset}(K,T_{\lambda^{-1}}\otimes\Psi_S^{\mathbf{c}-1})_S\to\mathrm{Sel}_{0,\emptyset}(K,T_{\lambda^{-1}})$$

which are injective with finite cokernel.

Proof. We only explain the case of r^*, as the case of r follows in the same manner. The map r^* fits into the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Sel}_{\emptyset,0}(K, A_{\lambda^{-\mathbf{c}}}) \arrow[r] \arrow[d, "r^*"] & \mathrm{Sel}^{\{p\}}(K, A_{\lambda^{-\mathbf{c}}}) \arrow[r] \arrow[d, "s^*"] & \mathrm{H}^1(K_{\mathfrak{p}}, A_{\lambda^{-\mathbf{c}}}) \arrow[d, "t^*"] \\ 0 \arrow[r] & \mathrm{Sel}_{\emptyset,0}(K, A_{\lambda^{-\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})|_{S=0} \arrow[r] & \mathrm{Sel}^{\{p\}}(K, A_{\lambda^{-\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})|_{S=0} \arrow[r] & \mathrm{H}^1(K_{\mathfrak{p}}, A_{\lambda^{-\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})|_{S=0}. \end{tikzcd}$$

It follows from Lubin–Tate theory that the local field extension $K_{\mathfrak{p}}(E[\overline{\mathfrak{p}}])/K_{\mathfrak{p}}$ has degree $p-1$ (see [dS87, Ch. I]), and so $\mathrm{H}^0(K_{\infty,\mathfrak{p}}, E[\overline{\mathfrak{p}}^\infty]) = 0$ since Γ^- is a pro-p group. By inflation-restriction, it follows that the map t^* is an isomorphism. On the other hand, letting Σ be any finite set of places of K containing ∞ and the primes dividing pN, the map s^* above fits into the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Sel}^{\{p\}}(K, A_{\lambda^{\mathbf{c}}}) \arrow[r] \arrow[d, "s^*"] & \mathrm{H}^1(K_\Sigma/K, A_{\lambda^{-\mathbf{c}}}) \arrow[r] \arrow[d, "u^*"] & \bigoplus_{w\in\Sigma, w\nmid p} \mathrm{H}^1(K_w, A_{\lambda^{-\mathbf{c}}}) \arrow[d, "v^*"] \\ 0 \arrow[r] & \mathrm{Sel}^{\{p\}}(K, A_{\lambda^{-\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})|_{S=0} \arrow[r] & \mathrm{H}^1(K_\Sigma/K, A_{\lambda^{-\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})|_{S=0} \arrow[r] & \bigoplus_{w\in\Sigma, w\nmid p} \mathrm{H}^1(K_w, A_{\lambda^{-\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})|_{S=0}, \end{tikzcd}$$

where K_Σ is the Galois group of the maximal extension of K unramified outside Σ. The vanishing of $\mathrm{H}^0(K_{\infty,\mathfrak{p}}, E[\overline{\mathfrak{p}}^\infty])$ implies that u^* is an isomorphism, and as shown in the proof of [dS87, Lem. IV.3.5] as consequence of [Maz72, Cor. 4.4], the kernel of the map v^* is finite. By the Snake Lemma applied to the previous two commutative diagrams, the result follows. $\square$

Specialised to $(W_1, W_2) = (S, 0)$, the decomposition of $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}^\dagger_{Q_0})$ given in Proposition 4.3.1 becomes

$$\begin{aligned} \mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger) \simeq\ & \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda^{-1}\psi^{\mathbf{c}-1}} \otimes \Psi_S^{1-\mathbf{c}}) \oplus \mathrm{Sel}_{0,\emptyset}(K, T_{\lambda^{-1}} \otimes \Psi_S^{\mathbf{c}-1}) \\ & \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda^{-1}\phi^{\mathbf{c}-1}} \otimes \Psi_S^{1-\mathbf{c}})_S \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda^{-1}\xi^{\mathbf{c}-1}} \otimes \Psi_S^{\mathbf{c}-1})_S. \end{aligned} \tag{6.2}$$

6.2. **Generalised Kato class.** We consider the following twisted variant of the generalised Kato classes introduced by Darmon–Rotger [DR16] (note that it is crucial for us to take $\varphi = \theta(\lambda\psi)$ for appropriate ψ as in §3.2, rather than the newform $\theta(\lambda)$ associated with $E_{/\mathbf{Q}}$ as in *loc. cit.*.)

Definition 6.2.1 (Generalised Kato class)**.** Let κ_p be the image of $\kappa(\varphi, \boldsymbol{gh})$ under the composition

$$\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger) \to \mathrm{Sel}_{0,\emptyset}(K, T_{\lambda^{-1}} \otimes \Psi_S^{\mathbf{c}-1}) \to \mathrm{Sel}_{0,\emptyset}(K, T_{\lambda^{-1}}),$$

where the first arrow is given by the projection onto the second direct summand in (6.2), and the second arrow is induced by the multiplication by S on $T_{\lambda^{-1}} \otimes \Psi_S^{\mathbf{c}-1}$.

Using $\mathrm{H}^1(K, V_{\lambda^{-1}}) \simeq \mathrm{H}^1(\mathbf{Q}, V_pE)$, we shall also view κ_p in the latter group.

Lemma 6.2.2. *Assume that $L(E,1) = 0$. Then $\kappa_p \in \mathrm{Sel}(\mathbf{Q}, V_pE)$.*

Proof. Viewing κ_p as a class in $\mathrm{Sel}_{0,\emptyset}(K, T_{\lambda^{-1}})$ it suffices to show that $\mathrm{res}_{\overline{\mathfrak{p}}}(\kappa_p) = 0$ (and therefore κ_p lies in the *fine Selmer group* of E). Since $L(E,1) = L(\lambda^{-1}, 0)$, by Theorem 5.0.2, Proposition 4.3.1, and Theorem 2.1.1 we see that

$$L(E,1) = 0 \quad \Longrightarrow \quad \exp^*_{\overline{\mathfrak{p}}}(\mathrm{res}_{\overline{\mathfrak{p}}}(\kappa_p)) = 0,$$

where $\exp^*_{\overline{\mathfrak{p}}}$ is the Bloch–Kato dual exponential map in (5.3). Since for $k = 2$ (i.e., $S = 0$) this map is an isomorphism, the result follows. $\square$

6.3. **Proof of Theorem A.**

Theorem 6.3.1. *Assume that $L(E,s)$ has sign $+1$ and $L(E,1)=0$. Then*

$$\mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}}\mathrm{Sel}_{0,\emptyset}(K,T_{\lambda^{-1}})=1 \quad\Longrightarrow\quad \kappa_p\neq 0,$$

where κ_p is any generalised Kato class associated with a triple $(\varphi,\boldsymbol{g},\boldsymbol{h})$ as in Choice 3.2.2.

Proof. Specialised to $(W_1,W_2)=(S,0)$, the factorisation in Proposition 3.4.1 reads as the equality

$$\mathscr{L}_p^{\varphi}(\varphi,\boldsymbol{gh})^2(S)=\mathcal{L}^-_{\mathfrak{p},\lambda^{-1}\psi^{\mathbf{c}-1}}(S)^{\iota}\cdot\mathcal{L}^-_{\mathfrak{p},\lambda^{-1}}(S)\cdot\mathcal{L}^-_{\mathfrak{p},\lambda^{-1}\psi^{\mathbf{c}-1}}(0)\cdot\mathcal{L}^-_{\mathfrak{p},\lambda^{-1}\xi^{\mathbf{c}-1}}(0)$$

up to a multiplication by a unit $u\in\mathcal{W}^{\times}$. On the other hand, the decomposition in Corollary 4.3.3 for the unbalanced Selmer group becomes

$$\begin{aligned}X^{\varphi}(\mathbf{Q},\mathbb{A}^{\dagger})\simeq{}& X_{0,\emptyset}(K,A_{\lambda^{-\mathbf{c}}\psi^{1-\mathbf{c}}}\otimes\Psi_S^{\mathbf{c}-1})\oplus X_{0,\emptyset}(K,A_{\lambda^{-\mathbf{c}}}\otimes\Psi_S^{1-\mathbf{c}})\\ &\oplus X_{0,\emptyset}(K,A_{\lambda^{-\mathbf{c}}\phi^{1-\mathbf{c}}}\otimes\Psi_S^{\mathbf{c}-1})_S\oplus X_{0,\emptyset}(K,A_{\lambda^{-\mathbf{c}}\xi^{1-\mathbf{c}}}\otimes\Psi_S^{1-\mathbf{c}})_S.\end{aligned}$$

Since clearly $X_{0,\emptyset}(K,A_{\lambda^{-\mathbf{c}}\psi^{1-\mathbf{c}}}\otimes\Psi_S^{\mathbf{c}-1})=X_{0,\emptyset}(K,A_{\lambda^{-\mathbf{c}}\psi^{1-\mathbf{c}}}\otimes\Psi_S^{1-\mathbf{c}})^{\iota}$, from Theorem 5.0.5 we deduce that $X^{\varphi}(\mathbf{Q},\mathbb{A}^{\dagger})$ is Λ-torsion with

$$\mathrm{char}_{\Lambda}\big(X^{\varphi}(\mathbf{Q},\mathbb{A}^{\dagger})\big)=\big(\mathscr{L}_p^{\varphi}(\varphi,\boldsymbol{gh})\big)^2$$

as ideals in $\Lambda_{\mathcal{W}}\otimes\mathbf{Q}_p$. By Proposition 5.0.3, it follows that

$$\mathrm{rank}_{\Lambda}\big(X^{\mathrm{bal}}(\mathbf{Q},\mathbb{A}^{\dagger})\big)=\mathrm{rank}_{\Lambda}\big(\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbb{V}^{\dagger})\big)=1,\tag{6.3}$$

and

$$\mathrm{char}_{\Lambda}\big(X^{\mathrm{bal}}(\mathbf{Q},\mathbb{A}^{\dagger})_{\mathrm{tors}}\big)=\mathrm{char}_{\Lambda}\left(\frac{\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbb{V}^{\dagger})}{\Lambda\cdot\kappa(\varphi,\boldsymbol{gh})}\right)^2\tag{6.4}$$

as ideals in $\Lambda_{\mathcal{W}}\otimes\mathbf{Q}_p$.

Now, from Theorem 2.1.1, Theorem 5.0.5, and Mazur's control theorem (for the usual Selmer groups) we have the implications

$$\begin{aligned}L(\lambda^{-1}\psi^{\mathbf{c}-1},0)\neq 0 \quad&\Longrightarrow\quad \mathcal{L}^-_{\mathfrak{p},\lambda^{-1}\psi^{\mathbf{c}-1}}(0)\neq 0\\ &\Longrightarrow\quad \big|X_{0,\emptyset}(K,A_{\lambda^{-\mathbf{c}}\psi^{1-\mathbf{c}}}\otimes\Psi_S^{\mathbf{c}-1})_S\big|<\infty.\end{aligned}$$

Similarly, the nonvanishing of $L(\lambda^{-1}\phi^{\mathbf{c}-1},0)$ and $L(\lambda^{-1}\xi^{\mathbf{c}-1},0)$ implies that

$$\big|X_{0,\emptyset}(K,A_{\lambda^{-\mathbf{c}}\phi^{1-\mathbf{c}}}\otimes\Psi_S^{\mathbf{c}-1})_S\big|<\infty,\quad \big|X_{0,\emptyset}(K,A_{\lambda^{-\mathbf{c}}\xi^{1-\mathbf{c}}}\otimes\Psi_S^{\mathbf{c}-1})_S\big|<\infty,$$

and so from the balanced Selmer group decomposition from Corollary 4.3.3:

$$\begin{aligned}X^{\mathrm{bal}}(\mathbf{Q},\mathbb{A}^{\dagger})\simeq{}& X_{0,\emptyset}(K,A_{\lambda^{-\mathbf{c}}\psi^{1-\mathbf{c}}}\otimes\Psi_S^{\mathbf{c}-1})\oplus X_{\emptyset,0}(K,A_{\lambda^{-\mathbf{c}}}\otimes\Psi_S^{1-\mathbf{c}})\\ &\oplus X_{0,\emptyset}(K,A_{\lambda^{-\mathbf{c}}\phi^{1-\mathbf{c}}}\otimes\Psi_S^{\mathbf{c}-1})_S\oplus X_{0,\emptyset}(K,A_{\lambda^{-\mathbf{c}}\xi^{1-\mathbf{c}}}\otimes\Psi_S^{1-\mathbf{c}})_S,\end{aligned}$$

we deduce that

$$\mathrm{rank}_{\Lambda}\big(X_{\emptyset,0}(K,A_{\lambda^{-\mathbf{c}}}\otimes\Psi_S^{1-\mathbf{c}})\big)=1.\tag{6.5}$$

Moreover, by Proposition 6.1.2 we have

$$\begin{aligned}\mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}}\big(X_{\emptyset,0}(K,A_{\lambda^{-\mathbf{c}}}\otimes\Psi_S^{1-\mathbf{c}})_S\big)&=\mathrm{corank}_{\mathcal{O}_{K,\mathfrak{p}}}\big(\mathrm{Sel}_{\emptyset,0}(K,A_{\lambda^{-\mathbf{c}}})\big)\\ &=\mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}}\big(\mathrm{Sel}_{\emptyset,0}(K,T_{\lambda^{-\mathbf{c}}})\big)\\ &=\mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}}\big(\mathrm{Sel}_{0,\emptyset}(K,T_{\lambda^{-1}})\big),\end{aligned}\tag{6.6}$$

using the isomorphism $\mathrm{Sel}_{\emptyset,0}(K, T_{\lambda^{-\mathbf{c}}}) \simeq \mathrm{Sel}_{0,\emptyset}(K, T_{\lambda^{-1}})$ given by the action of complex conjugation for the last equality.

On the other hand, denote by $\mathrm{pr}_\lambda(\kappa(\varphi, \boldsymbol{gh}))$ the projection of $\kappa(\varphi, \boldsymbol{gh})$ onto the second direct summand in the balanced Selmer groups decomposition from Proposition 4.3.1:

$$\begin{aligned}\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbb{V}^\dagger) \simeq\ & \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda^{-1}\psi^{\mathbf{c}-1}} \otimes \Psi_S^{1-\mathbf{c}}) \oplus \mathrm{Sel}_{0,\emptyset}(K, T_{\lambda^{-1}} \otimes \Psi_S^{\mathbf{c}-1}) \\ & \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda^{-1}\phi^{\mathbf{c}-1}} \otimes \Psi_S^{1-\mathbf{c}})_S \oplus \mathrm{Sel}_{\emptyset,0}(K, T_{\lambda^{-1}\xi^{\mathbf{c}-1}} \otimes \Psi_S^{\mathbf{c}-1})_S.\end{aligned}$$

From the above we see that (6.3) and (6.4) imply that $\mathrm{Sel}_{0,\emptyset}(K, T_{\lambda^{-1}} \otimes \Psi_S^{\mathbf{c}-1})$ has Λ-rank one, and we have the following equality of ideals in $\Lambda_{\mathcal{W}} \otimes \mathbf{Q}_p$:

$$\mathrm{char}_\Lambda\big(X_{\emptyset,0}(K, A_{\lambda^{-\mathbf{c}}} \otimes \Psi_S^{1-\mathbf{c}})_{\mathrm{tors}}\big) = \mathrm{char}_\Lambda\big(\mathfrak{Z}\big)^2, \tag{6.7}$$

where $\mathfrak{Z} = \mathrm{Sel}_{0,\emptyset}(K, T_{\lambda^{-1}} \otimes \Psi_S^{\mathbf{c}-1})/\Lambda \cdot \mathrm{pr}_\lambda(\kappa(\varphi, \boldsymbol{gh}))$.

The combination of (6.5), (6.6), and (6.7) shows that

$$\mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}}\big(\mathrm{Sel}_{0,\emptyset}(K, T_{\lambda^{-1}})\big) = 1 + 2\,\mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}}\big(\mathfrak{Z}_S\big).$$

Thus if $\mathrm{Sel}_{0,\emptyset}(K, T_{\lambda^{-1}})$ has $\mathcal{O}_{K,\mathfrak{p}}$-rank one then $\mathfrak{Z}_S$ is $\mathcal{O}_{K,\mathfrak{p}}$-torsion, and so the image of $\mathrm{pr}_\lambda(\kappa(\varphi, \boldsymbol{gh}))$ in $\mathrm{Sel}_{0,\emptyset}(K, T_{\lambda^{-1}} \otimes \Psi_S^{\mathbf{c}-1})_S$ is non-torsion. Since the injection

$$\mathrm{Sel}_{0,\emptyset}(K, T_{\lambda^{-1}} \otimes \Psi_S^{\mathbf{c}-1})_S \to \mathrm{Sel}_{0,\emptyset}(K, T_{\lambda^{-1}})$$

of Proposition 6.1.2 sends $(\mathrm{pr}_\lambda(\kappa(\varphi, \boldsymbol{gh})) \bmod S)$ into κ_p, this concludes the proof. □

The proof of Theorem A in the Introduction now follows:

Theorem 6.3.2. *Assume that $L(E, s)$ has sign $+1$ and $L(E, 1) = 0$. If $\mathrm{Sel}(\mathbf{Q}, V_pE) \neq \ker(\mathrm{loc}_p)$ then the following implication holds:*

$$\dim_{\mathbf{Q}_p} \mathrm{Sel}(\mathbf{Q}, V_pE) = 2 \quad \Longrightarrow \quad \kappa_p \neq 0,$$

where κ_p is any generalised Kato class associated with a triple $(\varphi, \boldsymbol{g}, \boldsymbol{h})$ as in Choice 3.2.2.

Proof. Since $\dim_{\mathbf{Q}_p} \mathrm{Sel}(\mathbf{Q}, V_pE) = \mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}} \mathrm{Sel}(K, T_{\lambda^{-1}})$, this is immediate from Theorem 6.3.1 and Lemma 6.1.1. □

Remark 6.3.3. Of course, by the theorem of Coates–Wiles [CW77] and the p-parity conjecture [Guo93], the first two conditions in Theorem 6.3.2 are superfluous.

Remark 6.3.4. The construction of κ_p depends on the choice of a triple $(\varphi, \boldsymbol{g}, \boldsymbol{h})$ as in Choice 3.2.2, but it follows from Lemma 6.1.1 and the proof of Lemma 6.2.2 that under the hypotheses of Theorem 6.3.2 they all differ by a nonzero scalar; in fact, under those hypotheses they all span the 1-dimensional subspace $\mathrm{Sel}_0(\mathbf{Q}, V_pE) \simeq \mathrm{Sel}_{0,\emptyset}(K, V_{\lambda^{-1}})$ inside the 2-dimensional $\mathrm{Sel}(\mathbf{Q}, V_pE)$.

7. Proof of Theorem C

We conclude with the proof of the rank 2 Kolyvagin theorem stated in the Introduction.

Theorem 7.0.1. *Assume that $L(E, s)$ has sign $+1$ and $L(E, 1) = 0$. If $\mathrm{Sel}(\mathbf{Q}, V_pE) \neq \ker(\mathrm{loc}_p)$ then the following implication holds:*

$$\kappa_p \neq 0 \quad \Longrightarrow \quad \dim_{\mathbf{Q}_p} \mathrm{Sel}(\mathbf{Q}, V_pE) = 2,$$

where κ_p is any generalised Kato class associated with a triple $(\varphi, \boldsymbol{g}, \boldsymbol{h})$ as in Choice 3.2.2.

Proof. By Lemma 6.1.1, it suffices to prove the implication

$$\kappa_p \neq 0 \quad \Longrightarrow \quad \mathrm{rank}_{\mathcal{O}_{K,\mathfrak{p}}} \mathrm{Sel}_{0,\emptyset}(K, T_{\lambda^{-1}}) = 1. \tag{7.1}$$

Let $\widetilde{\mathbf{V}}^{\dagger}_{Q_0}$ be as in (4.4). A straightforward modification of the constructions in [ACR21, §6] building on the big diagonal cycle classes of [BSV21, §8.1] give rise to a collection of cohomology classes

$$\boldsymbol{\kappa}(\varphi, \boldsymbol{g}, \boldsymbol{h}) = \left\{ \kappa(\varphi, \boldsymbol{g}, \boldsymbol{h})_n \in \mathrm{Sel}^{\mathrm{bal}}(K[n], \widetilde{\mathbf{V}}^{\dagger}_{Q_0}) \; : \; n \in \mathcal{S} \right\},$$

where n runs over the set $\mathcal{S}$ of all squarefree products of primes $q \nmid p\ell\ell' N$ split in K and $K[n]$ denotes the maximal p-extension inside the ring class field of K of conductor n, satisfying the obvious extension of the norm-compatibility relations of [ACR21, Prop. 6.3]. Projecting these classes to the cohomology of the direct summand $T_{\lambda^{-1}} \otimes \Psi_{W_1}^{\mathbf{c}-1}$ in (4.4) and specialising to $W_1 = 0$ we obtain a collection of cohomology classes

$$\left\{ \widetilde{\kappa}_\lambda(\varphi, g, h)_n \in \mathrm{Sel}_{0,\emptyset}(K[n], T_{\lambda^{-1}}) \; : \; n \in \mathcal{S} \right\}$$

such that, for all $nq \in \mathcal{S}$ with q a prime, we have

$$\mathrm{cor}_{K[nq]/K[n]} \left(\widetilde{\kappa}_\lambda(\varphi, g, h)_{nq} \right) = Q(\mathrm{Fr}_{\mathfrak{q}}^{-1}) \, \widetilde{\kappa}_\lambda(\varphi, g, h)_n,$$

where $\mathrm{Fr}_{\mathfrak{q}}$ is the Frobenius element in $\mathrm{Gal}(K[n]/K)$ for any of the primes above q, and

$$\begin{aligned} Q_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1}) = \frac{1}{q^2} \Bigg\{ & \chi_g(q)\chi_h(q) q \left(\frac{\lambda\psi(\mathfrak{q})}{q} \mathrm{Fr}_{\mathfrak{q}}^{-1} \right)^2 - a_q(g) a_q(h) q \left(\frac{\lambda\psi(\mathfrak{q})}{q} \mathrm{Fr}_{\mathfrak{q}}^{-1} \right) \\ & + \chi_g(q)^{-1} a_q(g)^2 q + \chi_h(q)^{-1} a_q(h)^2 - \frac{q^2+1}{q} \\ & - a_q(g) a_q(h) q \left(\frac{\lambda\psi(\bar{\mathfrak{q}})}{q} \mathrm{Fr}_{\mathfrak{q}} \right) + \chi_g(q)\chi_h(q) q \left(\frac{\psi(\bar{\mathfrak{q}})}{q} \mathrm{Fr}_{\mathfrak{q}} \right)^2 \Bigg\}. \end{aligned}$$

Moreover, by definition the generalised Kato class κ_p is related to this construction via

$$\mathrm{cor}_{K[1]/K}(\widetilde{\kappa}_\lambda(\varphi, g, h)_1) = \kappa_p.$$

Denote by $\mathbb{V}_S^{\dagger}$ the specialisation of $\widetilde{\mathbf{V}}^{\dagger}_{Q_0}$ at $(W_1, W_2) = (0, 0)$ so by (4.4) we have

$$\mathbb{V}_S^{\dagger} = T_{\lambda^{-1}\psi^{\mathbf{c}-1}} \oplus T_{\lambda^{-1}} \oplus T_{\lambda^{-1}\phi^{\mathbf{c}-1}} \oplus T_{\lambda^{-1}\xi^{\mathbf{c}-1}}.$$

Then as in [ACR21, Thm. 4.6] we find the congruence

$$\widetilde{P}_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1}) \chi_g(q)\chi_h(q)\lambda\psi(\bar{\mathfrak{q}})^2 \mathrm{Fr}_{\mathfrak{q}}^2 \equiv Q_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1}) \pmod{q-1},$$

where $\widetilde{P}_{\mathfrak{q}}(X) = \det(1 - \mathrm{Fr}_{\mathfrak{q}}^{-1} X \,|\, (\mathbb{V}_S^{\dagger})^{\vee}(1))$. Thus by [Rub00, Lem. 9.6.1] and after restricting to a subset $\mathcal{S}' \subset \mathcal{S}$ consisting of all squarefree products of primes in a positive density set $\mathcal{P}'$ as in [ACR22, §5.1], the classes $\widetilde{\kappa}_\lambda(\varphi, g, h)_n$ can be modified similarly as in Theorem 5.1 of *op. cit.* to yield an anticyclotomic Euler system for $T_{\lambda^{-1}}$ with base class κ_p, i.e., a collection of classes

$$\left\{ \kappa_\lambda(\varphi, g, h)_n \in \mathrm{Sel}_{0,\emptyset}(K[n], T_{\lambda^{-1}}) \; : \; n \in \mathcal{S}' \right\}$$

satisfying, for all $nq \in \mathcal{S}'$ with q a prime,

$$\mathrm{cor}_{K[nq]/K[n]} \left(\kappa_\lambda(\varphi, g, h)_{nq} \right) = P_{\mathfrak{q}}(\mathrm{Fr}_{\mathfrak{q}}^{-1}) \, \kappa_\lambda(\varphi, g, h)_n,$$

where $P_{\mathfrak{q}}(X) = \det(1 - \mathrm{Fr}_{\mathfrak{q}}^{-1} X \,|\, (T_{\lambda^{-1}})^{\vee}(1))$ and such that

$$\mathrm{cor}_{K[1]/K}(\kappa_\lambda(\varphi, g, h)_1) = \kappa_p.$$

Therefore, by Kolyvagin's methods in the form extended in [JNS] to general anticyclotomic Euler systems (see also [ACR21, Thm. 8.3]), the proof of (7.1) follows and this gives the result. □

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