ON THE NON-VANISHING OF GENERALIZED KATO CLASSES FOR
ELLIPTIC CURVES OF RANK TWO

FRANCESC CASTELLA AND MING-LUN HSIEH

Abstract. A conjecture of Darmon–Rotger predicts that a certain analogue of Kato’s classes for elliptic curves $E/\mathbb{Q}$ generate a non-trivial subspace of the $p$-adic Selmer group of $E$ if and only if the rank of $E$ (either algebraic or analytic, by the Birch–Swinnerton-Dyer conjecture) is two. These generalized Kato classes are attached to the weight two eigenform $f$ associated with $E$ and two classical $p$-stabilized eigenforms $g$ and $h$ of weight one, with $g = h^*$. In this paper, we consider the case in which $g$ is the theta series of an auxiliary imaginary quadratic field $K$ in which $p$ splits, and prove the following results about the corresponding generalized Kato class $\kappa_{E,K}$:

1. The non-triviality of $\kappa_{E,K}$ implies the $p$-adic Selmer group of $E$ is two-dimensional.
2. If $\text{III}(E/\mathbb{Q})[p^\infty]$ is finite, then the converse to (1) holds.

The first result can be regarded as a rank two analogue of Kolyvagin’s theorem for Heegner points, although our proof also shows that $\kappa_{E,K}$ is not the base class of an Euler system in the usual sense. A key new ingredient in the proof of these results is a formula for the leading coefficient at $T = 0$ of the anticyclotomic $p$-adic $L$-function attached to $E$ over $K$, with which we obtain an expression for $\kappa_{E,K}$ in terms of derived $p$-adic heights and an enhanced $p$-adic regulator.

1. Introduction

1.1. Motivating question. Let $E/\mathbb{Q}$ be an elliptic curve, and let $L(E, s)$ be its Hasse–Weil $L$-series. After the pioneering work of Coates–Wiles [CW77] in the CM case, a major advance towards the Birch and Swinnerton-Dyer conjecture was the proof by Gross–Zagier [GZ86] and Kolyvagin [Kol88] of the implication

$$r_{an} := \text{ord}_{s=1} L(E, s) \in \{0, 1\} \implies \text{rank}_E \mathbb{Q} = r_{an}. \quad (1.1)$$

The proof of (1.1) resorts to choosing an imaginary quadratic field $K$ for which Heegner points on $E$ (over ring class extensions of $K$) becomes available and such that $\text{ord}_{s=1} L(E/K, s) = 1$.

Date: September 13, 2019.
2010 Mathematics Subject Classification. Primary 11G05; Secondary 11G40.
Key words and phrases. Elliptic curves, Birch and Swinnerton-Dyer conjecture, $p$-adic families of modular forms, $p$-adic $L$-functions, Euler systems.

The first author was partially supported by NSF grant DMS-1801385, 1946136.
The second author was partially supported by a MOST grant 108-2628-M-001-009-MY4.
By the Gross–Zagier formula, the basic Heegner point $y_K \in E(K)$ is then non-torsion, which by Kolyvagin’s work implies that $\text{rank}_z E(K) = 1$. Since $y_K$ descends to $E(Q)$ precisely when $L(E, s)$ vanishes to odd order at $s = 1$, the conclusion of (1.1) follows.

Two more recent majors advances towards BSD are the works by Kato [Kat04] and Skinner–Urban [SU14] on the Iwasawa main conjecture and Skinner’s converse [Sk14] to the theorem of Gross–Zagier and Kolyvagin, which combined proved the implication

\[
\text{rank}_z E(K) = \text{rank}_z E(Q) = 2 \quad \text{and} \quad \# \text{III}(E/Q)[p^{\infty}] < \infty,
\]

for certain primes $p$ of good ordinary reduction for $E$.

It is natural to ask about the extension of these results to elliptic curves $E/Q$ of rank $r > 1$. As a first step in this direction, in this paper for good ordinary primes $p$ and for the choice of a suitable imaginary quadratic field $K$ such that

\[
\text{rank}_z E(K) = \text{rank}_z E(Q) = 2 \quad \text{and} \quad \# \text{III}(E/Q)[p^{\infty}] < \infty,
\]

we prove analogues in this setting of the implication

\[
y_K \notin E(Q)_{\text{tors}} \implies \dim_{Q_p} \text{Sel}(Q, V_p E) = 1
\]

and the implication

\[
\text{rank}_z E(K) = \text{rank}_z E(Q) = 1 \quad \# \text{III}(E/Q)[p^{\infty}] < \infty \implies y_K \notin E(Q)_{\text{tors}}
\]

appearing in the course of the proof of (1.1) and of (1.2), respectively, with the above Heegner point $y_K$ replaced by certain generalized Kato classes introduced by Darmon–Rotger.

1.2. Darmon–Rotger conjecture for rank two elliptic curves. Following their groundbreaking work on the equivariant Birch–Swinnerton-Dyer conjecture, Darmon–Rotger [DR17a] formulated a conjectural non-vanishing criterion for the generalized Kato classes

\[
\kappa(f, g, h)
\]

introduced in [DR16]. The classes $\kappa(f, g, h)$ are associated to triples consisting of an eigenform $f$ of weight 2 and classical $p$-stabilized eigenforms $g$ and $h$ of weight 1, corresponding to odd two-dimensional Artin representations $\rho_g$ and $\rho_h$, and they are germane to the BSD conjecture for the Mordell–Weil group $A_f(H)$ of the abelian variety $A_f/Q$ attached to $f$ over the number field $H$ cut out by $\rho_g \otimes \rho_h$. In this paper, we consider Darmon–Rotger’s conjecture in cases where the classes $\kappa(f, g, h)$ are predicted to have a bearing on the arithmetic of elliptic curves $E/Q$ with rank $Z E(Q) = 2$.

Let $E$ be an elliptic curve over $Q$ of conductor $N_E$, and fix a prime $p > 2$ of good ordinary reduction for $E$. Let $K$ be an imaginary quadratic field of discriminant prime to $N_E$ in which $p = p\bar{p}$ splits, and let $\chi : G_K = \text{Gal}(Q/K) \to \mathbb{L}^\times$ be a ring class character of conductor prime to $pN_E$ valued in a number field $L$. For simplicity, we assume here that the image of $L$ under a fixed embedding $\bar{Q} \to \bar{Q}_p$ is contained in $Q_p$.

Let $f \in S_2(\Gamma_0(N_E))$ be the newform attached to $E$ by modularity, so that $L(E, s) = L(f, s)$, and let $g$ and $h$ be the weight 1 theta series of $\chi$ and $\chi^{-1}$, respectively. Suppose $\chi(p) \neq \pm 1$, and set $\alpha := \chi(\bar{p})$ and $\beta := \chi(p)$. As explained in [DR16], attached to the triple $(f, g, h)$ one has four (a priori distinct) generalized Kato classes

\[
\kappa(f, g_\alpha, h_{\alpha^{-1}}), \ k(f, g_\alpha, h_{\beta^{-1}}), \ k(f, g_\beta, h_{\alpha^{-1}}), \ k(f, g_\beta, h_{\beta^{-1}}) \in H^1(Q, V_{fgh}),
\]

1In which $g$ and $h$ can be more general weight 1 eigenform with their $p$-th Hecke polynomials having distinct roots.
where $V_{fgh} \simeq V_p E \otimes V_q \otimes V_h$ is the tensor product of the p-adic representations associated to $f$, $g$, and $h$ (so in particular $V_q$ realized $q_g \otimes 1, \mathbb{Q}_p$, and similarly for $h$). The class $\kappa(f, g_\alpha, h_{\alpha-1})$ (and similarly the other three) arises as the p-adic limit

$$\kappa(f, g_\alpha, h_{\alpha-1}) = \lim_{\ell \to 1} \kappa(f, g_\ell, h_\ell)$$

as $(g_\ell, h_\ell)$ runs over the classical weight $\ell \geq 2$ specializations of Hida families $g$ and $h$ passing through the p-stabilizations

$$g_\alpha := g(q) - \beta g(q^p), \quad h_{\alpha-1} := h(q) - \beta^{-1} h(q^p),$$

in weight 1, and where $\kappa(f, g_\ell, h_\ell)$ is obtained from the p-adic étale Abel–Jacobi image of certain higher-dimensional Gross–Kudla–Schoen diagonal cycles $\text{GK92, GS95}$ on triple products of modular curves. The construction of $\kappa(f, g_\ell, h_\ell)$ takes place at level $N := \text{lcm}(N_E, N_g, N_h)$, where $N_g$ and $N_h$ are the Artin conductors of $q_g$ and $q_h$, respectively, and the construction of the classes in (1.3) further depends on the choice of a $G\mathbb{Q}$-equivariant projection

$$\pi : V_{fgh}(N) \to V_{fgh},$$

where $V_{fgh}(N)$ is isomorphic to several copies of $V_{fgh}$. Note that any choice of $\pi$ amounts to a triplet $(\tilde{f}, \tilde{g}, \tilde{h})$ of Hecke eigenforms whose associated primitive forms are $(f, g, h)$ (see [4.5]).

For a good choice of $\pi$, a key result in [DR17a] (see also [BSV19a]) is an explicit reciprocity law of the form

$$\exp^*(\text{res}_p(\kappa(f, g_\alpha, h_{\alpha-1}))) = (\text{nonzero constant}) \cdot L(f \otimes g \otimes h, 1)$$

whereby the classes in (1.3) land in the Bloch–Kato Selmer group $\text{Sel}(\mathbb{Q}, V_{fgh}) \subset H^1(\mathbb{Q}, V_{fgh})$ precisely when the triple product $L$-series $L(f \otimes g \otimes h, s)$ vanishes at $s = 1$. One of the main conjectures of [DR16] then went further to formulate the following non-vanishing criterion.

**Conjecture 1.1 ([DR16, Conj. 3.2]).** The generalized Kato classes in (1.3) generate a non-trivial subspace of $\text{Sel}(\mathbb{Q}, V_{fgh})$ for a suitable choice of $\pi$ if and only if the following equivalent conditions are satisfied:

- (a) The $L$-series $L(f \otimes g \otimes h, s)$ has a double zero at $s = 1$.
- (b) The Mordell–Weil group $E(H)_L \otimes q_{gh} \mathbb{G}_m$ is two-dimensional over $L$, where $E(H)_L := E(H) \otimes \mathbb{Z} L$ and $q_{gh} := q_g \otimes q_h$.
- (c) The Selmer group $\text{Sel}(\mathbb{Q}, V_{fgh})$ is two-dimensional over $\mathbb{Q}_p$.

As noted in [DR17], Rem. 3.3], the equivalence of conditions (a), (b) and (c) is part of the equivariant BSD conjecture, the main novelty of Conjecture 1.1 being in providing a criterion for the non-triviality of the space generated by the generalized Kato classes. For the above $g$ and $h$, the decomposition

$$V_{fgh} \simeq (V_p E \otimes \text{Ind}_K^Q 1) \oplus (V_p E \otimes \text{Ind}_K^Q \chi^2)$$

gives rise to the factorization

$$L(f \otimes g \otimes h, s) = L(E, s) \cdot L(E^K, s) \cdot L(E/K, \chi^2, s),$$

where $E^K$ is the twist of $E$ by the quadratic character associated with $K$. Thus Conjecture 1.1 specialized to the case in which $E$ has rank 2 may be stated as follows, where we let

$$\kappa_{\alpha, \alpha^{-1}}, \kappa_{\alpha, \beta^{-1}}, \kappa_{\beta, \alpha^{-1}}, \kappa_{\beta, \beta^{-1}} \in H^1(\mathbb{Q}, V_p E)$$

be the natural image of the classes (1.3) under the projection $H^1(\mathbb{Q}, V_{fgh}) \to H^1(\mathbb{Q}, V_p E)$.

**Conjecture 1.2 ([DR16, §4.5.3]).** Assume that $L(E^K, s)$ and $L(E/K, \chi^2, s)$ are both nonzero at $s = 1$. Then the generalized Kato classes (1.3) generate a non-trivial subspace of $\text{Sel}(\mathbb{Q}, V_p E)$ for a suitable choice of $\pi$ if and only if the following equivalent conditions are satisfied:

- (a) $\text{ord}_{s=1} L(E, s) = 2$. 
(b) \( \text{rank}_Z E(\mathbb{Q}) = 2 \).
(c) \( \dim_{\mathbb{Q}_p} \text{Sel}(Q,V_p E) = 2 \).

Note that the non-vanishing assumptions in Conjecture 1.2 imply that \( E \) has root number +1. Also, here \( \text{Sel}(Q,V_p E) \) denotes the Selmer group fitting in the exact sequence

\[
0 \to E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}_p \to \text{Sel}(Q,V_p E) \to T_p \text{III}(E/\mathbb{Q}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to 0,
\]

and of course, the equivalence of condition (b) and (c) amounts to the finiteness of \( \text{III}(E/\mathbb{Q})[p^\infty] \), and the equivalence of (a) and (b) is the rank 2 case of the Birch–Swinnerton-Dyer conjecture.

### 1.3. Main result.

Let \( \bar{\rho}_{E,p} : G_\mathbb{Q} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{Aut}_{\mathbb{F}_p}(E[p]) \) be the residual Galois representation associated to \( E \), and write

\[
N_E = N^+N^-\n\]

with \( N^+ \) (resp. \( N^- \)) divisible only by primes which are split (resp. inert) in \( K \).

**Theorem A.** Assume that \( L(E^K,s) \) and \( L(E/K,\chi^2,s) \) are both nonzero at \( s = 1 \), and that:

- \( \bar{\rho}_{E,p} \) is irreducible,
- \( N^- \) is squarefree,
- \( \bar{\rho}_{E,p} \) is ramified at every prime \( q|N^- \).

Then \( \kappa_{\alpha,\alpha^{-1}} = \kappa_{\beta,\alpha^{-1}} = 0 \) for any choice of \( \pi \) and the following hold:

(i) If \( \text{rank}_Z E(\mathbb{Q}) = 2 \) and \( \text{III}(E/\mathbb{Q})[p^\infty] \) is finite, then \( \kappa_{\alpha,\alpha^{-1}} \neq 0 \) for a suitable choice of \( \pi \).

(ii) If \( \kappa_{\alpha,\alpha^{-1}} \) is a nonzero class in \( \text{Sel}(Q,V_p E) \), then \( \dim_{\mathbb{Q}_p} \text{Sel}(Q,V_p E) = 2 \).

In particular, if \( \text{III}(E/\mathbb{Q})[p^\infty] \) is finite, Conjecture 1.2 holds.

As before, note that the non-vanishing assumptions in Theorem A imply that \( E \) has root number +1 (and being squarefree, that \( N^- \) is the product of an odd number of primes), and either of the conditions in part (i) or (ii) imply that \( L(E,1) = 0 \) by [Kat04]. Thus the elliptic curves \( E \) in Theorem A all satisfy \( \text{ord}_{s=1} L(E,s) \geq 2 \). On the other hand, if the root number of \( E/\mathbb{Q} \) is +1 and \( \bar{\rho}_{E,p} \) is irreducible and ramified at some prime \( q \), by [BFH90] and [Vat03] there exist infinitely many imaginary quadratic fields \( K \) and ring class characters \( \chi \) of \( p \)-power conductor such that:

- \( p \) splits in \( K \),
- \( q \) is inert in \( K \),
- every prime factor of \( N/q \) splits in \( K \),
- \( L(E^K,1) \neq 0 \) and \( L(E/K,\chi^2,1) \neq 0 \).

Therefore, by Theorem A the generalized Kato classes in \([1,7]\) provide an explicit construction of non-trivial Selmer classes for rank 2 elliptic curves analogous to the construction of Heegner classes for rank 1 elliptic curves.

Moreover, our proof shows that up to multiplication by a nonzero scalar in \( \overline{\mathbb{Q}}^\times \), the classes \( \kappa_{\alpha,\alpha^{-1}} \) depend only on \( K \), not on the auxiliary choice of ring class character \( \chi \).

### 1.4. Outline of the proof.

We conclude this Introduction with a sketch of the proof of part (i) in Theorem A, establishing the non-vanishing of

\[
\kappa_{E,K} := \kappa_{\alpha,\alpha^{-1}} \in H^1(\mathbb{Q},V_p E)
\]

for a suitable choice of \( \pi \).

Let \( \Gamma \) be the Galois group of the anticyclotomic \( \mathbb{Z}_p \)-extension of \( K \). Building on Gross’s refinement of Waldspurger’s special value formula [Wal85, Gro87], one can construct a \( p \)-adic \( L \)-function \( \Theta_f/K \in \mathbb{Z}_p[\Gamma] \) interpolating “square-roots” of the central critical values \( L(E/K,\phi,1) \), as \( \phi \) runs over finite order characters of \( \Gamma \). Since its original construction [BD99], the element \( \Theta_f/K \) has been widely studied in the literature, but its place in Perrin-Riou’s influential vision
whereby $p$-adic $L$-functions arise as the image of $p$-adic families of special cohomology classes under generalized Coleman power series maps, remained mysterious.

Letting $\kappa(f, gh)$ be the $p$-adic family of diagonal cycle classes giving rise to $\kappa(f, g_\alpha, h_{\alpha^{-1}})$ at $\ell = 1$, in Section 4 we prove that, for a suitable choice of $\pi$,

$$\text{Col}^p(\text{loc}_p(\kappa(f, gh))) = \Theta_{f/K} \cdot (\text{explicit nonzero constant}),$$

where $\text{Col}^p$ is a generalized Coleman power series map defined in terms of an anticyclotomic variant of Perrin-Riou’s big exponential map. Viewing (1.8) as an identity in the power series ring $\mathbb{Z}_p[[T]] \simeq \mathbb{Z}_p[[T]]$, we are led to study the leading coefficient of (1.8) at $T = 0$.

Consider the filtration

$$\text{Sel}(K, V_pE) = S^{(1)} \supset S^{(2)} \supset \cdots \supset S^{(r)} \supset \cdots \supset S^{(\infty)}$$

defined by Bertolini–Darmon [BD95] and Howard [How04], and the associated anticyclotomic derived $p$-adic height pairings

$$h^{(r)} : S^{(r)} \times S^{(r)} \to \mathbb{Q}_p.$$  

From the properties of $h^{(r)}$, we deduce that under the assumption in part (i) of Theorem A the filtration (1.9) reduces to

$$\text{Sel}(\mathbb{Q}, V_pE) = S^{(1)} = S^{(2)} = \cdots = S^{(r)} = \cdots = S^{(\infty)} = \{0\}$$

for some $r \geq 2$. On the other hand, in Section 3 we establish a formula in the style of [Rnb94] for derived $p$-adic height pairings, which combined with (1.8) shows that

$$\kappa_{E,K} \in S^{(\rho)}, \quad \text{where } \rho := \text{ord}_{T=0} \Theta_{f/K}(T),$$

and that for every $x \in S^{(\rho)}$ we have

$$h^{(\rho)}(\kappa_{E,K}, x) = \left( \frac{d}{dt} \right)^\rho \Theta_{f/K}(T)|_{T=0} \cdot \log_p(x) \cdot (\text{nonzero constant in } \mathbb{Q}_p^\times),$$

from where the non-vanishing of $\kappa_{E,K}$ follows easily.

The proof of part (ii) of Theorem A follows from similar ideas, and in fact the above analysis yields the following stronger result.

Consider the strict Selmer group defined by

$$\text{Sel}_{\text{str}}(\mathbb{Q}, V_pE) := \ker (\log_p : \text{Sel}(\mathbb{Q}, V_pE) \to \mathbb{Q}_p)$$

where $\log_p$ denotes the restriction map $\text{Sel}(\mathbb{Q}, V_pE) \to E(\mathbb{Q}_p) \hat{\otimes} \mathbb{Q}_p$ composed with the formal group logarithm $E(\mathbb{Q}_p) \hat{\otimes} \mathbb{Q}_p \to \mathbb{Q}_p$.

**Theorem B.** Assume that $L(E^K, s)$ and $L(E/K, \chi^2, s)$ are both nonzero at $s = 1$, and that:

- $\bar{\rho}_{E,p}$ is irreducible,
- $N^-$ is square-free,
- $\bar{\rho}_{E,p}$ is ramified at every prime $q | N^-.$

Then $\kappa_{\alpha,\beta^{-1}} = 0$ for any choice of $\pi$ and the following statements are equivalent:

(a) \text{The class } $\kappa_{\alpha,\beta^{-1}}$ \text{ is a non-trivial element in } $\text{Sel}(\mathbb{Q}, V_pE)$ \text{ for a suitable choice of } $\pi$.

(b) $\dim_{\mathbb{Q}_p} \text{Sel}_{\text{str}}(\mathbb{Q}, V_pE) = 1$.

Moreover, if $\text{rank}_\mathbb{Z} E(\mathbb{Q}) = 2$ and $\text{III}(E/\mathbb{Q})[p^\infty]$ is finite, then up to $\mathbb{Q}_p^\times$ we have:

$$\kappa_{\alpha,\alpha^{-1}} = \log_p(Q) \cdot P - \log_p(P) \cdot Q, \quad \kappa_{\beta,\alpha^{-1}} = 0,$$

$$\kappa_{\beta,\beta^{-1}} = \log_p(Q) \cdot P - \log_p(P) \cdot Q,$$

where $(P, Q)$ is any basis of $E(\mathbb{Q}) \otimes \mathbb{Q}_p$.

In addition to easily implying Theorem A (see [5.1]), this yields some evidence towards the refinement of the “elliptic Stark conjecture” in [DLR15] that was also formulated in [DR16].
Remark 1.3. Part (ii) of Theorem A (which follows easily from the implication (b) \(\implies\) (a) in Theorem B) can be regarded as a rank two analogue of Kolyvagin’s theorem for Heegner points. However, by [Rub00, Thm. 2.2.3] the conclusion that \(\Sel_{str}(K,V_pE)\) is one-dimensional (as opposed to trivial) implies that \(\kappa_{\alpha,\alpha^{-1}}\) does not into an Euler system for \(V_pE\).

Remark 1.4. Dealing with elliptic curves \(E/Q\) of rank 2, the finiteness assumption on \(\III(E/Q)\) in Theorems A and B seems very difficult to remove at present. In the Appendix we list elliptic curves \(E/Q\) of rank 2 satisfying the hypotheses of these theorems (for suitably chosen \(p, K\) and \(\chi\)) and for which the finiteness of \(\III(E/Q)[p^\infty]\) can be verified, thus providing by virtue of Theorem A the first instances (as far as we know) of such elliptic curves with non-vanishing generalized Kato classes (cf. [DR16, §4.5.3]).

Acknowledgements. We would like to thank John Coates, Dick Gross, and Barry Mazur for their comments on earlier drafts of this paper.

2. Derived \(p\)-adic heights

In this section, we review the definition of the sequence of “derived” \(p\)-adic height pairings given in [How04, §4.5.3], and Howard’s generalization of Rubin’s formula, which will be our starting point in Section 3.

2.1. Notation and definitions. Let \(p > 2\) be a prime, let \(K\) be a number field, and let \(\Sigma\) be a finite set of places of \(K\) containing all archimedean places and all primes above \(p\). Let \(K_\Sigma\) be the maximal algebraic extension of \(K\) unramified outside \(\Sigma\) and set \(G_{K,\Sigma} = \Gal(K_\Sigma/K)\). Let \(K_\infty/K\) be a \(\Z_p\)-extension in \(K_\Sigma\). Denote by \(K_n\) the subfield with of \(K_\infty\) with \([K_n : K] = p^n\), and set

\[
\Gamma_n = \Gal(K_n/K), \quad \Gamma_\infty = \Gal(K_\infty/K), \quad \Lambda = \Z_p[[\Gamma_\infty]].
\]

Let \(\kappa_\Lambda : G_{K,\Sigma} \to \Gal(K_\infty/K) \to \Lambda^\times\) be the tautological character given by \(\kappa_\Lambda(\sigma) = \sigma|_{K_\infty}\), and for \(k \in \Z\) and a \(\Lambda\)-module \(M\) on which \(G_{K,\Sigma}\) acts, let \(M\{k\}\) be the \(G_{K,\Sigma}\)-module \(M\) with the \(G_{K,\Sigma}\)-action twisted by \(\kappa_\Lambda^k\).

Let \(O\) be a local ring finitely generated over \(\Z_p\) with maximal ideal \(m\), and denote by \(\Mod_O\) the category of finite free \(O\)-modules equipped with continuous \(O\)-linear action of \(G_{K,\Sigma}\). Put \(\Lambda_O = \Lambda \otimes_{\Z_p} O\). For an object in \(\Mod_O\) we put

\[
\hat{H}^1(K_\infty, T) := \varprojlim_n H^1(K_n, T),
\]

where the limit is with respect to corestriction, and denote by \(\text{pr}_{K_n} : \hat{H}^1(K_\infty, T) \to H^1(K_n, T)\) the canonical projection map. By Shapiro’s lemma and [How04 Lem. 1.4], there is a canonical identification

\[
\hat{H}^1(K_\infty, T) \simeq H^1(K, T_\Lambda),
\]

where \(T_\Lambda = T \otimes_O \Lambda_O\{1\} - \{1\}\).

2.2. Derived \(p\)-adic heights. Suppose now that \(O\) is Artinian. Let \(K\) be the localization of \(\Lambda_O\) at the prime \(m\Lambda_O\), and define \(P\) by the exactness of the sequence

\[
0 \to \Lambda_O \to K \to P \to 0.
\]

For an object in \(\Mod_O\), define the \(G_{K,\Sigma}\)-modules \(T_K := T_\Lambda \otimes_{\Lambda_O} K\) and \(T_P := T_\Lambda \otimes_{\Lambda_O} P\). By [How04 Lem. 1.5], the choice of a topological generator \(\gamma\) of \(\Gamma_\infty\) determines an isomorphism

\[
\eta_\gamma : H^1(K, T_P) \simeq H^1(K_\infty, T)
\]

with the property that

\[
\eta_\gamma(z \otimes (\gamma - 1)^{-1}) = \text{pr}_K(z) \in H^1(K, T)
\]

for \(z \in H^1(K, T_\Lambda)\).
Let $\mathcal{F}$ be a Selmer structure on $T_K$, i.e., a choice of $K$-submodule $H^1_{\mathcal{F}}(K_v, T_K) \subset H^1(K_v, T_K)$ for every $v \in \Sigma$, and define the Selmer module $H^1_{\mathcal{F}}(K, T_K)$ to be the kernel of the map

$$H^1(G_{K, \Sigma}, T_K) \to \prod_{v \in \Sigma} H^1(K_v, T_K)/H^1_{\mathcal{F}}(K_v, T_K).$$

The natural images of $H^1_{\mathcal{F}}(K_v, T_K)$ under the maps induced by (2.1) gives Selmer structures on $T_\lambda$ and $T_P$, which we use to define the Selmer modules $H^1_{\mathcal{F}}(K, T_\lambda)$ and $H^1_{\mathcal{F}}(K, T_P)$, respectively. We then let $H^1_{\mathcal{F}}(K, T)$ be the module consisting of the classes $s \in H^1(K, T)$ whose image in $H^1(K_{\infty}, T)$ belongs to $\eta_\gamma(H^1_{\mathcal{F}}(K, T_P))$, and set

$$H^1_{\mathcal{F}}(K, T) = \lim_{\rightarrow} H^1_{\mathcal{F}}(K_n, T) = \eta_\gamma(H^1_{\mathcal{F}}(K, T_P)),$$

which does not depend on the choice of $\gamma$.

Let $T^* := \text{Hom}(T, \mathcal{O}(1)) \in \text{Mod}_\mathcal{O}$, and denote by $e: T \times T^* \to \mathcal{O}(1)$ the canonical $G_{K, \Sigma}$-equivariant perfect paring, which naturally extends to a perfect $G_{K, \Sigma}$-equivariant pairing $e_\lambda: T_\lambda \times T^*_\lambda \to \mathcal{O}(1)$ satisfying

$$e_\lambda(t \otimes \lambda_1, s \otimes \lambda_2) = \lambda_1 \lambda_2 e_\lambda(t, s),$$

for all $\lambda_1, \lambda_2 \in \mathcal{O}$, where $\nu$ is the involution of $\mathcal{O}$ given by $\gamma \mapsto \gamma^{-1}$ on group-like elements. Tensoring $e_\lambda$ with $K$ we obtain the perfect pairing $e_K: T_K \times T_K \to K(1)$.

For any prime $v$ of $K$, let $(\cdot, \cdot)_v: H^1(K_v, T) \times H^1(K_v, T^*) \to \mathcal{O}$ be the perfect paring given by $(z, w)_v := \text{inv}_v(e(z \cup w))$, where $\text{inv}_v : H^2(K_v, \mathcal{O}(1)) \simeq \mathcal{O}$ is the invariant map. Similarly, define the bilinear pairing

$$(\cdot, \cdot)_{K_\infty, v}: H^1(K_v, T_\lambda) \times H^1(K_v, T^*_\lambda) \to H^2(K_v, \mathcal{O}(1)) \simeq \mathcal{O},$$

by $\langle z, w \rangle_{K_\infty, v} = \text{inv}_v(e_\lambda(z \cup w))$.

Let $\mathcal{F}^\perp$ be the Selmer structure on $T_K$ with $H^1_{\mathcal{F}^\perp}(K_v, T_K^\perp)$ given for every $v$ by the orthogonal complement of $H^1_{\mathcal{F}}(K_v, T_K)$ under local Tate pairing induced by $e_K$, and let

$$[-, -]_{\text{CT}}: H^1_{\mathcal{F}}(K, T_P) \times H^1_{\mathcal{F}^\perp}(K, T^*_P) \to P$$

be the generalized Cassels–Tate pairing constructed in [How04, Thm. 1.8].

Let $J$ be the augmentation ideal of $\mathcal{O}$, i.e., the principal ideal of $\mathcal{O}$ generated by $\gamma - 1$, and for $r > 0$ put

$$(2.4)\quad Y^{(r)}_T := H^1_{\mathcal{F}}(K_\infty, T)[J] \cap (\gamma - 1)^{r-1}H^1_{\mathcal{F}}(K_\infty, T)[J'],$$

which defines a decreasing filtration $H^1_{\mathcal{F}}(K_\infty, T)[J] = Y^{(1)}_T \supset Y^{(2)}_T \supset Y^{(3)}_T \supset \cdots$, and similarly for $T^*$.

Denote by $\phi_\gamma$ the composition

$$P \simeq \mathcal{O} \{1 \} \overset{\mathbb{Z}}{\longmapsto} \left( \lim_{\rightarrow} \text{Ind}_{K_n/K} \mathcal{O} \right) \{1\} \overset{\text{ev}}{\longrightarrow} \mathcal{O},$$

where the last arrow is given by evaluation at the identity element of $G_{K, \Sigma}$. By construction, for any $\lambda \in \Lambda$ one has

$$(2.5)\quad \phi_\gamma((\lambda \otimes (\gamma - 1)^{-1})^\ell) = \phi_\gamma(-\lambda^\ell \otimes (\gamma - 1)^{-1}) = \text{ev}(-\lambda^\ell) = -\text{ev}(\lambda)$$

(see [How04, p. 1324]).

**Definition 2.1.** The height pairing $h_\mathcal{O}: H^1_{\mathcal{F}}(K_\infty, T) \times H^1_{\mathcal{F}^\perp}(K_\infty, T^*) \to J/J^2$ is defined by $h_\mathcal{O}(z, w) := (\gamma - 1) \cdot \phi_\gamma([\eta_\gamma^{-1}(z), \eta_\gamma^{-1}(w)]_{\text{CT}}),$
and the $r$-th derived height pairing $h_{O}^{(r)}(-,-): Y_{T}^{(r)} \times Y_{T}^{(r)} \to J'/J'^{+1}$ is defined by
\[
h_{O}^{(r)}(z, w) := (\gamma - 1)^{-r} \cdot h_{O}(u, w),
\]
where $u \in H^{1}_{F}(K_{\infty}, T)[J']$ is any class such that $z = (\gamma - 1)^{-r}u$.

In particular, $h_{O}^{(1)}$ is the restriction of $h_{O}$ to $Y_{T}^{(1)} \times Y_{T}^{(1)}$. It is easy to see that these pairings are independent of the choice of $\gamma$.

The following result is a restatement of part (c) of Theorem 2.5 in [How04], which generalizes a formula of Rubin [Rub94] (cf. [Nek06, Prop. 11.5.11]).

**Proposition 2.2.** Let $z \in Y_{T}^{(r)}$ and $w \in Y_{T}^{(r)}$. Suppose that there exist a class $z \in H^{1}(K, T_{\Lambda})$ and a semi-local class $w_{\Sigma} = (w_{v}) \in \bigoplus_{v \in \Sigma} H^{1}_{F_{v}}(K_{v}, T_{\Lambda}^{1})$ such that $pr_{K}(z) = z$ and $pr_{K_{v}}(w_{v}) = loc_{v}(w)$ for all $v \in \Sigma$. Then
\[
h_{O}^{(r)}(z, w) = \sum_{v \in \Sigma} \langle z, w_{v} \rangle_{K_{\infty}, v} \pmod{J'^{+1}}.
\]

**Proof.** Write $z = (\gamma - 1)^{-r}u$ with $u \in H^{1}_{F}(K_{\infty}, T)$, and let $s$ and $t$ be cocycles representing $\eta_{\gamma}^{-1}(u) \in H^{1}_{F}(K, T_{P})$ and $\eta_{\gamma}^{-1}(w) \in H^{1}_{F_{p}}(K, T_{P})$, respectively. Choose cochains
\[
\tilde{s} \in C^{1}(G_{K}, \Sigma, T_{K}), \quad \tilde{t} \in C^{1}(G_{K}, \Sigma, T_{K}^{*})
\]
lifting $s$ and $t$ under the maps induced by the projections $T_{K} \to T_{P}$ and $T_{K}^{*} \to T_{P}^{*}$, respectively. The image of $\tilde{s} \cup \tilde{t}$ in $C^{3}(G_{K}, \Sigma, P(1))$ is then easily seen to be expressible as $de_{0}$ for some $e_{0} \in C^{2}(G_{K}, \Sigma, P(1))$. After choosing a class $\tilde{t}_{\Sigma} \in \bigoplus_{v \in \Sigma} H^{1}_{F_{v}}(K_{v}, T_{\Lambda})$ lifting $loc_{\Sigma}(t)$, according to the definition of the generalized Cassels–Tate pairing in [How04] (2), page 1321, we have
\[
h_{O}^{(r)}(z, w) = (\gamma - 1)^{r} \cdot \phi_{\gamma}(\text{inv}_{\Sigma}(loc_{\Sigma}(\tilde{s}) \cup \tilde{t}_{\Sigma} - loc_{\Sigma}(e_{0}))),
\]
where
\[
\text{inv}_{\Sigma} : \bigoplus_{v \in \Sigma} H^{2}(K_{v}, P(1)) \to P
\]
is the sum of the local invariants. Now let $\tilde{z} \in C^{1}(G_{K}, \Sigma, T_{\Lambda})$ and $\tilde{w}_{\Sigma} \in \bigoplus_{v \in \Sigma} C^{1}(G_{K_{v}}, T_{\Lambda}^{*})$ be cocycles representing $z$ and $w_{\Sigma}$. Then [2.3] shows that $\tilde{z} \otimes (\gamma - 1)^{-r}t$ and $\tilde{w}_{\Sigma} \otimes (\gamma - 1)^{-r}$ are liftings of $s$ and $loc_{\Sigma}(t)$, respectively, and with these choices of $\tilde{s}$ and $\tilde{t}_{\Sigma}$ in (2.6) (with $e_{0} = 0$, since $d\tilde{z} = 0$), we obtain
\[
h_{O}^{(r)}(z, w) = (\gamma - 1)^{r} \cdot \phi_{\gamma}(\text{inv}_{\Sigma}(\tilde{z}((\gamma - 1)^{r}) \cup \tilde{w}_{\Sigma}((\gamma - 1)^{r}))) \in J'/J'^{+1}
\]
\[
= -\text{inv}_{\Sigma}(e_{0}(loc_{\Sigma}(\tilde{z}))) - \sum_{v \in \Sigma} \langle z, w_{v} \rangle_{K_{\infty}, v} \pmod{J'^{+1}},
\]
using (2.5) for the second equality. This completes the proof. \qed

2.3. Derived $p$-adic heights for ordinary elliptic curves. Let $E$ be an elliptic curve over the number field $K$ with good ordinary reduction at all primes of $K$ above $p$ and $T = T_{p}E$ be the $p$-adic Tate module of $E$, and assume that $\Sigma$ contains the archimedean places, the primes of $K$ above $p$, and the primes at which $E$ has bad reduction.

Let $T_{k} = E[p^{k}]$, and consider the modules $Y_{T_{k}}^{(r)}$ in (2.4) equipped with the ordinary Selmer structure $F$ in [How04, Def. 3.2]. The Weil pairing $(,)_{\text{Weil}} : T_{k} \times T_{k} \to \mu_{p^{k}}$ yields an identification $T_{k}^{*} \simeq T_{k}$ under which $F$ is its own orthogonal complement. The discussion of [2.2] thus yields derived $p$-adic height pairings $h_{Z/p^{k}Z}^{(r)}$ on $Y_{T_{k}}^{(r)}$. The constructions are compatible under the natural maps and $Z/p^{k+1}Z \to Z/p^{k}Z$ and $T_{k+1} \to T_{k}$, and in the limit they define
\[
h^{(r)} := \lim_{\leftarrow} h_{Z/p^{k}Z}^{(r)}, \quad Y^{(r)} := \lim_{\leftarrow} Y_{T_{k}}^{(r)}.
\]
Let \( \text{Sel}(K, V_p E) = (\lim \text{Sel}_p(E/K)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) be the usual pro-\( p \) Selmer group with \( p \) inverted. As shown in [How04, Lem. 4.1], there is canonical isomorphism

\[
Y_T^{(1)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \text{Sel}(K, V_p E).
\]

Letting \( S_p^{(r)}(E/K) \) be the subspace of \( \text{Sel}(K, V_p E) \) spanned by the image of \( Y_T^{(r)} \subset Y_T^{(1)} \) under the isomorphism \((2.7)\), we thus obtain derived \( p \)-adic height pairings

\[
h^{(r)} : S_p^{(r)}(E/K) \times S_p^{(r)}(E/K) \to (J^r/J^{r+1}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,
\]

where \( J \) is the augmentation ideal of \( \Lambda = \mathbb{Z}_p[[\Gamma_\infty]] \).

**Corollary 2.3.** Let \( z, w \in S_p^{(r)}(E/K) \). Suppose that there exist a class \( z \in \hat{H}^1(K_\infty, T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) and classes \( (w_v) \in \bigoplus_v \hat{H}^1(K_{\infty,v}, T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) such that \( \text{pr}_K(z) = z \) and \( \text{pr}_{K_v}(w_v) = \text{loc}_v(w) \). Then

\[
h^{(r)}(z, w) = - \sum_{v | p} \langle \text{loc}_v(z), w_v \rangle_{K_{\infty,v}} \quad \text{(mod } J^{r+1})\text{.}
\]

**Proof.** This follows from Proposition 2.2 and the fact that \( H^1(K_{\infty,v}, T) \otimes \mathbb{Q}_p = 0 \) for \( v \nmid p \). \( \square \)

We conclude this section with Theorem 2.4 below, relating the degeneracies of \( h^{(r)} \) to the \( \Lambda \)-module structure of

\[
X_\infty := \text{Hom}_{\mathbb{Z}_p}(\text{Sel}_{p\infty}(E/K_\infty), \mathbb{Q}_p/\mathbb{Z}_p)),
\]

where \( \text{Sel}_{p\infty}(E/K_\infty) \subset H^1(E/\mathbb{Q}[p^{\infty}]) \) is the usual \( p^{\infty} \)-Selmer group.

From now on, we assume that \( p \) is unramified in \( K \) and that the primes of \( K \) above \( p \) ramify in the \( \mathbb{Z}_p \)-extension \( K_{\infty}/K \). Let

\[
\hat{\rho}_{E,p} : G_K \to \text{Aut}_{\mathbb{F}_p}(E[p])
\]

be the two-dimensional Galois representation on the \( p \)-torsion of \( E \).

**Theorem 2.4.** There is a filtration

\[
\text{Sel}(K, V_p E) = S_p^{(1)}(E/K) \supset S_p^{(2)}(E/K) \supset \cdots
\]

and a sequence of height pairings

\[
h^{(r)} : S_p^{(r)}(E/K) \times S_p^{(r)}(E/K) \to (J^r/J^{r+1}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p
\]

with the following properties:

1. \( S_p^{(r+1)}(E/K) \) is the null-space of \( h^{(r)} \).
2. \( S_p^{(\infty)}(E/K) := \bigcap_{r=1}^{\infty} S_p^{(r)}(E/K) \) is the subspace of universal norms for \( K_\infty/K \).
3. \( h^{(r)}(x, y) = (-1)^{r+1} h^{(r)}(y, x) \).
4. If \( K' \) is a subextension of \( K \) with \( K_{\infty}/K \) and the \( G_K \)-action on \( T = T_p E \) extends to an action of \( G_{K'} \), then

\[
h^{(r)}(x^\tau, y^\tau) = \omega(\tau)^r h^{(r)}(x, y)
\]

for all \( \tau \in \text{Gal}(K/K') \), where \( \omega \) is the \( \mathbb{Z}_p^{\times} \)-valued character defined by \( \tau \gamma \tau^{-1} = \gamma^{\omega(\tau)} \) for \( \gamma \in \Gamma_\infty \) and \( \tau \in \text{Gal}(K_{\infty}/K') \).

Moreover, assume that for every prime \( v \) of \( K \) such that:

- \( E \) has bad reduction at \( v \), and
- \( v \) is infinitely decomposed in \( K_{\infty}/K \),
the reduction type of $E$ at $v$ is multiplicative, and $\hat{\rho}_{E,p}$ is ramified at $v$. Then, fixing a $\Lambda$-module pseudo-isomorphism
\[ X_\infty \sim \Lambda^{\mathbb{F}_p} \oplus M \oplus M' \]
with $M'$ a torsion $\Lambda$-module with characteristic ideal prime to $M$, and writing $M$ in the form
\[ M \simeq (\Lambda/J)^{\mathbb{F}_p} \oplus (\Lambda/J^2)^{\mathbb{F}_p} \oplus \cdots, \]
we have
\[ (5) \ e_\infty = \dim_{\mathbb{F}_p} S_p^{(\infty)} (E/K) \quad \text{and} \quad e_r = \dim_{\mathbb{F}_p} (S_p^{(r)} (E/K) / S_p^{(r+1)} (E/K)) \quad \text{for all} \ r \geq 1. \]

Proof. Assuming that the primes of bad reduction of $E$ are finitely decomposed in $K_\infty$ (but no conditions on $\hat{\rho}_{E,p}$), the result is shown in [How04] Thm. 4.2, Cor. 4.3. Since for our purposes we need to allow primes of bad reduction that split completely in $K_\infty$, we explain how to adapt the arguments in [How04] under the above hypothesis on $\hat{\rho}_{E,p}$. For notational simplicity, we assume below that the primes of bad reduction of $E$ that are infinitely decomposed in $K_\infty$ split completely in $K_\infty$ (an assumption that on the other hand will hold in our application).

Following the notations in [How04] p. 1331, let
\[ S := \lim_{\rightarrow k} \lim_{\rightarrow n} \text{Ind}_{K_n/K} E[p^k], \]
so by Shapiro’s lemma $H^1(K, S) \simeq H^1(K_\infty, E[p^\infty])$ and
\[ H^1(K_v, S[p^k]) \simeq \bigoplus_{n, w | v} H^1(K_n, w, E[p^k]) \]
for any place $v$ of $K$, where $w$ runs over the primes of $K_n$ lying above $v$.

Let $Y_k = H^1_\infty(K_\infty, E[p^k])$ be as in [How04] p. 1334, so $Y_k[J] = Y_k^{(1)}$ in the notations above. The assumption made in [How04] Thm. 4.2, Cor. 4.3 that the primes of bad reduction of $E$ are finitely decomposed in $K_\infty$ is only used to show that the natural map
\[ Y_k \rightarrow \text{Sel}_{p,\infty} (E/K_\infty)[p^k] \]
has finite kernel and cokernel bounded independently of $k$, which follows from Propositions 3.4 and 3.5 of [How04], as shown in [loc.cit., p. 1334]. Let $v$ be a prime of bad reduction of $E$ (so in particular, $v \nmid p$ is inert in $K$) which splits completely in $K_\infty$ and let
\[ (2.8) \ H^1_{\text{ord}}(K_v, S[p^k]) = H^1_{\text{ur}}(K_v, S[p^k]) := \ker(H^1(K_v, S[p^k]) \rightarrow H^1(K_{\text{ur}}, S[p^k])). \]
To adapt the proof of those propositions for such $v$, it suffices to show that the module $H^1_{\text{ord}}(K_v, S) := \lim_{\twoheadrightarrow} H^1_{\text{ord}}(K_v, S[p^k])$ vanishes, and that the kernel of the natural map
\[ (2.9) \ H^1(K_v, S[p^k]) / H^1_{\text{ord}}(K_v, S[p^k]) \rightarrow H^1(K_v, S) / H^1_{\text{ord}}(K_v, S) \]
is bounded independently of $k$. (Assuming $v$ is finitely decomposed in $K_\infty$, Howard shows that both (2.8) and the kernel of (2.9) are trivial using [How04] Lem. 1.7.) By Shapiro’s lemma and inflation-restriction, we have identifications
\[ H^1_{\text{ur}}(K_v, S[p^k]) \simeq \ker(H^1(K_v, E[p^k]) \otimes \Lambda^\vee \rightarrow H^1(K_{\text{ur}}, E[p^k]) \otimes \Lambda^\vee) \]
\[ \simeq H^1(F_v, E[p^k]^{I_v}) \otimes \Lambda^\vee \]
\[ = (E[p^k]^{I_v} / (F_v - 1) E[p^k]^{I_v}) \otimes \Lambda^\vee, \]
where $F_v$ is the residue field of $K_v$, $F_v$ is a Frobenius element at $v$, and $\Lambda^\vee = \text{Hom}_{\mathbb{Z}_p}(\Lambda, \mathbb{Q}_p / \mathbb{Z}_p)$. Since we assume that any $v$ as above is a prime of multiplicative reduction for $E$, by Tate’s uniformization we have
\[ E[p^\infty] \sim \begin{pmatrix} \varepsilon_{\text{cyc}} & * \\ 0 & 1 \end{pmatrix} \]
as $G_{K_v}$-modules, where $\varepsilon_{\text{cyc}}$ is the $p$-adic cyclotomic character. Since we also assume that $\bar{\rho}_{E,p}$ is ramified at $v$, the image of $\ast^v$ in the above matrix generates $Q_p/\mathbb{Z}_p$. Thus we see that

$$E[p^\infty]^I_v/(Fr_v - 1)E[p^\infty]^I_v = 0,$$

which by (2.10) implies the vanishing of $H^1_{\text{ord}}(K_v, S)$.

On the other hand, from the preceding calculation we see that to obtain the desired bound on the kernel of (2.9) it suffices to show that the natural map $H^1(K_{ur}^w, S[p^k]) \to H^1(K_v^w, S)$ is injective. A similar argument as before shows that this map is identified with the natural map

$$H^1(K_{ur}^w, E[p^k]) \otimes \Lambda^v \to H^1(K_v^w, E[p^\infty]) \otimes \Lambda^v = E[p^\infty]^I_v \otimes \Lambda^v,$$

where $E[p^k]^I_v \simeq p^{-k} \mathbb{Z}_p/\mathbb{Z}_p$ is the $I_v$-coinvariants of $E[p^k]$ for $k \leq \infty$. It follows that the map (2.9) is injective. □

### 3. Rubin’s formula for derived $p$-adic heights

In this section we explicitly compute the derived $p$-adic height pairings for $p$-ordinary elliptic curves via Perrin-Riou’s big exponential maps. The main result of this section is Theorem 3.7.

#### 3.1. Preliminaries

We begin by reviewing the generalization of Perrin-Riou’s theory [PR94] to Lubin–Tate formal groups developed in [Kob18]. Fix a completed algebraic closure $C_p$ of $Q_p$. Let $Q_{ur} \subset C_p$ be the maximal unramified extension of $Q_p$, and let $Fr \in \text{Gal}(Q_{ur}/Q_p)$ be the absolute Frobenius. Let $F \subset Q_{ur}$ be a finite unramified extension of $Q$ with valuation ring $\mathcal{O} = \mathcal{O}_F$ and set

$$R = \mathcal{O}[X].$$

Let $\mathcal{F} = \text{Spf } R$ be a relative Lubin–Tate formal group of height one defined over $\mathcal{O}$, and for each $n \in \mathbb{Z}$ set

$$\mathcal{F}^{(n)} := \mathcal{F} \times_{\text{Spec } \mathcal{O}_F - n \text{ Spec } \mathcal{O}} \text{Spec } \mathcal{O}.$$  

The Frobenius morphism $\varphi_{\mathcal{F}} \in \text{Hom}(\mathcal{F}, \mathcal{F}^{(-1)})$ induces a homomorphism $\varphi_{\mathcal{F}} : R \to R$ defined by

$$\varphi_{\mathcal{F}}(f) := f^{Fr} \circ \varphi_{\mathcal{F}},$$

where $f^{Fr}$ is the conjugate of $f$ by Fr. Let $\psi_{\mathcal{F}}$ be the left inverse of $\varphi_{\mathcal{F}}$ satisfying

$$\varphi_{\mathcal{F}} \circ \psi_{\mathcal{F}}(f) = p^{-1} \sum_{x \in F[p]} f(X) \psi_{\mathcal{F}}(X).$$

Let $F_\infty = \bigcup_{n \geqslant -1} F(F[p^n])$ be the Lubin–Tate $\mathbb{Z}_p^{\times}$-extension associated with $\mathcal{F}$, and for every $n \geqslant -1$, let $F_n$ be the subfield of $F_\infty$ with $\text{Gal}(F_n/F) \simeq (\mathbb{Z}/p^{n+1}\mathbb{Z})^{\times}$ (so $F_{-1} = F$). Letting $G_\infty = \text{Gal}(F_\infty/F)$, there is a unique decomposition $G_\infty = \Delta \times \Gamma_\infty^F$, where $\Delta \simeq \text{Gal}(F_0/F)$ is the torsion subgroup of $G_\infty$ and $\Gamma_\infty^F \simeq \mathbb{Z}_p$.

For every $a \in \mathbb{Z}_p^\times$, there is a unique formal power series $[a] \in R$ such that

$$[a]^{Fr} \circ \varphi_{\mathcal{F}} = \varphi_{\mathcal{F}} \circ [a] \quad \text{and} \quad [a](X) \equiv aX \pmod{X^2}.$$  

Letting $\varepsilon_{\mathcal{F}} : G_\infty \to \mathbb{Z}_p^\times$ be the Lubin–Tate character, we let $\sigma \in G_\infty$ act on $f \in R$ by

$$\sigma.f(X) := f([\varepsilon_{\mathcal{F}}(\sigma)](X)),$$

thus making $R$ into an $\mathcal{O}[G_{\infty}]$-module.

**Lemma 3.1.** $R^{\psi_{\mathcal{F}} = 0}$ is free of rank one over $\mathcal{O}[G_{\infty}]$.

**Proof.** This is a standard fact. See [Kob18, Prop. 5.4]. □
Let $L$ be a finite extension of $\mathbb{Q}_p$, and $V$ be a crystalline representation of $G_{\mathbb{Q}_p}$ defined over $L$. Let $D(V) = D_{\text{cris}} \mathbb{Q}_p(V)$ be the filtered $\varphi$-module associated with $V$ over $\mathbb{Q}_p$, and set

$$\mathcal{D}_\infty(V) := D(V) \otimes_{\mathbb{Z}_p} R_{\varphi=0} \simeq D(V) \otimes_{\mathbb{Z}_p} \mathcal{O}[G_\infty].$$

Fix an invariant differential $\omega_F \in \Omega_R$, and let $\log_F \in R \otimes \mathbb{Q}_p$ be the logarithm map satisfying

$$\log_F(0) = 0 \quad \text{and} \quad d \log_F = \omega_F,$$

where $d : R \to \Omega_R$ be the standard derivation. Let $\partial : R \to R$ be defined by $df = \partial f \cdot \omega_F$.

Let $\epsilon = (\epsilon_n) \in T_p \mathcal{F} = \lim_{\leftarrow} \mathcal{F}^{(n+1)}[p^{n+1}]$ be a basis of the $p$-adic Tate module of $\mathcal{F}$, where the limit is with respect to the maps $\varphi^{\text{Fr}-(n+1)} : \mathcal{F}^{(n+1)}[p^{n+1}] \to \mathcal{F}^{(n)}[p^n]$. As in [Kob18, p. 42], we associate to $\epsilon$ and $\omega_F$ a $p$-adic period $t_\epsilon \in B^+_{\text{cris}}$ as follows. For each $n$, there exists a unique isomorphism $\varphi_n^\epsilon : \mathcal{F}^{(n)} \to \mathcal{F}$ such that

$$\varphi^{\text{Fr}^{-1}} \circ \cdots \circ \varphi^{\text{Fr}-(n-1)} \circ \varphi^{\text{Fr}^{-n}} = [p^n] \circ \varphi_n^\epsilon.$$

Put $w_n := \varphi_n^\epsilon(\epsilon_n-1) \in \mathcal{F}[p^n]$, so that $[p](w_n) = w_n-1$ by definition. Let $A_{\text{inf}} = A_{\text{inf}}(\mathcal{O}_{\mathbb{C}_p}/\mathcal{O}_F)$ and $\theta : A_{\text{inf}} \to \mathcal{O}_{\mathbb{C}_p}$ be as defined in [Fon91]. It is not difficult to show that there is a unique sequence $(\tilde{w}_n)$ of elements in $\mathcal{F}(A_{\text{inf}})$ such that $[p](\tilde{w}_n) = \tilde{w}_n-1$ and $\theta(\tilde{w}_n) = w_n$, and we set $t_\epsilon := \log_F(\tilde{w}_0) \in B^+_{\text{cris}}$. This satisfies

$$\log_F(0) = 0 \quad \text{and} \quad \varphi_{t_\epsilon} = \omega_F t_\epsilon,$$

where $\omega$ is the uniformizer in $F$ such that $\varphi^* (\omega_F^2) = \omega \cdot \omega_F$.

Fix an extension $\tilde{\epsilon}_F : \text{Gal}(F_{\text{cyc}}/\mathbb{Q}_p) \to L^\times$ of the Lubin–Tate character $\epsilon_F$, and for each $j \in \mathbb{Z}$ let $V(j) := V \otimes_L \tilde{\epsilon}_F^j$ denote the $j$-th Lubin–Tate twist of $V$. Then

$$D_{\text{cris}}(V(j)) = D(V) \otimes_{\mathbb{Q}_p} F t_\epsilon^{-j}.$$

Define the derivation $d_\epsilon : \mathcal{D}_\infty(V(j)) \to \mathcal{D}_\infty(V(j-1))$ by

$$d_\epsilon f := \eta t_\epsilon \otimes \partial g,$$

writing $f = \eta \otimes g \in D_{\text{cris}}(V(j)) \otimes \mathcal{O} R_{\varphi=0}^{\psi=0}$, and the map

$$\tilde{\Delta} : \mathcal{D}_\infty(V) \to \bigoplus_{j \in \mathbb{Z}} \mathcal{D}_{\text{cris}}(V(-j)) / \mathcal{O}$$

by $f \mapsto (\partial^j f(0)t_\epsilon^j \mod (1-\varphi))$.

Remark 3.2. If $\mathcal{F} = \hat{G}_m$, then $F_{\text{cyc}} = F(\zeta_{p^\infty})$, $\epsilon_F$ is the cyclotomic character $\epsilon_{\text{cyc}} : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$, $\varphi_F(f) = f^{\text{Fr}}((1+X)p^{-1})$, and $\psi_F(f)$ is given by the unique power series such that

$$\varphi_{\hat{G}_m} \circ \psi_{\hat{G}_m}(f) = p^{-1} \sum_{\zeta_{p^n} - 1} f(\zeta(1+X)-1).$$

If we take $\omega_{\hat{G}_m}$ to be the invariant differential $(1+X)^{-1}dX$, then $\partial = (1+X)^{-d}$ and $\log_{\hat{G}_m}$ is the usual logarithm $\log(1+X)$.

In the following, we fix a sequence $\{\zeta_{p^n}\}_{n=1,2,3,\ldots}$ of primitive $p^n$-th roots of unity with $\zeta_{p^{n+1}} = \zeta_{p^n}$, and let $t \in B^+_{\text{cris}}$ be the period $t_\epsilon$ corresponding to $\omega_{\hat{G}_m}$ and the basis $(\zeta_{p^{n+1}}-1) \in T_p \hat{G}_m$. 

3.2. Perrin-Riou’s big exponential map and the Coleman map. For a finite extension \( K \) over \( \mathbb{Q}_p \), let
\[
\exp_{K,V} : D(V) \otimes_{\mathbb{Q}_p} K \to H^1(K,V)
\]
be Bloch–Kato’s exponential map [BK90, §3]. In this subsection, we recall the main properties of Perrin-Riou’s map \( \Omega_{V,h} \) interpolating \( \exp_{K,V(j)} \) as \( j \) runs over non-negative integers \( j \).

Let \( V^* := \text{Hom}_L(V, L(1)) \) be the Kummer dual of \( V \) and denote by
\[
[-,-]_V : D(V^*) \otimes K \times D(V) \otimes K \to L \otimes K
\]
the \( K \)-linear extension of the de Rham pairing
\[
\langle , \rangle_{dR} : D(V^*) \times D(V) \to L.
\]
Let \( \exp_{K,V}^* : H^1(K,V) \to D(V) \otimes K \) be the Bloch–Kato dual exponential map, which characterized uniquely by
\[
\text{Tr}_{K/Q_p}([x, \exp_{K,V}^*(y)]_V) = \langle \exp_{K,V}(x), y \rangle_{dR},
\]
for all \( x \in D(V^*) \otimes K, y \in H^1(K,V) \).

Choose a \( G_{Q_p} \)-stable \( \mathcal{O}_L \)-lattice \( T \subset V \), and set \( \hat{H}^1(F_{\infty}, T) = \lim_{\leftarrow} H^1(F_n, T) \) and
\[
\hat{H}^1(F_{\infty}, V) = \hat{H}^1(F_{\infty}, T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.
\]
(This does not depend on the choice of \( T \).) Denote by
\[
\text{Tw}_j : \hat{H}^1(F_{\infty}, V) \simeq \hat{H}^1(F_{\infty}, V(j))
\]
the twisting map by \( \tilde{e}_F^j \). For a non-negative real number \( r \), put
\[
\mathcal{H}_{r,K}(X) = \left\{ \sum_{n \geq 0, \tau \in \Delta} c_{n,\tau} \cdot \tau \cdot X^n \in K[\Delta][X] \mid \sup_n |c_{n,\tau}|_p n^{-\tau} < \infty \text{ for all } \tau \in \Delta \right\},
\]
where \( \cdot |_p \) is the normalized valuation of \( K \) with \( |p|_p = p^{-1} \). Let \( \gamma \) be a topological generator of \( \Gamma_F^\infty \), and denote by \( \mathcal{H}_{r,K}(G_{\infty}) \) the ring of elements \( \{ f(\gamma - 1) : f \in \mathcal{H}_{r,K}(X) \} \), so in particular \( \mathcal{H}_{0,K}(G_{\infty}) = \mathcal{O}_K [G_{\infty}] \otimes \mathbb{Q}_p \).

Put
\[
\mathcal{H}_{\infty,K}(G_{\infty}) = \bigcup_{r > 0} \mathcal{H}_{r,K}(G_{\infty}).
\]

Let \( F \subset F_n \subset F_{\infty} \) be as in [3.1] and define a map
\[
\Xi_{n,V} : D(V) \otimes_{\mathbb{Q}_p} \mathcal{H}_{\infty,F}(X) \to D(V) \otimes_{\mathbb{Q}_p} F_n
\]
by
\[
\Xi_{n,V}(G) := \begin{cases} p^{-(n+1)} \varphi^{-(n+1)}(G F^{-(n+1)}(\epsilon_n)) & \text{if } n \geq 0, \\ (1 - p^{-1} \varphi^{-1})(G(0)) & \text{if } n = -1. \end{cases}
\]

Let \( h \) be a positive integer such that \( D(V) = \text{Fil}^{-h} D(V) \) and assume that \( H^0(F_{\infty}, V) = 0 \).

**Theorem 3.3** (Perrin-Riou, Colmez, Kobayashi, Shaowei Zhang). Let \( \tilde{\Lambda} := \mathbb{Z}_p [G_{\infty}] \). There exists a big exponential map
\[
\Omega_{V,h} : \mathcal{D}_\infty(V)_{\tilde{\Lambda}=0} \to \hat{H}^1(F_{\infty}, T) \otimes_{\tilde{\Lambda}} \mathcal{H}_{\infty,F}(G_{\infty})
\]
which is \( \tilde{\Lambda} \)-linear and characterized by the following interpolation property. Let \( g \in \mathcal{D}_\infty(V)_{\tilde{\Lambda}=0} \). If \( j \geq 1 - h \), then
\[
\text{pr}_{F_n}(\text{Tw}_j \circ \Omega_{V,h}(g)) = (-1)^{h+j-1}(h+j-1)! \cdot \exp_{F_n,V(j)}(\Xi_{n,V}(j)(d_{\epsilon}^{-j}G)) \in H^1(F_n, V(j)),
\]
and if \( j \leq -h \), then
\[
\exp_{F_n,V(j)}^*(pr_{F_n}(Tw_j \circ \Omega_{V,j}^*(g))) = \frac{1}{(-h - j)!} : \Xi_n \cdot V_{j}(d_{e,j}^{-1}G) \in D(V(j)) \otimes \mathbb{Q}_p F_n,
\]
where \( G \in D(V) \otimes \mathbb{Q}_p, \mathcal{H}_{h,F}(X) \) is a solution of the equation
\[
(1 - \varphi \otimes \varphi_F)G = g.
\]
Moreover, if \( D_{[s]} \subset D(V) \) is a \( \varphi \)-invariant in which all eigenvalues of \( \varphi \) have \( p \)-adic valuation \( s \), then \( \Omega_{V,j}^*(\mathcal{H}_{h,F}(G)) = \hat{H}^1(F_{\infty},T) \otimes \mathcal{H}_{s+h,F}(G_{\infty}) \)
\[
\square
\]
Proof. For \( \mathcal{F} = \hat{G}_m \), the construction of \( \Omega_{V,j}^* \) and the proof of its interpolation property for \( j \geq 1 - h \) is due to Perrin-Riou \([PR94, \S 3.2.3 \text{ Théorème}, \S 3.2.4(i)] \), while the interpolation formula for \( j \leq -h \) is a consequence of the proof of Perrin-Riou’s “explicit reciprocity law” by Colmez \([Col98 \text{ Théorème IX.4.5}] \). The extension of these results to general relative Lubin–Tate formal groups of height one is well understood. For example, the details for the construction of \( \Omega_{V,j}^* \) and the interpolation property at \( j \geq 1 - h \) are given in \([Kob18, \text{ Appendix}] \), and the extension of Colmez’s proof of the explicit reciprocity law is given in \([Zha04 \text{ Theorem 6.2}] \).

To introduce the Coleman map, we further assume the following hypothesis:
\[
\mathcal{D}_\infty(V) \hat{\Delta} = \mathcal{D}_\infty(V).
\]
For simplicity, we shall write \( \mathcal{H}_K \) for \( \mathcal{D}_\infty(K(G_\infty)) \) in the sequel. We let \([-,-]|_V : D(V^*) \otimes \mathbb{Q}_p \mathcal{H}_F \times D(V) \otimes \mathbb{Q}_p \mathcal{H}_F \to L \otimes \mathbb{Q}_p \mathcal{H}_F\)
\[
\text{be the pairing defined by}
\]
\[
[\eta_1 \otimes \lambda_1, \eta_2 \otimes \lambda_2]|_V = \langle \eta_1, \eta_2 \rangle_{dR} \otimes \lambda_1 \lambda_2
\]
for all \( \lambda_1, \lambda_2 \in \mathcal{H}_F \). For any \( e \in R^{\psi = 0} \) and \( \psi \) a generator of \( T_p \mathcal{F} \), there is unique \( \mathcal{O}_L[\mathcal{G}_\infty] \)-linear \( \text{Coleman map} \) \( \text{Col}_e^*: \hat{H}^1(F_\infty, V^*) \to D(V^*) \otimes \mathbb{Q}_p \mathcal{H}_F \) characterized by
\[
\text{Tr}_{F/Q_p}((\text{Col}_e^*(\mathcal{L}), \eta)|_V) = \langle \mathcal{L}, \Omega_{V,j}^*(\eta \circ e) \rangle_{F_\infty} \in L \otimes \mathbb{Q}_p \mathcal{H}_F
\]
for all \( \eta \in D(V) \).

Let \( \mathcal{Q} \) be the completion of \( \mathbb{Q}_p^{ur} \) in \( \mathbb{C}_p \), with ring of integers \( \mathcal{W} \), and set \( \mathbb{F}_n^{ur} = \mathbb{F}_n \mathbb{Q}_p^{ur} \) for \(-1 \leq n \leq \infty \) (so \( \mathbb{F}_1^{ur} = \mathbb{F}_p^{ur} \)). Let \( \sigma_0 \in \text{Gal}(\mathbb{F}_\infty^{ur}/\mathbb{Q}_p) \) be such that \( \sigma_0|_{\mathbb{Q}_p^{ur}} = \text{Fr} \) is the absolute Frobenius. Fix an isomorphism \( \rho : \hat{G}_m = \mathcal{F} \) defined over \( \mathcal{O} \) and let \( \rho : \mathcal{W}[T] \approx R \otimes \mathcal{O} \mathcal{B} \) be the map defined by \( \rho(f) = f \circ \rho^{-1} \), so \( \varphi F \circ \rho = \rho \circ \varphi_{G_m} \).

Let \( e \in R^{\psi = 0} \) be a generator over \( \mathcal{O}[\mathcal{G}_\infty] \) and write \( \rho(1+X) = h_e \cdot e \) for some \( h_e \in \mathcal{W}[\mathcal{G}_\infty] \).

This implies that \( \epsilon(0) \in O^\times \). Let \( \epsilon = (\epsilon_n)_{n=0,1,2,...} \) be the generator of \( T_p \mathcal{F} \) given by
\[
\epsilon_n = \rho\text{Fr}^{-(n+1)}(\zeta_{p,n+1} - 1) \in \mathcal{F}(n+1)[p^{n+1}].
\]

Let \( \eta \in D(V) \) be such that \( \varphi \eta = \alpha \eta \) and of slope \( s \) (i.e. \( \alpha|_p = p^{-s} \)). For every \( \mathcal{L} \in \hat{H}^1(F_\infty, V^*) \), we define
\[
\text{Col}^s(\mathcal{L}) := \sum_{j=1}^{[F:Q_p]} \text{Col}_e^*(\zeta^\sigma_j^{ \epsilon}), \eta \cdot h_e \cdot \sigma_0^i \in \mathcal{H}_{s+h,LQ}(\hat{G}_\infty), \quad \hat{G}_\infty := \text{Gal}(\mathcal{F}/\mathbb{Q}_p),
\]
where \([-,-] : D(V^*) \otimes \mathcal{H}_Q \times D(V) \otimes \mathcal{H}_Q \to \mathcal{H}_{LQ} \) is the image of \([-,-]|_V \) under the natural map \( L \otimes \mathbb{Q}_p \mathcal{H}_Q \to \mathcal{H}_{LQ} \).

For any integer \( j \), put
\[
z_{-j,n} := pr_{F_n}(Tw_{-j}(z)) \in H_1(F_n, V^*_{-j}).
We say that a finite order character \( \chi \) of \( \tilde{G}_\infty \) has conductor \( p^{n+1} \) if \( n \) is the smallest integer \( \geq -1 \) such that \( \chi \) factors through \( \text{Gal}(F_n/Q_p) \).

**Theorem 3.4.** Suppose that \( \text{Fil}^{-1}D(V) = D(V) \) and take \( h = 1 \). Let \( \psi \) be a \( p \)-adic character of \( G_\infty \) such that \( \psi = \chi^{\varepsilon_f} \) with \( \chi \) a finite order character of conductor \( p^{n+1} \). If \( j < 0 \), then

\[
\text{Col}^j(\mathfrak{z})(\psi) = \frac{(-1)^{j-1}}{(-j-1)!} \times \sum_{\tau \in \text{Gal}(F_n/Q_p)} \chi^{-1}(\tau) \left[ \log_{F_n,V}(\zeta) \right] \]

\[
\times \left\{ \begin{array}{ll}
\log_{F_n,V}(\zeta) z_j \otimes t^{-j}, (1 - p^{-1}, \varphi(1)(1 - p^{-1})^{-1} \eta) & \text{if } n = -1, \\
p^{(n+1)(j-1)} \tau(\psi) \sum_{\tau \in \text{Gal}(F_n/Q_p)} \chi^{-1}(\tau) \left[ \log_{F_n,V}(\zeta) z_j \otimes t^{-j}, \varphi(1) \eta \right] & \text{if } n \geq 0.
\end{array} \right.
\]

If \( j \geq 0 \), then

\[
\text{Col}^j(\mathfrak{z})(\psi) = \frac{j!(-1)^j}{\text{Gal}(F_n/Q_p)} \times \sum_{\tau \in \text{Gal}(F_n/Q_p)} \chi^{-1}(\tau) \left[ \log_{F_n,V}(\zeta) z_j \otimes t^{-j}, (1 - p^{-1}, \varphi(1)(1 - p^{-1})^{-1} \eta) \right] \]

\[
\times \left\{ \begin{array}{ll}
\exp_{F_n,V}(\zeta) z_j \otimes t^{-j}, (1 - p^{-1}, \varphi(1)(1 - p^{-1})^{-1} \eta) & \text{if } n = -1, \\
p^{(n+1)(j-1)} \tau(\psi) \sum_{\tau \in \text{Gal}(F_n/Q_p)} \chi^{-1}(\tau) \left[ \exp_{F_n,V}(\zeta) z_j \otimes t^{-j}, \varphi(1) \eta \right] & \text{if } n \geq 0.
\end{array} \right.
\]

Here \( \tau(\psi) \) is the Gauss sum defined by

\[
\tau(\psi) := \sum_{\tau \in \text{Gal}(F_n/Q_p)} \psi^{\varepsilon_f}(\tau \varsigma)^{n+1} \varsigma^{n+1}.
\]

**Proof.** This follows from Theorem 3.3 and the explicit computation in [Kob18, Thm. 5.10]. \( \square \)

### 3.3. Rubin’s formula for derived \( p \)-adic heights

Let \( E \) be an elliptic curve over \( Q \) with good ordinary reduction at \( p \), and let \( V = T_pE \otimes_{Z_p} Q_p \), so \( \text{Fil}^{-1}D(V) = D(V) \) and \( V \) has good ordinary reduction at \( p \). Let \( \omega_E \) be the Neron differential of \( E \), regarded as an element in \( D(\Gamma_\infty) \). Fix an embedding \( \eta\mathbb{Q} \hookrightarrow \mathbb{C} \), and let \( \mathfrak{P} \) be the prime of \( \mathbb{Q} \) induced by \( \eta\mathbb{Q} \). For any subfield \( H \subset \mathbb{Q} \), denote by \( \mathbb{H} = \mathbb{H}_\mathfrak{P} \) the completion of \( H \) with respect to \( \mathfrak{P} \).

Let \( K \) be an imaginary quadratic field such that \( p = \mathfrak{P} \) splits in \( K \), with \( \mathfrak{P} \) inducing the prime \( p \). Let \( K_\infty \) be the anticyclotomic \( \mathfrak{P} \)-extension of \( K \), and set

\[
\Gamma_\infty = \text{Gal}(K_\infty/K), \quad \hat{\Gamma}_\infty = \text{Gal}(\hat{K}_\infty/Q_p).
\]

Then \( \hat{\Gamma}_\infty \subset \Gamma_\infty \) is the decomposition group of \( \mathfrak{P} \).

Let \( H_\mathfrak{P} \) be the ring class field of \( K \) of conductor \( c \), and put \( F = H_\mathfrak{P} \) for a fixed \( c \) prime to \( p \). Let \( \xi \in K \) be a generator of \( p^{[F:Q_p]} \) and let \( F_\infty/F \) be the Lubin–Tate \( \mathfrak{P} \)-extension associated with the uniformizer \( \xi/\xi \in \mathcal{O}_F \) (see [Kob18, §3.1]). As is well-known (c.f. [Shn16, Prop. 8.3]), we have \( F_\infty = \bigcup_{n=0} F_{\mathfrak{P}^n} \), and hence \( F_\infty \) is a finite extension of \( K_\infty \). Moreover, hypothesis (3.2) holds since \( D(V)/\rho(\mathfrak{P}) = \{\xi/\xi \} \) is a \( 1 \)-Weil number.

Let \( \alpha_p \in \mathfrak{Z}_p^\times \) be the \( p \)-adic unit eigenvalue of the Frobenius map \( \varphi \) acting on \( D(V) \), and let \( \eta \in D(V) = D(H_\mathfrak{P}(E/Q_p)) \otimes D(Q_p(1)) \) be a \( \varphi \)-eigenvector of slope \(-1\) such that

\[
\varphi \eta = p^{-1} \alpha_p \cdot \eta \quad \text{and} \quad \langle \eta, \omega_E \otimes t^{-1} \rangle_{\mathfrak{P}} = 1.
\]

Let \( e \in R^{[F:Q_p]} \) be a generator over \( \mathcal{O}_F[\mathfrak{P}] \) such that \( e(0) = 1 \). Applying the big exponential map \( \Omega_{V,1} \) in Theorem 3.3, we define

\[
\omega^\eta = \Omega_{V,1}(\eta \otimes e) \in \hat{H}(F_\infty, V).
\]
Lemma 3.5. We have

$$\text{pr}_F(w^\eta) = \exp_{F,V} \left( \frac{1-p^{-1} \varphi^{-1}}{1-\varphi} \right) \in \text{H}^1(F,V).$$

Proof. Let $g = \eta \otimes e$ and let $G(X) \in D(V) \otimes \mathcal{H}_{1,Q}(X)$ such that $(1-\varphi \otimes \varphi X)G = g$. Then we have

$$G(\epsilon_0) = \eta \otimes e(\epsilon_0) - \eta + (1-\varphi)^{-1}\eta.$$

The equation $\psi_x e(X) = 0$ implies

$$\sum_{\zeta \in H^{pr-1}[F]} e^{Fr^{-1}}(X \oplus F \zeta) = 0.$$

It follows that

$$\text{Tr}_{F_0/F}(G^{Fr^{-1}}(\epsilon_0)) = \sum_{\tau \in \text{Gal}(F_0/F)} \eta \otimes e(\epsilon_0) - \eta + (1-\varphi)^{-1}\eta = \frac{p^r-1}{1-\varphi} \eta,$$

and hence

$$\text{pr}_F(w^\eta) = \text{cor}_{F_0/F}(\Xi_0,V(G)) = \exp_{F,V} \text{Tr}_{F_0/F} \left( \frac{p^{-1} \varphi^{-1}}{1-\varphi} \right) \exp_{F,V} \left( (1-p^{-1} \varphi^{-1})(1-\varphi)^{-1}\eta \right).$$

This completes the proof. $\square$

Lemma 3.6. Let $Q_p^{cyc}$ be the cyclotomic $Z_p^{cyc}$-extension of $Q_p$. Let $\sigma_{cyc} \in \text{Gal}(F_\infty^\text{ur}/Q_p)$ be the Frobenius such that $\sigma_{cyc}|Q_p^{cyc} = 1$ and $\sigma_{cyc}|Q_p^\text{ur} = Fr$. For each $z \in \hat{H}^1(K_\infty,V)$, we have

$$\langle z, \text{cor}_{F_{\infty}/K_{\infty}}(w^\eta) \rangle_{K_{\infty}} = \text{pr}_{K_{\infty}} \left( \text{Col}_e(z) \right) \left[ \frac{F_{\infty}}{[F_{\infty} : K_{\infty}]} \cdot h_{Fr} \right] \in \text{W}[\hat{\Gamma}_\infty] \otimes Q_p.$$

Proof. We first recall that for every $e \in (R \otimes \mathcal{O} W)^{Fr=0}$, the big exponential map $\Omega_{V,1}(\eta \otimes e)$ in Theorem 3.3 is given by

$$\Omega_{V,1}(\eta \otimes e) = (\exp_{F_{n,V}}(\Xi_{n,V}(G_e)))_{n=0,1,2,\ldots},$$

where $G_e \in D(V) \otimes \mathcal{H}_{1,Q}(X)$ is a solution of $(1-\varphi \otimes \varphi X)G_e = \eta \otimes e$. By the definition of $G_e$, we verify that

$$\Xi_{n,V}(G_e) = p^{-(n+1)}(\varphi^{-(n+1)} \otimes \epsilon_{n+1}) \text{Fr}^{-(n+1)}(\epsilon_n)$$

$$= \sum_{m=0}^{\infty} (p^r)^{-n+1} \varphi^m \eta \otimes e_{F_{n-m-n}^{-1}}(\epsilon_{n-m})$$

$$= \sum_{m=0}^{n+1} (p^r)^{-n+1} \varphi^m \eta \otimes e_{F_{n-m-n}^{-1}}(\epsilon_{n-m}) + p^{-(n+1)}(1-\varphi \otimes Fr)^{-1}(\eta \otimes e(0)).$$

Put $z_n = \text{pr}_{K_{n}}(z)$ and $G_n = \text{Gal}(F_n/F)$. Following the computation in [Kob18 Thm. 5.10], we find that $\left[ \text{pr}_{K_n}(\text{Col}_e(z)), \eta \right]$ is given by

$$\sum_{\gamma \in G_n} \left[ \sum_{r \in G_n} \exp_{F_{n,V}}(\gamma \eta_0 \sigma_0^{+m-1}) \gamma, \sum_{\tau \in G_n} (p^r)^{-n+1} \varphi^m \eta \otimes e_{F_{n-m-n}^{-1}}(\epsilon_{n-m}) \sigma_0^{+m-1} \tau |_{K_n} \right].$$
On the other hand,

\[ \text{pr}_{K_n}(\langle z, \text{cor}_{F_n/K_n}(w) \rangle|_{K_n}) = \frac{1}{[F_n : K_n]} \sum_{j=1}^{[F_n:Q_p]} \text{pr}_{K_n}(\langle z^{\sigma_j^{-1}}, w \rangle|_{F_n}) \sigma_0^j |_{K_n}, \]

and \( \text{pr}_{K_n}(\langle z^{\sigma_j^{-1}}, w \rangle|_{F_n}) \) equals

\[ \sum_{\gamma \in G_n} \langle z_n^{\sigma_j^{-1} \gamma^{-1}}, \exp_{F_n,V}(\Xi_n,V(G_E))|F_n \rangle \gamma |_{K_n} = \text{Tr}_{F_n/Q_p} \left( \left[ \sum_{\gamma \in G_n} \exp_{K_n,V}(z_n^{\sigma_j^{-1} \gamma^{-1}}) \gamma |_{K_n}, \Xi_n,V(G_E) \right] \right) \]

\[ = \sum_{m=0}^{[F_n:Q_p]} \left[ \sum_{\gamma \in G_n} \exp_{K_n,V}(z_n^{\sigma_j^{-1} \gamma^{-1}}) \gamma, \sum_{\tau \in G_n} (p \varphi)^{-(n+1)} \epsilon \eta \otimes e^{F_n} \epsilon^{-(n+1)} (\epsilon_{n-m}) \tau |_{K_n} \right] \]

\[ = \sum_{i=1}^{[F_n:Q_p]} \left[ \text{pr}_{K_n}(\text{Col}_e(z^{\sigma_j^{-1}}) \sigma_0^i), \eta \right]. \]

From this, it follows immediately that

\[ \text{pr}_{K_n}(\langle z, \text{cor}_{F_n/K_n}(w) \rangle|_{K_n}) = \frac{1}{[F_n : K_n]} \sum_{j=1}^{[F_n:Q_p]} \sum_{i=1}^{[F_n:Q_p]} \left[ \text{pr}_{K_n}(\text{Col}_e(z^{\sigma_j^{-1}}) \sigma_0^i), \eta \right] \sigma_0^j \]

(3.6)

\[ = \frac{1}{[F_n : K_n]} \sum_{i=1}^{[F_n:Q_p]} (\text{Col}_p(z)) \sigma_0^i \cdot \frac{1}{h_e^i} \cdot \sigma_0^j. \]

On the other hand, by definition,

\[ \text{Col}_p(z) = \sum_{j=1}^{[F_n:Q_p]} \left[ \text{Col}_g_p(z^{\sigma_j^{-1}}), \eta \right] \sigma_0^j \]

with \( g_p = \rho(1 + X) \). From [3.5] with \( e = g_p \) and the fact that \( g_p^{\sigma_0^m - n - 1} (\epsilon_{n-m}) = \zeta_p^{n+1-m} \in Q_p^c \), we deduce that

\[ \left[ \text{Col}_g_p(z^{\sigma_j^{-1}}), \eta \right] = \left[ \text{Col}_g_p(z^{\sigma_j^{-1}}), \sigma_0^i \right], \]

so \( (\text{Col}_p(z))^\sigma_0^i = \text{Col}_p(z) \cdot \sigma_0^i \). Now the lemma follows from (3.6).

For every prime \( v \) above \( p \), let \( H^1_{\text{lin}}(K_v, V) \subset H^1(K_v, V) \) be the Bloch–Kato finite subspace, and set

\[ \log_{\omega_E,v} = \langle \log_{K_v,V}(-), \omega_E \otimes t^{-1} \rangle_{dR} : H^1_{\text{lin}}(K_v, V) \to Q_p. \]

Since \( p \) is a prime of good reduction for \( E \), we have \( H^1_{\text{exp}}(K_v, V) = H^1_{\text{lin}}(K_v, V) \) by [BK90, Cor. 3.8.4], where \( H^1_{\text{exp}}(K_v, V) \subset H^1(K_v, V) \) is the image of \( \exp_{K_v,V} \).

The following result gives a formula for the derived \( p \)-adic height pairing

\[ h^{(r)} : S^{(r)}_p(E/K) \times S^{(r)}_p(E/K) \to (J^r/J^{r+1}) \otimes \mathbb{Z}_p Q_p \]

of Theorem 2.4, where \( J \) is the augmentation ideal of \( \mathbb{Z}_p[\Gamma], \) in terms of the Coleman map. For a global Iwasawa cohomology class \( z \in H^1(K_v, V) \), we put

\[ (3.8) \quad \text{Col}_p(\text{loc}_p(z)) := \sum_{\sigma \in \Gamma_v / \Gamma_v} \text{pr}_{K_v}(\text{Col}_p(\text{loc}_p(z^{\sigma^{-1}}))) \sigma \in \mathcal{W}[\Gamma_v]. \]
Theorem 3.7. Let \( z \in \hat{H}^1(K_\infty, V) \) and \( z = \text{pr}_K(z) \). Suppose that \( \text{Col}^0(\text{loc}_p(z)) \) and \( \text{Col}^0(\text{loc}_p(z)) \) both belong to \( J^r \mathcal{W}[\Gamma_\infty] \otimes \mathbb{Q}_p \). Then \( z \in S_p^{(r)}(E/K) \) and for any \( x \in S_p^{(r)}(E/K) \) we have
\[
\sum_{1 \leq \gamma \leq (\mathbb{Q}/\mathbb{Z})^*} \text{log}_E(x) \cdot \langle \text{loc}_p(z^{-1}), w_\gamma \rangle_{K_\infty} \sigma + \log_{\text{loc}_p(z)}(\mathbb{Q}_p) \cdot \langle \text{loc}_p(z^{-1}), \omega_\eta \rangle_{K_\infty} \sigma \quad (\mod J^{r+1})
\]
where \( \sigma \) and \( \tilde{\sigma} \) are the complex conjugates of \( \sigma \) and \( \tilde{\sigma} \).

Proof. Suppose first that \( z = \text{pr}_K(z) \in S_p^{(r)}(E/K) \) and fix \( x \in S_p^{(r)}(E/K) \). Let
\[
\omega_\eta := \text{cor}_{F_{\infty}/\hat{K}_\infty}(w_\eta) \in \hat{H}^1_{\text{fin}}(\hat{K}_\infty, V) := (\lim_{\chi} H^1_{\text{fin}}(\hat{K}_n, T)) \otimes \mathbb{Q}_p.
\]
Since \( \dim_{\mathbb{Q}_p} H^1_{\text{fin}}(Q_p, V) = 1 \), we can write
\[
\langle \text{loc}_p(x), \omega_\eta \rangle_{F_\infty} \otimes t^{-1}) dR = c[F : Q_p] \cdot \left( \frac{1 - p^{-1} \varphi^{-1} \eta, \omega_\eta \otimes t^{-1}}{1 - \varphi} \right) dR.
\]
Since \( \varphi \eta = p^{-1} \alpha_p \cdot \eta \), this shows that
\[
c = \frac{1 - p^{-1} \alpha_p}{1 - \alpha_p} \cdot [F : Q_p]^{-1} \cdot \log_{\text{loc}_p(x)}(\mathbb{Q}_p).
\]
Applying Corollary 2.3 we find that that
\[
h^{(r)}(z, x) = -(1 - p^{-1} \alpha_p)(1 - \alpha_p)^{-1} [F : Q_p]^{-1}
\]
\[
\times \left( \sum_{\gamma \in \Gamma_{\infty}/F_\infty} \text{log}_E(x) \cdot \langle \text{loc}_p(z^{-1}), w_\gamma \rangle_{\hat{K}_\infty} \sigma + \log_{\text{loc}_p(z^{-1})} \cdot \langle \text{loc}_p(z^{-1}), \omega_\eta \rangle_{\hat{K}_\infty} \sigma \right) \quad (\mod J^{r+1}).
\]
Since \( \rho(1 + X) = h_\text{x} \cdot e \) and \( e(0) = 1 \), we find that \( 1 = e(0) \cdot (h_\text{x} |_{\gamma = 1}) \) and hence \( h_\text{x} \equiv 1 \) \( (\mod J) \). The assertion now follows from the above equation, Lemma 3.6 and the definition 3.8.

To conclude the proof of the result it remains to show that \( z \in S_p^{(r)}(E/K) \), which with the formula 3.9 for \( h^{(r)}(z, x) \) at hand follows easily by induction on \( r \), using that by Theorem 2.4 \( S_p^{(r)}(E/K) \) is the left kernel of \( h^{(r-1)} \) (see [How04, p. 1329]).

4. Diagonal cycles and theta elements

In this section we prove Theorem 4.7 recovering the square-root anticyclotomic \( p \)-adic \( L \)-functions of Bertolini–Darmon [BD96] (in the definite case) as the image of a \( p \)-adic family of diagonal cycles [DR17a] under the Coleman map constructed in 3.2.

4.1. Ordinary \( \Lambda \)-adic forms. Fix a prime \( p > 2 \). Let \( \mathbb{B} \) be a normal domain finite flat over \( \Lambda := \mathcal{O}\left[1 + p\mathbb{Z}_p\right] \), where \( \mathcal{O} \) is the ring of integers of a finite extension \( L/Q_p \). We say that a point \( x \in \text{Spec} \left[\mathbb{B}(\mathbb{Q}_p^\infty)\right] \) is locally algebraic if its restriction to \( 1 + p\mathbb{Z}_p \) is given by \( x(\gamma) = \gamma^k \cdot \epsilon_x(\gamma) \) for some integer \( k_x \), called the weight of \( x \), and some finite order character \( \epsilon_x : 1 + p\mathbb{Z}_p \to \mu_{p^\infty} \); we say that \( x \) is arithmetic if it has weight \( k_x \geq 2 \). Let \( \mathbb{X}_+^r \) be the set of arithmetic points.

Fix a positive integer \( N \) prime to \( p \), and let \( \chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{O}^\times \) be a Dirichlet character modulo \( Np \). Let \( S^0(N, \chi, \mathbb{B}) \) be the space of ordinary \( \mathbb{B} \)-adic cusp forms of tame level \( N \) and
branch character \(\chi\), consisting of formal power series
 \[ f(q) = \sum_{n=1}^{\infty} a_n(f)q^n \in \mathbb{Z}[q] \]

such that for every \(x \in \mathcal{X}_+^1\) the specialization \(f_x(q)\) is the \(q\)-expansion of a \(p\)-ordinary cusp form \(f_x \in S_k(S(N) \otimes \chi)\), \(\chi = \chi_{p^{2-k+\epsilon}}\). Here \(r_x \geq 0\) is such that \(\epsilon_x(1+p)\) has exact order \(p^r_x\), and \(\omega : (\mathbb{Z}/p\mathbb{Z})^\times \to \mu_{p-1}\) is the Teichmüller character.

We say that \(f \in S^0(N,\chi,\mathfrak{1})\) is a \emph{primitive Hida family} if for every \(x \in \mathcal{X}_+^1\) we have that \(f_x\) is an ordinary \(p\)-stabilized newform (in the sense of [Hid86 Def. 2.4]) of tame level \(N\). Given a primitive Hida family \(f \in S^0(N,\chi,\mathfrak{1})\), and writing \(\chi = \chi'\chi_p\) with \(\chi'\) (resp. \(\chi_p\)) a Dirichlet modulo \(N\) (resp. \(p\)), there is a primitive \(f' \in S^0(N,\chi_p\mathfrak{1},\mathfrak{1})\) with Fourier coefficients

\[ a_{\ell}(f') = \begin{cases} \chi(\ell)a_{\ell}(f) & \text{if } \ell \nmid N, \\ a_{\ell}(f) \chi(\ell) & \text{if } \ell | N, \end{cases} \]

having the property that for every \(x \in \mathcal{X}_+^1\) the specialization \(f_x'\) is the \(p\)-stabilized newform attached to the character twist \(f_x \otimes \chi'_\ell\).

By [Hid86] (cf. [Wil88 Thm. 2.2.1]), attached to every primitive Hida family \(f \in S^0(N,\chi,\mathfrak{1})\) there is a continuous \(\mathbb{Z}\)-adic representation \(\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{F}_p)\) which is unramified outside \(Np\) and such that for every prime \(\ell \nmid Np\)

\[ \mathrm{tr} \rho_f(\mathrm{Frob}_\ell) = a_{\ell}(f), \quad \det \rho_f(\mathrm{Frob}_\ell) = \chi(\ell) \ell^{-1}, \]

where \((\ell)\) is the image of \(\ell \mathbb{Z}/\mathbb{Z}\) under the natural map \(1+p\mathbb{Z}/(1+p\mathbb{Z})\). In particular, letting \(\langle \epsilon_{\text{cyc}} \rangle : G_{\mathbb{Q}} \to \mathbb{Z}\) be defined by \(\langle \epsilon_{\text{cyc}} \rangle(\sigma) = \langle \epsilon_{\text{cyc}}(\sigma) \rangle_1\), it follows that \(\rho_f\) has determinant \(\chi_1 \epsilon_{\text{cyc}}\), where \(\chi_1 : G_{\mathbb{Q}} \to \mathbb{Z}\) is given by \(\chi_1 := \sigma \langle \epsilon_{\text{cyc}} \rangle^{-2}\langle \epsilon_{\text{cyc}} \rangle_2\), with \(\sigma\) the Galois character sending \(\mathrm{Frob}_\ell \mapsto \chi(\ell)\). Moreover, by [Wil88 Thm. 2.2.2] the restriction of \(\rho_f\) to \(G_{\mathbb{Q}_p}\) is given by

\[ \rho_f|_{G_{\mathbb{Q}_p}} \sim \begin{pmatrix} \psi_f & 0 \\ 0 & \psi_f^{-1} \chi_1^{-1} \epsilon_{\text{cyc}}^{-1} \end{pmatrix}, \]

where \(\psi_f : G_{\mathbb{Q}_p} \to \mathbb{Z}\) is the unramified character with \(\psi_f(\mathrm{Frob}_p) = a_p(f)\).

### 4.2. Triple product \(p\)-adic \(L\)-function.

Let

\[ (f, g, h) \in S^0(N_f, \chi_f, \mathfrak{1}_f) \times S^0(N_g, \chi_g, \mathfrak{1}_g) \times S^0(N_h, \chi_h, \mathfrak{1}_h) \]

be a triple of primitive Hida families. Set

\[ \mathcal{R} := \mathfrak{I}_f \otimes \mathfrak{I}_g \otimes \mathfrak{I}_h, \]

which is a finite extension of the three-variable Iwasawa algebra \(\mathcal{R}_0 := \Lambda \otimes \mathcal{O}_\Lambda \otimes \mathcal{O}_\Lambda\), and define the weight space \(\mathcal{X}_R^0\) for the triple \((f, g, h)\) in the \emph{\(f\)-dominated unbalanced range} by

\[ \mathcal{X}_R^0 := \left\{ (x, y, z) \in \mathcal{X}_R^1 : x^{\mathfrak{1}}_y \times x^{\mathfrak{1}}_h \in \mathcal{X}_x^{k_x} : k_x > k_y + k_z \text{ and } k_x \equiv k_y + k_z \pmod{2} \right\}, \]

where \(\mathcal{X}_x^{k} \supset \mathcal{X}_y^{k} \) (and similarly \(\mathcal{X}_x^{k}\)) is the set of locally algebraic points in \(\mathfrak{I}_g(\mathcal{Q}_{\mathbb{Q}})\) for which \(g_x(q)\) is the \(q\)-expansion of a classical modular form.

For \(\phi \in \{ f, g, h \}\) a positive integer \(N\) prime to \(p\) and divisible by \(N_\phi\), define the space of \(\Lambda\)-adic test vectors \(S^0(N, \chi_\phi, \mathfrak{1}_\phi)|\phi\) to be the \(\mathfrak{I}_\phi\)-submodule of \(S^0(N, \chi_\phi, \mathfrak{1}_\phi)\) generated by \(\{\phi(q^d)\}\), as \(d\) ranges over the positive divisors of \(N/N_\phi\).

For the next result, set \(N := \text{lcm}(N_f, N_g, N_h)\), and consider the following hypothesis:

\[ (\Sigma^-) \quad \text{for some } (x, y, z) \in \mathcal{X}_R^0, \text{ we have } \epsilon_q(f_x^q, g_y^q, h_z^q) = +1 \text{ for all } q \mid N. \]

Here \(\epsilon_q(f_x^q, g_y^q, h_z^q)\) denotes the local root number of the Kummer self-dual twist of the Galois representations attached to the newforms \(f_x, g_y, h_z\).
Theorem 4.1. Assume that the residual representation $\bar{\rho}_f$ satisfies

- (CR) $\bar{\rho}_f$ is absolutely irreducible and $p$-distinguished,
- and that, in addition to $(\Sigma^-)$, the triple $(f, g, h)$ satisfies

  - (ev) $\chi_f \chi_g \chi_h = \omega^{2a}$ for some $a \in \mathbb{Z}$,
  - (sq) $\gcd(N_f, N_g, N_h)$ is squarefree.

Then there exist $\Lambda$-adic test vectors $(\tilde{f}^*, \tilde{g}^*, \tilde{h}^*)$ and an element

$$\mathcal{L}_p^f(\tilde{f}^*, \tilde{g}^*, \tilde{h}^*) \in \mathcal{R}$$

such that for all $(x, y, z) \in \mathcal{X}_p^f$ of weight $(k, \ell, m)$:

$$\nu(x, y, z)(\mathcal{L}_p^f(\tilde{f}^*, \tilde{g}^*, \tilde{h}^*))^2 = \frac{\Gamma(k, \ell, m)}{2^n(k, \ell, m)} \cdot \frac{\mathcal{E}(f_x, g_y, h_z)^2}{\mathcal{E}_0(f_x)^2 \cdot \mathcal{E}_1(f_x)^2} \cdot \prod_{q \mid N} c_q \cdot \frac{L(f_x^\sigma \otimes g_y \otimes h_z^\sigma, c)}{\pi^{2(k-2)} \cdot \|f_x^\sigma\|^2},$$

where:

- $c = (k + \ell + m - 2)/2$,
- $\Gamma(k, \ell, m) = (c - 1)! \cdot (c - m)! \cdot (c - \ell)! \cdot (c + 1 - \ell - m)!$,
- $\alpha(k, \ell, m) \in \mathcal{R}$ is a linear form in the variables $k, \ell, m$,
- $\mathcal{E}(f_x, g_y, h_z) = (1 - \frac{\beta f_x \alpha_g \alpha_h}{p}) (1 - \frac{\beta f_x \alpha_g \alpha_h}{p}) (1 - \frac{\beta f_x \alpha_g \alpha_h}{p}) (1 - \frac{\beta f_x \alpha_g \alpha_h}{p})$,
- $\mathcal{E}_0(f_x) = (1 - \frac{\beta f_x}{\alpha f_x})$, $\mathcal{E}_1(f_x) = (1 - \frac{\beta f_x}{\alpha f_x})$,

and $\|f_x^\sigma\|^2$ is the Petersson norm of $f_x^\sigma$ on $\Gamma_0(N_f)$.

Proof. See [Hsi19] Thm. A. More specifically, the construction of $\mathcal{L}_p^f(\tilde{f}^*, \tilde{g}^*, \tilde{h}^*)$ under hypotheses (CR), (ev), and (sq) is given in [Hsi19] §3.6 (where it is denoted $\mathcal{L}_p^f$), and the proof of its interpolation property assuming $(\Sigma^-)$ is contained in [Hsi19] §7. □

4.3. Triple tensor product of big Galois representations. Let $(f, g, h)$ be a triple of primitive Hida families with $\chi_f \chi_g \chi_h = \omega^{2a}$ for some $a \in \mathbb{Z}$. For $\phi \in \{f, g, h\}$, let $V_{\phi}$ be the natural lattice in $(\text{Frac} \, \mathbb{I}_p)^2$ realizing the Galois representation $\rho_{\phi}$ in the étale cohomology of modular curves (see [Oht00]), and set

$$\forall_{fgh} := V_f \otimes V_g \otimes V_h.$$

This has rank 8 over $\mathcal{R}$, and by hypothesis its determinant can be written as $\det \forall_{fgh} = X^2 \varepsilon_{\text{cyc}}$ for a $p$-ramified Galois character $\chi$ taking the value $(-1)^a$ at complex conjugation. Similarly as in [How07] Def. 2.1.3], we define the critical twist

$$\forall_{fgh}^\dagger := \forall_{fgh} \otimes X^{-1}.$$

More generally, for any multiple $N$ of $N_\phi$ one can define Galois modules $V_{\phi}(N)$ by working in tame level $N$; these split non-canonically into a finite direct sum of the $\mathbb{I}_p$-adic representations $V_{\phi}$ (see [DR17a] §1.5.3]), and they define $\forall_{fgh}^N(N)$ for any $N$ divisible by $\text{lcm}(N_f, N_g, N_h)$.

If $f$ is a classical specialization of $\phi$ with associated $p$-adic Galois representation $V_f$, we let $\forall_{f, gh}$ be the quotient of $\forall_{fgh}$ given by

$$\forall_{f, gh} := V_f \otimes V_g \otimes V_h.$$

Denote by $\forall_{f, gh}^\dagger$ the corresponding quotient of $\forall_{fgh}^\dagger$, and by $\forall_{f, gh}^\dagger(N)$ its level $N$ counterpart.
4.4. Theta elements and factorization. We recall the factorization proven in [Hsi19 §8]. Let $f \in S_2(pN_f)$ be a $p$-stabilized newform of tame level $N_f$ defined over $\mathcal{O}$, let $f^o \in S_2(N_f)$ be the associated newform, and let $\alpha_p = \alpha_p(f) \in \mathcal{O}^\times$ be the $U_p$-eigenvalue of $f$. Let $K$ be an imaginary quadratic field of discriminant $D_K$ prime to $N_f$. Write

$$N_f = N^+N^-$$

with $N^+$ (resp. $N^-$) divisible only by primes which are split (resp. inert) in $K$, and choose an ideal $\mathfrak{N}^+ \subset \mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{N}^+ \simeq \mathbb{Z}/N^+\mathbb{Z}$.

Assume that $p = p\mathfrak{N}^+$ splits in $K$, with our fixed embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ inducing the prime $p$. Let $\Gamma_{\infty}$ be the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension of $K_{\infty}/K$, fix a topological generator $\gamma \in \Gamma_{\infty}$, and identity $\mathcal{O}[\Gamma_{\infty}]$ with the power series ring $\mathcal{O}[T]$ via $\gamma \mapsto 1 + T$. For any prime-to-$p$ ideal $\mathfrak{a}$ of $K$, let $\sigma_{\mathfrak{a}}$ be the image of $\mathfrak{a}$ in the Galois group of the ray class field $K(p^\infty)/K$ of conductor $p^\infty$ under the geometrically normalized reciprocity law map.

**Theorem 4.2.** Let $\chi$ be a ring class character of $K$ of conductor $c\mathcal{O}_K$ with values in $\mathcal{O}$, and assume that:

(i) $(pN_f,cD_K) = 1$,

(ii) $N^-$ is the squarefree product of an odd number of primes,

(iii) $\widehat{\rho}_p$ is absolutely irreducible and $p$-distinguished,

(iv) if $q | N^+$ is a prime with $q \equiv 1 \pmod{p}$, then $\widehat{\rho}_p$ is ramified at $q$.

There exists a unique element $\Theta_{f/K,\chi}(T) \in \mathcal{O}[T]$ such that for every $p$-power root of unity $\zeta$:

$$\Theta_{f/K,\chi}(\zeta - 1)^2 = \frac{p^n}{\alpha_p^n} \cdot \varepsilon_p(f,\chi,\zeta)^2 \cdot \frac{L(f^o/K \otimes \chi_{\zeta},1)}{(2\pi)^2} \cdot \Omega_{f^o,N^-} \cdot \varepsilon_p^{\zeta} \cdot \varepsilon_p,$$

where:

- $n \geq 0$ is such that $\zeta$ has exact order $p^n$,
- $\epsilon : \Gamma_{\infty} \rightarrow \mu_{p^n}$ be the character defined by $\epsilon_\chi(\gamma) = \zeta$,
- $\varepsilon_p(f,\chi,\zeta) = \begin{cases} (1 - \alpha_p^{-1}\chi(p))(1 - \alpha_p\chi(\overline{p})) & \text{if } n = 0, \\ 1 & \text{if } n > 0, \end{cases}$
- $\Omega_{f^o,N^-} = 4 \cdot \|f^o\|_{\Gamma_0(N_f)}^2 \cdot \eta_{f^o,N^-}^{-1}$ is the Gross period of $f^o$,
- $\sigma_{\mathfrak{a}^+} \in \Gamma_{\infty}$ is the image of $\mathfrak{N}^+$ under the geometrically normalized Artin’s reciprocity map,
- $\mu_K = |\mathcal{O}_K^\times|/2$, and $\varepsilon_p \in \{\pm 1\}$ is the local root number of $f^o$ at $p$.

**Proof.** See [BD96] for the first construction, and [CH18] Thm. A] for the stated interpolation property. \hfill \square

When $\chi$ is the trivial character, we write $\Theta_{f/K,\chi}(T)$ simply as $\Theta_{f/K}(T)$. Suppose now that $f$ is the specialization of a primitive Hida family $f \in S^o(N_f,\mathbb{I})$ with branch character $\chi_f = 1$ at an arithmetic point $x_1 \in \mathbb{X}_K^+$ of weight 2. Let $\ell \nmid pN = \Gamma_{\infty}$ be a prime split in $K$, and let $\chi$ be a ring class character of $K$ of conductor $\ell^m\mathcal{O}_K$ for some even $m > 0$. Set $C = D_K\ell^{2m}$ and let

$$g = \theta_{\chi}(S_2) \in S^o(C,\omega^{-1}\eta_{K/Q},\mathcal{O}[S_2]), \quad h = \theta_{\chi^{-1}}(S_3) \in S^o(C,\omega^{-1}\eta_{K/Q},\mathcal{O}[S_3]).$$

be the primitive CM Hida families constructed in [Hsi19 §8.3], where $\eta_{K/Q}$ is the quadratic character associated to $K$. The $p$-adic triple product $L$-function of Theorem 4.1 for this triple $(f,g,h)$ is an element in $\mathcal{R} = \mathbb{I}[S_2,S_3]$; in the following we let

$$\mathcal{L}_p^f(\tilde{\ell}^*,\tilde{g}^*,\tilde{h}^*) \in \mathcal{O}[S]$$

denote the restriction to the “line” $S = S_2 = S_3$ of its image under the specialization map at $x_1$. 
Let $\mathbb{K}_\infty$ be the $\mathbb{Z}_p^2$-extension of $K$, and let $K_p^\infty$ denote the $p$-ramified $\mathbb{Z}_p^e$-extension in $\mathbb{K}_\infty$, with Galois group $\Gamma_{p,\infty} = \text{Gal}(K_p^\infty/K)$. Let $\gamma_p \in \Gamma_{p,\infty}$ be a topological generator, and for the formal variable $T$ let $\Psi_T : \text{Gal}(\mathbb{K}_\infty/K) \to \mathcal{O}[T]^\times$ be the universal character defined by

\[
\Psi_T(\sigma) = (1 + T)^{l(\sigma)}, \quad \text{where } \sigma|_{K_p^\infty} = \gamma_p^{l(\sigma)}.
\]

Denoting by the superscript $\tau$ the action of the non-trivial automorphism of $K/\mathbb{Q}$, the character $\Psi_{1-\tau}$ factors through $\Gamma_\infty$ and yields an identification $\mathcal{O}[\Gamma_\infty] \cong \mathcal{O}[T]$ corresponding to the topological generator $\gamma_p^{-1} \in \Gamma_\infty$. Let $p^b$ be the order of the $p$-part of the class number of $K$. Hereafter, we shall fix $v \in \mathbb{Z}_p^\times$ such that $\nu = \varepsilon_{\text{cyc}}(\gamma_p^b) \in 1 + p\mathbb{Z}_p$. Let $K(\chi, \alpha_p)/K$ (resp. $K(\chi)/K$) be the finite extension obtained by adjoining to $K$ the values of $\chi$ and $\alpha_p$ (resp. the values of $\chi$).

**Proposition 4.3.** Set $T = v^{-1}(1 + S) - 1$. Then

\[
\mathcal{L}_p^I(f^*, \hat{g}^* \hat{h}^*) = \pm \Psi_{1-\tau}^{-1}(\sigma_{\mathcal{R}^+}) \cdot \Theta_{f/K}(T) \cdot C_{f, \chi} \cdot \sqrt{L_{\text{alg}}(f/K \otimes \chi^2, 1)},
\]

where $C_{f, \chi} \in K(\chi, \alpha_p)^\times$ and

\[
L_{\text{alg}}(f/K \otimes \chi^2, 1) := \frac{L(f/K \otimes \chi^2, 1)}{\pi^2 \Omega_{f^*, \mathcal{R}^-}} \in K(\chi).
\]

**Proof.** This is the factorization formula of [Hsi19, Prop. 8.1] specialized to $S = S_2 = S_3$, using the interpolation property of $\Theta_{f/K, \chi^2}(T)$ at $\zeta = 1$. \hfill $\square$

**Remark 4.4.** The factorization of Proposition 4.3 reflects the decomposition of Galois representations

\[
\nu_{f, gh}^I = (V_f(1) \otimes \text{Ind}_{\mathbb{K}}^\mathbb{Q} \psi_{1-\tau}^{-1}) \oplus (V_f(1) \otimes \text{Ind}_K^\mathbb{Q} \chi^2).
\]

4.5. **Diagonal cycles and theta elements.** Assume that $f, g = \theta_{\chi}(S)$, and $h = \theta_{\chi^{-1}}(S)$ are as in §4.4 viewing the latter two in $S^3(C, \omega^{-1} \eta_{K/\mathbb{Q}}, \mathcal{O}[S])$. Keeping the notations from §4.3 by [DR16, §1] there exists a class

\[
\kappa(f, gh) \in H^1(\mathbb{Q}, V_{f, gh}^I(N))
\]

constructed from twisted diagonal cycles on the triple product of modular curves of tame level $N$ (we shall briefly recall the construction of this class in Theorem 4.6 below), where we may take $N = \text{lcm}(N_f, C)$.

Every triple of test vectors $\tilde{F} = (\tilde{f}, \tilde{g}, \tilde{h})$ defines a $\mathbb{Q}$-equivariant projection $V_{f, gh}^I(N) \to V_{f, gh}^I$, and hence a map on cohomology

\[
\text{pr}_F : H^1(\mathbb{Q}, V_{f, gh}^I(N)) \to H^1(\mathbb{Q}, V_{f, gh}^I),
\]

and we let

\[
\kappa(\tilde{f}, \tilde{g}, \tilde{h}) := \text{pr}_F(\kappa(f, gh)) \in H^1(\mathbb{Q}, V_{f, gh}^I).
\]

Since $\psi_{1-\tau}$ gives the universal character of $\Gamma_\infty = \text{Gal}(K_\infty/K)$, by [4.4] and Shapiro’s lemma we have the identifications

\[
H^1(\mathbb{Q}, V_{f, gh}^I) \simeq H^1(\mathbb{Q}, V_f(1) \otimes \text{Ind}_{\mathbb{K}}^\mathbb{Q} \psi_{1-\tau}^{-1}) \oplus H^1(\mathbb{Q}, V_f(1) \otimes \text{Ind}_K^\mathbb{Q} \chi^2)
\]

\[
\simeq \hat{H}^1(K_\infty, V_f(1)) \oplus H^1(K, V_f(1) \otimes \chi^2).
\]

Let $g$ and $h$ be the weight 1 eigenform $\theta_{\chi}$ and $\theta_{\chi^{-1}}$, respectively, so that the specialization of $(\tilde{g}, \tilde{h})$ at $T = 0$ (or equivalently, $S = v - 1$) is a $p$-stabilization of the pair $(g, h)$.

**Lemma 4.5.** Assume that $L(f \otimes g \otimes h, 1) = 0$ and that $L(f/K \otimes \chi^2, 1) \neq 0$. Then for every choice of test vectors $\tilde{F} = (\tilde{f}, \tilde{g}, \tilde{h})$ we have:
We thus conclude that loc_\text{\textit{p}}(\kappa(f, g\hat{h})) = 0 \in H^1(K, \mathbb{Z}_p(1)).

Proof. Let $\kappa = \kappa(f, g\hat{h})$ and for every $? \in \{f, g, h\}$, let $F^0V_f$ be the rank one subspace of $V_f$ fixed by the inertia group at $p$. By (4.7), in order to prove (1) it suffices to show that some specialization of $\kappa$ has trivial image in $H^1(K, V_f(1) \otimes \chi^2)$. Let

$$\kappa_{g\hat{h}} := \kappa|_{s = -1} \in H^1(Q, V_f) = H^1(K, V_f(1) \oplus 1, V_f(1) \otimes \chi^2),$$

where $V_f := V_f(1) \otimes V_g \otimes V_h$. As noted in [DR17a, p. 634], the Selmer group $\text{Sel}(Q) \subset V_f(1)$ is given by

$$\text{Sel}(Q, V_f) = \ker \left( \text{H}^1(Q, V_f) \to \text{H}^1(Q, V_f)^{-1} \otimes V_g \otimes V_h \right),$$

where $\partial_p$ is the natural map induced by the projection $V_f \to V_f^+ := V_f/F^0V_f$, and so

$$\text{Sel}(Q, V_f) = \text{Sel}(K, V_f(1)) \oplus \text{Sel}(K, V_f(1) \otimes \chi^2).

The implications $L(f \otimes g \otimes h, 1) = 0 \implies \kappa_{g\hat{h}} \in \text{Sel}(Q, V_f)$ and $L(f/K \otimes \chi^2, 1) \neq 0 \implies \text{Sel}(K, V_f(1) \otimes \chi^2) = 0$, which follow from [DR17a, Thm. C] and [CH15, Thm. 1], respectively, thus yield assertion (1).

We proceed to prove (2). We know that the local class $\text{loc}_p(\kappa)$ belongs to $\text{H}^1(Q, p, F^+V_f)$, where

$$F^+V_f := (F^0V_f(1) \otimes F^0V_g \otimes V_h + F^0V_f(1) \otimes V_g \otimes F^0V_h + V_f(1) \otimes F^0V_g \otimes F^0V_h) \otimes \chi^{-1}$$

is a rank four subspace of $V_f$ (see [DR17a, Cor. 2.3]). In our case where $(g, h) = (\theta, \theta^{-1})$, we have

$$F^+V_f = V_f(1) \otimes \Psi^{-1}_T + F^0V_f(1) \otimes (\chi^2 \otimes \chi^{-2}),$$

where $\Psi_T$ is viewed as a character of $G_\mathbb{Q}$ via the embedding $K \hookrightarrow Q$ induced by $p$. From part (1) of the lemma, it follows that

$$\text{loc}_p(\kappa) = (\text{loc}_p(\kappa), \text{loc}_p(\kappa)) \in \text{H}^1(K_p, V_f(1) \otimes \Psi^{-1}_T) \oplus 0 \oplus \text{H}^1(K_p, V_f(1) \otimes \Psi^{-1}_T) = H^1(Q, V_f(1) \otimes \text{Im}_K \Psi^{-1}_T).$$

We thus conclude that $\text{loc}_p(\kappa) = 0$, and hence $\text{loc}_p(\kappa) = 0$.

From now on, assume that $f^o \in S_2(N_f)$ is the newform corresponding to an elliptic curve $E/\mathbb{Q}$ with good ordinary reduction at $p$. In particular, $V_f(1) \simeq V_pE$, and under the conditions in Lemma 4.3, we have the class

$$\kappa(f, g\hat{h}) \in \hat{H}^1(K, \mathbb{Z}_p(1) \otimes \mathbb{L}).$$

The following key theorem recasts [DR17a, Thm. 5.3] in terms of the Coleman map of (4.2).

Theorem 4.6 (Darmon–Rotger). Assume that $L(f \otimes g \otimes h, 1) = 0$ and that $L(f/K \otimes \chi^2, 1) \neq 0$. Then $\text{loc}_p(\kappa(f, g\hat{h})) = 0$ and

$$L_p^f(f, g\hat{h}) = \alpha_p/2 \cdot (1 - \alpha_p^{-1}a_p(g)a_p(h)^{-1}) \cdot \text{Col}^p(\text{loc}_p(\kappa(f, g\hat{h}))),$$

where $\mathcal{F}^* = (\mathcal{F}^*, \mathcal{G}^*, \mathcal{H}^*)$ is the triple of test vectors from Theorem 4.1.

Proof. The first claim is contained Lemma 4.5. For the proof of the second, we begin by briefly recalling from [DR17a, §1] the construction of the class $\kappa(f, gh)$ in (4.5). In the following, all references are to [DR17a] unless otherwise stated.

Consider the triple product of modular curves over $\mathbb{Q}$:

$$W_{s,s} := X_0(Np) \times X_s \times X_s,$$
where $X_0(Np)$ and $X_s$ are the classical modular curves attached to the congruence subgroups $\Gamma_0(Np)$ and $\Gamma_1(Np^s)$, respectively, and the model for the latter is the one for which the cusp $\infty$ is defined over $\mathbb{Q}$. The group $G_s^{(N)} := (\mathbb{Z}/Np^s\mathbb{Z})^\times$ acts on $X_s$ by the diamond operators $\langle a; b \rangle$ ($a \in (\mathbb{Z}/N\mathbb{Z})^\times$, $b \in (\mathbb{Z}/p^s\mathbb{Z})^\times$), and we let

$$W_s := W_{s,s}/D_s$$

be the quotient of $W_{s,s}$ by the action of the subgroup $D_s \subset G_s^{(N)} \times G_s^{(N)}$ consisting of elements of the form $(\langle a; b \rangle, \langle a; b^{-1} \rangle)$. Let $^b\Delta_{s,s} \in \text{CH}^2(W_{s,s})(\mathbb{Q}(\zeta_s))$ be the class in the Chow group defined by the “twisted diagonal cycle” defined in (41), and let $^b\Delta_s \in \text{CH}^2(W_s)(\mathbb{Q}(\zeta_s))$ denote its natural image under the projection $\text{pr}_s : W_s \to W_s$. By Proposition 1.4, after applying the correspondence $\varepsilon_{s,s}$ in (47) the cycle $\Delta_{s,s}$ becomes null-homologous, and so

$$\Delta_s := \varepsilon_{s,s}(^b\Delta_s) \in \text{CH}^2(W_s)_0(\mathbb{Q}(\zeta_s)),$$

letting $\varepsilon_{s,s}$ still denote the linear endomorphism of $\text{CH}^2(W_s)$ defined by the above correspondence. Let $\varepsilon_s : G_Q \to (\mathbb{Z}/p^s\mathbb{Z})^\times$ be the mod $p^s$ cyclotomic character, and let $X_s^\dagger$ be the twist of $X_s$ by the cocycle $\sigma \in G_Q \to \langle 1; \varepsilon_s(\sigma) \rangle$. By Proposition 1.6, we may alternatively view

$$\Delta_s \in \text{CH}^2(W_s)_0(\mathbb{Q}),$$

where $W_s^\dagger$ the quotient of $W_{s,s} := X_0(Np) \times X_s \times X_s^\dagger$ be a diamond action defined as before. Consider the $p$-adic étale Abel–Jacobi map

$$\text{AJ}_{et} : \text{CH}^2(W_s)_0(\mathbb{Q}) \to H^1(\mathbb{Q}, H^3_{et}(W_s/\overline{\mathbb{Q}}, \mathbb{Z}_p)(2)).$$

Let $e_{ord} = \lim_n U_p^n$ be Hida’s ordinary projector. Set

$$V_{s,s}^{\text{ord}} := H^1_{et}(X_0(Np)/\overline{\mathbb{Q}}, \mathbb{Z}_p) \otimes e_{ord}(H^1(X_s/\overline{\mathbb{Q}}, \mathbb{Z}_p)(1)) \otimes e_{ord}(H^1(X_s^\dagger/\overline{\mathbb{Q}}, \mathbb{Z}_p)(1)),$$

and let $V_{s,s}^{\text{ord}} := (V_{s,s}^{\text{ord}})_{D_s}$ denote the $D_s$-coinvariants. Let $\varpi_2 : X_{s+1}^\dagger \to X_s$ be the degeneracy map given by $z \mapsto pz$ on the complex upper half plane, which naturally defines

$$\varpi_{2,s} : (1, \varpi_2, \varpi_2)_s : V_{s+1,s+1}^{\text{ord}} \to V_{s,s}^{\text{ord}}.$$

Let $\kappa_s \in H^1(\mathbb{Q}, V_{s,s}^{\text{ord}})$ denote the image of $\text{AJ}_{et}(\Delta_s)$ under the composite map

$$H^1(\mathbb{Q}, H^3_{et}(W_s/\overline{\mathbb{Q}}, \mathbb{Z}_p)(2)) \xrightarrow{\varepsilon_s \cdot \varpi_{2,s}^{\ast} \cdot e_{ord}} H^1(\mathbb{Q}, H^3_{et}(W_s/\overline{\mathbb{Q}}, \mathbb{Z}_p)(2)) \xrightarrow{(1, e_{ord}, e_{ord})_{pr_1,1,1}} H^1(\mathbb{Q}, (V_{s,s}^{\text{ord}})_{D_s}) = H^1(\mathbb{Q}, V_{s,s}^{\text{ord}}),$$

where the first arrow is defined by Lemma 1.8, and $pr_{1,1,1}$ is the projection onto the $(1, 1, 1)$-component in the Künneth decomposition for $H^3_{et}(W_s/\overline{\mathbb{Q}}, \mathbb{Z}_p)$. By Proposition 1.9, we have $(\varpi_{2,s})_{s}(\kappa_{s+1}) = (1, U_p, 1)(\kappa_s)$, and hence we obtain the compatible family

$$\kappa_\infty := \lim_s (1, U_p, 1)^{-s}(\kappa_s) \in H^1(\mathbb{Q}, V_{s,s}^{\text{ord}}), \quad \text{where } V_{s,s}^{\text{ord}} := \lim_s V_{s,s}^{\text{ord}},$$

with limit with respect to the maps induced by (4.11). The triple $(f, g, h)$ defines a natural projection $\varpi_{f,g,h : V_{s,s}^{\text{ord}} \to V_{s,s}^{f,g,h}(N)}$, and following Definition 1.15 one sets

$$\kappa(f, g, h) := \varpi_{f,g,h}(\kappa_\infty) \in H^1(\mathbb{Q}, V_{s,s}^{f,g,h}(N)),$$

this is the class in (4.15). Now, to prove the equality (4.19) in the theorem, it suffices to show that both sides agree at infinitely many points. Let $x \in X_s^\dagger$ have weight 2 with $\zeta := \varepsilon_x(1+p) \in \mu_{p^n}$ a primitive $p^n$-th root of unity, and set

$$\kappa(f, g, h_x) := \kappa(f, gh)|_{x = \zeta = -1}.$$
Directly from the definitions (cf. Proposition 2.5), we have
\begin{equation}
\kappa(f, g, h_x) = a_p(g_x)^{-s} \cdot \varpi_{f, g, h_x}(\text{AJ}_{et}(\Delta_x)) \in H^1(Q, V_{fg, h_x}(N)),
\end{equation}
where $V_{fg, h_x}(N)$ is the $(f, g, h_x)$-isotypic component of $\left(4.10\right)$, and $\varpi_{f, g, h_x}$ is the projection to that component. By Corollary 2.3 and (77), the image of $\kappa(f, g, h_x)$ in the local cohomology group $H^1(Q_p, V_{fg, h_x}(N))$ lands in the Bloch–Kato finite subspace $H^1_{\text{fin}}(Q_p, V_{fg, h_x}(N)) \subset H^1(Q, V_{fg, h_x}(N))$, and so we may consider the image $\log_p(\kappa(f, g, h_x))$ of this restriction under the Bloch–Kato logarithm map
\[
\log_p : H^1_{\text{fin}}(Q_p, V_{fg, h_x}(N)) \to (\text{Fil}^0D_{fg, h_x}(N))^\vee,
\]
where $D_{fg, h_x}(N) := (B_{\text{cris}} \otimes V_{fg, h_x}(N))G_{Q_p}(\xi)$, and the dual is with respect to the de Rham pairing $\langle \ , \rangle_{\text{dR}}$. By the de Rham comparison isomorphism, we have
\[
D_{fg, h_x}(N) \simeq H^1_{\text{dR}}(X_0(Np)/Q_p)[f] \times H^1_{\text{dR}}(X_1(Q_p)(\xi))(1)[g_x] \times H^1_{\text{dR}}(X_0(Q_p)(\xi))(1)[h_x].
\]
As in p. 639, attached to the test vectors $(\tilde{f}, \tilde{g}, \tilde{h}_x)$ one has the de Rham classes $(\eta_f \otimes \omega_{g, h_x}^\circ, \omega_{g, h_x}^\circ)$, and comparing Proposition 2.10 and Corollary 2.11 we deduce from (4.12) that
\[
\langle \log_p(\kappa(f, g, h_x)), \eta_f \otimes \omega_{g, h_x}^\circ \rangle_{\text{dR}} = \langle \mathcal{E}(f, g, h_x) \cdot \mathfrak{g}(\epsilon_x) \cdot \alpha_p^{-1}a_p(g_x)^{-s}a_p(h_x)^{-s} \cdot \tilde{f}^*(\tilde{g}, \tilde{H}_x'),
\]
where $\tilde{H}_x' = d^{-1}\tilde{h}_x'$ is the primitive of $\tilde{h}_x'$ given by part (3) of Corollary 4.5, and $\mathcal{E}(f, g, h_x) = -2(1 - \alpha_p^{-1}a_p(g_x)a_p(h_x)^{-1})^{-1}$. Consider the formal $q$-expansion
\[
\mathcal{H}^*(q) := \sum_{p \mid n}(n^{-1})a_n(h)q^n.
\]
Taking $(\tilde{f}, \tilde{g}, \tilde{h})$ to be the test vectors $\tilde{F}^*$ from Theorem 4.1 above, the construction in [Hsi19, §3.6] yields $\mathcal{L}_p'(\tilde{f}, \tilde{g}, \tilde{h}) = \tilde{f}^*(\tilde{g}, \tilde{H}^*)$. Since by construction $\tilde{H}^*$ specializes at $x$ to $\tilde{H}_x'$, we thus see as in the proof of Proposition 4.16 that
\begin{equation}
\langle \log_p(\kappa(f, g, h_x)), \eta_f \otimes \omega_{g, h_x}^\circ \rangle_{\text{dR}} = \mathcal{E}(f, g, h_x) \cdot \mathfrak{g}(\epsilon_x) \cdot \alpha_p^{-1}a_p(g_x)^{-s}a_p(h_x)^{-s} \cdot \mathcal{L}_p'(\tilde{f}, \tilde{g}, \tilde{h})(x).
\end{equation}

On the other hand, letting $\psi_\T := \Psi_T|_{T = \psi^{-1}}$, we obtain that $(g_x, h_x)$ is a pair of theta series attached to the character $(\chi \psi^{-1}, \chi \psi^{-1})$ of $G_K$ with $a_p(g_x) = \chi \psi^{-1}(\sigma_\T)$ and $a_p(h_x) = \chi \psi^{-1}(\sigma_\T)$. Moreover, we have
\[
\epsilon_x|_{G_{Q_p}} = \psi_\T^{1+s}\cdot \psi^{-1}, \quad \psi_\T^{-1} = \phi \epsilon^{-1}
\]
for some finite order character $\phi$ of Gal($F_{\infty}/Q_p$), viewing the character in the left-hand side of this equality as character on Gal($F_{\infty}/Q_p$) by composition with Gal($F_{\infty}/Q_p$) $\rightarrow$ Gal($K_{\infty}, Q_p/K_p$) $\subset \Gamma_\infty$. Setting $\eta = \eta_f \otimes t^{-1}$ and $z_x = \log_p(\kappa(\tilde{f}^*, \tilde{g}^* h^*))_x$, we thus see that
\begin{equation}
\langle \log_p(\kappa(f, g, h_x)), \eta_f \otimes \omega_{g, h_x}^\circ \rangle_{\text{dR}} = \langle \log_p(z_x) \otimes t, \eta \rangle_{\text{dR}}
\end{equation}
\[
= \mathfrak{g}(\epsilon_x) \cdot \alpha_p a_p(g_x)^{-s}a_p(h_x)^{-s} \cdot \text{Col}_p(z_x)(\psi^{-1})
\]
using Theorem 3.4 with $j = -1$ for the last equality. Comparing (4.13) with (4.14) and letting $s$ vary, the result follows. \hfill \qed

We can now immediately deduce the following key cohomological construction of $\Theta_{f/K}$:

**Theorem 4.7.** With notations and assumptions as in Theorem 2.6, we have
\[
\text{Col}_p(\log_p(\kappa(\tilde{f}_*^*, \tilde{g}_*^* h^*))) = \pm \Psi_T^{1+s}(\sigma_\T) \cdot \Theta_{f/K}(T) \cdot \sqrt{L_{\text{alg}}(E/K \otimes \chi^2, 1)} \cdot \frac{2C_f \chi}{\alpha_p(1 - \alpha_p) \chi(-\bar{p})^2},
\]
Proof. Note that \( a_p(g) a_p(h)^{-1} = \chi(\bar{\mathfrak{p}})^2 \). The theorem thus follows immediately from Proposition 4.3 and Theorem 4.6. \( \square \)

### 4.6. Generalized Kato classes.

Set \( \alpha = \chi(\bar{\mathfrak{p}}) \), and denote by \((g_\alpha, h_{\alpha^{-1}})\) the weight 1 forms obtained by specializing the Hida families \((g, h)\) at \( S = v - 1 \). Thus \( g_\alpha \) (resp. \( h_{\alpha^{-1}} \)) is the \( p \)-stabilization of the theta series \( g = \theta_\chi \) (resp. \( h = \theta_{\chi^{-1}} \)) having \( U_p \)-eigenvalue \( \alpha \) (resp. \( \alpha^{-1} \)).

By specialization, for every the choice of a triple of test vectors \((f, \tilde{g}, \tilde{h})\) the \( \mathcal{O}[S] \)-adic class \( \kappa(f, \tilde{g}, \tilde{h}) \) in \((4.6)\) yields the generalized Kato class

\[
\kappa(f, g_\alpha, h_{\alpha^{-1}}) := \kappa(f, \tilde{g}, \tilde{h})|_{S=v-1} \in H^1(\mathbb{Q}, V_{fgh}),
\]

where \( V_{fgh} := V_f \otimes V_g \otimes V_h \). Setting \( \beta = \chi(\mathfrak{p}) \) and alternatively changing the roles of \( \mathfrak{p} \) and \( \overline{\mathfrak{p}} \) in the construction \( g \) and \( h \) we thus obtain the four generalized Kato classes

\[
(4.15) \quad \kappa(f, g_\alpha, h_{\alpha^{-1}}), \; \kappa(f, g_\beta, h_{\beta^{-1}}), \; \kappa(f, g_\beta, h_{\alpha^{-1}}), \; \kappa(f, g_\beta, h_{\beta^{-1}}) \in H^1(\mathbb{Q}, V_{fgh}).
\]

If we assume that \( \chi(\overline{\mathfrak{p}}) \neq \pm 1 \), as we shall do from now one, then the four classes \((4.15)\) are a priori distinct. We also assume now that \( f \) is the \( p \)-stabilization of the newform associated to an elliptic curve \( E/\mathbb{Q} \), so that \( V_f(1) \simeq V_p E \), and let

\[
\kappa_{\alpha, \alpha^{-1}}, \kappa_{\alpha, \beta^{-1}}, \kappa_{\beta, \alpha^{-1}}, \kappa_{\beta, \beta^{-1}} \in H^1(K, V_p E \otimes L)
\]

be the image of the classes in \((4.15)\) under the natural map \( H^1(\mathbb{Q}, V_{fgh}) \to H^1(K, V_p E \otimes L) \) (see \((1.5)\)).

**Corollary 4.8.** Assume that \( L(E/K, 1) = 0 \) and that \( L(f/K \otimes \chi^2, 1) \neq 0 \). Then:

1. \( \kappa_{\alpha, \alpha^{-1}}, \kappa_{\beta, \beta^{-1}} \in \text{Sel}(K, V_p E \otimes L) \).
2. \( \kappa_{\alpha, \beta^{-1}} = \kappa_{\beta, \alpha^{-1}} = 0 \).

**Proof.** By the factorization \((1.6)\), the inclusions in part (1) follow from the proof of Lemma 4.5. To see part (2), we make use of the three-variable generalized Kato class

\[
\kappa := \kappa(f, g, h')(S_1, S_2, S_3) \in H^1(\mathbb{Q}, \mathbb{V}_{fgh}')
\]

defined in [DR17b, §3.7 (119)] (see also [BSV19b, Thm. A]) attached to the triple \( f = f(S_1), g = \theta_\chi(S_2) \) and \( h' = \theta_\chi(S_3) \). Thus \( \kappa(f, g_\alpha, h_{\beta^{-1}}) = \kappa((1+p)^2-1, v-1, v-1) \). Let

\[
\kappa' := \kappa((1+p)^2-1, v(1+T)-1, v(1+T)^{-1}-1) \in H^1(\mathbb{Q}, \mathbb{V}_{fgh}'),
\]

where \( \mathbb{V}_{fgh}' \simeq V_p E \otimes (\text{Ind}_K^Q \chi^2 \oplus \text{Ind}_K^Q \Psi_{T}^{1-\tau}) \). As in Lemma 4.5 by [DR17b, Prop. 3.28] the class \( \text{loc}_p(\kappa') \) belongs to \( H^1(Q_p, F^{+\mathbb{V}_{fgh}'}) \), where

\[
F^{+\mathbb{V}_{fgh}'} = V_p E \otimes \chi^{-2} + F^0 V_p E \otimes (\Psi_{T}^{1-\tau} \oplus \Psi_{T}^{1-\tau}).
\]

It follows that the projection \( \kappa'_V \) of \( \kappa' \) to \( \hat{H}^1(K\infty, V_p E \otimes L) \) is crystalline at \( p \), and hence \( \kappa'_V \) is a Selmer class for \( V_p E \otimes L \) over the anticyclotomic \( Z_p \)-extension \( K_{\infty}/K \). Since the space of such universal norms is trivial by Cornut–Vatsal [CV05] (the sign of \( E/K \) is \( +1 \) in our case), this shows that \( \kappa'_V = 0 \) and therefore \( \kappa(f, g_\alpha, h_{\beta^{-1}}) = \kappa_{\alpha, \beta^{-1}} = 0 \). The vanishing of \( \kappa_{\beta, \alpha^{-1}} \) is shown in the same manner. \( \square \)
5. Proof of the Main Result

5.1. Theorem B implies Theorem A. Suppose first that \( \text{rank}_\Z E(Q) = 2 \) and \( \text{III}(E/Q)[p^\infty] \) is finite, so in particular \( \text{Sel}(Q, V_p E) \) is two-dimensional. Since \( E(Q) \) injects into \( E(Q_p) \), we have \( \text{Sel}(Q, V_p E) \neq \ker(\log_p) \), and so \( \text{dim}_{Q_p} \text{Sel}_{str}(Q, V_p E) = 1 \). Part (i) of Theorem A thus follows from the implication (b) \( \Longrightarrow \) (a) in Theorem B.

On the other hand, suppose \( \kappa_{\alpha, \alpha^{-1}} \) is a nonzero class in \( \text{Sel}(Q, V_p E) \). Since the hypotheses in Theorem A imply that \( E/Q \) has root number +1, by the \( p \)-parity conjecture \( \text{Nek01} \). As a result, letting \( V = V_p E \otimes_{Q_p} L \), the composite map

\[
\log_{\omega_{E,p}} : \text{Sel}(K, V) \to H^1_{\text{fin}}(K_p, V) \to L
\]

is nonzero, where the second arrow is given by the logarithm map in (5.7). Since by Kolyvagin’s work \( \text{Koi88} \) (or Kato’s \( \text{Kat04} \)) the non-vanishing \( L(E^K, 1) \) implies that \( \text{Sel}(Q, V_p E^K) = \{0\} \), we conclude that

\[
(5.1) \quad \text{dim}_L \text{Sel}(K, V) = \text{dim}_L \text{Sel}(Q, V). = 2.
\]

Consider the filtration

\[
(5.2) \quad \text{Sel}(K, V) = S^{(1)} \supset S^{(2)} \supset \cdots \supset S^{(r)} \supset S^{(r+1)} \supset \cdots \supset S^{(\infty)}
\]

deduced from that of Theorem B. i.e., \( S^{(r)} := S_p^{(r)}(E/K) \otimes_{Q_p} L \), and continue to denote by \( h^{(r)} \) the induced derived \( p \)-adic height pairings

\[
h^{(r)} : S^{(r)} \times S^{(r)} \to (J^r/J^{r+1}) \otimes_{Z_p} L.
\]

Since the non-trivial element \( \tau \in \text{Gal}(K/Q) \) acts on \( J/J^2 \cong \Gamma \) as multiplication by \(-1\), by part (4) Theorem 2.4 we have

\[
(5.3) \quad h^{(r)}(x^\tau, y^\tau) = (-1)^r h^{(r)}(x, y).
\]

The non-vanishing in the statement of Theorem B also imply that \( E/K \) has root number +1 (so \( N^- \) is the squarefree product of an odd number of primes), and hence \( S^{(\infty)} = \bigcap_{n=1}^{\infty} S^{(n)} \) vanishes by part (5) of Theorem 2.4 and \( \text{CV05} \). Thus there are only two possibilities for the filtration (5.2):

(i) there are exactly two jumps, each of rank 1;

(ii) there is exactly one jump, of rank 2.

We argue that case (i) is impossible. By (5.3) for \( r = 1 \), the \( \tau \)-eigenspaces of \( \text{Sel}(K, V) = S^{(1)} \) are isotropic under \( h^{(1)} \). Letting \( r^\pm \) be the \( L \)-dimension of the \( \pm \)-eigenspace of \( S^{(1)} \) under the action of \( \tau \), we thus have

\[
\text{dim}_L S^{(2)} \geq |r^+ - r^-|,
\]

since \( S^{(2)} \) is the null-space of \( h^{(1)} \) by part (1) of Theorem 2.4. But (5.1) shows that \( r^+ = 2 \) and \( r^- = 0 \), and so

\[
S^{(1)} = S^{(2)}.
\]

By the same argument, if \( r \) is odd and \( S^{(r)} \) is two-dimensional, then \( S^{(r)} = S^{(r+1)} \). On the other hand, since by part (3) of Theorem 2.4 the jumps in the filtration (5.2) for even \( r \) are
of dimension $e_0 \equiv 0 \pmod{2}$ (since then $h^{(r)}$ induces a non-degenerate alternating pairing on $S^{(r)} / S^{(r+1)}$), this shows that only case (ii) is possible, and hence \[(5.2)\]

$$\text{Sel}(K, V) = S^{(1)} = S^{(2)} = \cdots = S^{(r)} \supseteq S^{(r+1)} = \cdots = S^{(\infty)} = \{0\}$$

for some (even) $r \geq 2$; in particular, the $r$-th derived $p$-adic height $h^{(r)}$ is a non-degenerate pairing on $S^{(r)} = \text{Sel}(K, V)$.

Letting $X_\infty$ be the Pontryagin dual of Sel$_{p, \infty}(E/K_\infty)$, by part (5) of Theorem \[2.4\] this shows that

$$X_\infty \sim (\Lambda/J')^\oplus 2 \oplus M',$$

where $M'$ is a torsion $\Lambda$-module with characteristic ideal prime to $J$, and so letting $L_p \in \Lambda$ be a generator of the characteristic ideal of $X_\infty$, we have

$$\text{ord}_J(L_p) = 2r.$$

On the other hand, let $\rho := \text{ord}_J(\Theta_{f/K})$ (which is finite integer by [Yat03]), and denote by $\bar{\theta}$ the image of $\Theta_{f/K}$ in $J^r/J^{r+1}$. Note that $\bar{\theta} \neq 0$ by definition. Letting $\kappa_{\alpha, -1} \in H^1(K, V)$ be the generalized Kato class constructed as in \[4.6\] taking the triple of test vectors $(\bar{f}, \bar{g}, \bar{h})$ coming from the triple $(\bar{f}^*, \bar{g}^*, \bar{h}^*)$ in Theorem \[4.1\] by part (2) of Lemma \[4.5\] Theorem \[4.7\] and Theorem \[3.7\] we conclude that

\[(5.5)\] $$\kappa_{\alpha, -1} \in S^{(\rho)},$$

and for every for every $x \in S^{(\rho)}$ we have

\[(5.6)\] $$h^{(\rho)}(\kappa_{\alpha, -1}, x) = \frac{1 - p^{-1} \alpha_p \cdot \bar{\theta} \cdot \log_{\omega_{E, p}}(x) \cdot C,}$$

where $\alpha_p$ is the $p$-adic unit root of $X^2 - \alpha_p(E)X + p = 0$ and $C$ is a non-zero algebraic number with $C^2 \in K(\chi, \alpha_p)^\times$. By [SU14] §3.6.3 we have the divisibility $(\Theta^2_{f/K}) \supset (L_p)$, which implies $\rho \leq r$, and so $S^{(\rho)} = \text{Sel}(K, V)$ by \[5.4\]. Since as noted above the map $\log_{\omega_{E, p}}$ is non-zero on $\text{Sel}(K, V)$ the non-vanishing of $\kappa_{\alpha, -1}$ follows from \[5.6\].

Now we prove the implication (a) $\implies$ (b). Thus we keep the above notations and assume that $\kappa_{\alpha, -1} \neq 0$. By part (2) of Lemma \[4.5\] Theorem \[4.7\] and Theorem \[3.7\] we conclude that the inclusion \[5.5\] holds. In particular, $S^{(\rho)} \neq \{0\}$, so also $\text{Sel}(K, V) = \text{Sel}(Q, V) \neq \{0\}$, where the equality follows from the non-vanishing of $L(E^K, 1)$ as before.

We next note that the non-trivial jumps in \[5.2\] can only occur at even values of $r$. Indeed, for odd $r$ the same argument as before shows that the null-space of $h^{(r)}$ (that is, $S^{(r+1)}$) has dimension at least $|d^+ - d^-|$, where $d^\pm$ is the dimension of the $\pm$-eigenspace of $S^{(r)}$ under the action of $\tau$; but $d^- = 0$, since $\text{Sel}(K, V)^- \simeq \text{Sel}(Q, V_\rho E^K) \otimes_{Q_\rho} L = \{0\}$, and so $S^{(r+1)} = S^{(r)}$.

Thus letting $r_0$ be the last non-trivial jump in \[5.2\] (and using that $S^{(\infty)} = \{0\}$) we have

\[(5.7)\] $$e_{r_0} = \dim_L(S^{(r_0)}) \geq 2$$

by parts (3) and (5) of Theorem \[2.4\]. Now, by [BD05] (and its refinement in [PW11]) we have the divisibility $(\Theta^2_{f/K}) \subset (L_p)$, which by part (5) of Theorem \[2.4\] implies that

$$2\rho \geq e_1 + 2e_2 + \cdots + r_0 e_{r_0} \geq 2r_0,$$

using \[5.7\] for the second inequality. Since $\rho \leq r_0$ by the non-vanishing of $S^{(\rho)}$ shown above, we conclude that $\rho = r_0$ and

$$e_r = \begin{cases} 2 & \text{if } r = \rho, \\ 0 & \text{if } r \neq \rho. \end{cases}$$
In particular, by part (5) of Theorem 2.4 this shows that $S^{(1)} = \text{Sel}(K, V)$ is two-dimensional, and so also $\dim \text{Sel}(\mathbb{Q}, V) = 2$, and that $h^{(p)}$ is a non-degenerate pairing on $S^{(p)} = \text{Sel}(\mathbb{Q}, V)$ which is alternating. Hence by (5.6) we conclude that the map $\log_{\omega_{E,p}}$ is nonzero, so
$$
\dim_{\mathbb{Q}} \text{Sel}_{\text{str}}(\mathbb{Q}, V_p E) = 1,
$$
and taking $x = \kappa_{\alpha,\alpha - 1}$ if follows that $\log_{\omega_{E,p}}(\kappa_{\alpha,\alpha - 1}) = 0$, so $\kappa_{\alpha,\alpha - 1} \in \text{Sel}_{\text{str}}(\mathbb{Q}, V_p E) \otimes_{\mathbb{Q}} L$.

Since on the other hand the vanishing of $\kappa_{\alpha,\beta - 1}$ and $\kappa_{\beta,\alpha - 1}$ follows from Corollary 4.8, this concludes the proof of Theorem B.

5.3. Application to the refined elliptic Stark conjecture. The following is an immediate consequence of the height formula (5.6):

**Corollary 5.1.** The class $\kappa_{\alpha,\alpha - 1}$ (mod $\overline{\mathbb{Q}}^\times$) depends only on $K$, not on the auxiliary choice of ring class character $\chi$. Moreover, as elements in $E(\mathbb{Q}) \otimes_{\mathbb{Z}} L$, we have
$$
\kappa_{\alpha,\alpha - 1} = C \cdot \frac{1 - p - \alpha_p}{1 - \alpha_p} \cdot \frac{\hat{\theta}}{h^{(p)}(P,Q)} \cdot (P \otimes \log_p Q - Q \otimes \log_p P)
$$
for any basis $(P,Q)$ of $E(\mathbb{Q}) \otimes_{\mathbb{Z}} Q$, where $C$ is nonzero and such that $C^2 \in K(\chi, \alpha_p)^\times$.

**Remark 5.2.** Given the expected equivalence between the derived $p$-adic height pairings constructed by Bertolini–Darmon [BD95] and Howard [How04], Corollary 5.1 shows that (in the cases considered in this paper) the refinement of the “elliptic Stark conjecture” [DLR15] given by [DR16], Conj. 3.12] follows from the expression for $\hat{\theta}$ predicted by [BD96 Conj. 4.3].

APPENDIX. Numerical examples

In this section, we exhibit the first examples of elliptic curves of rank 2 having non-vanishing generalized Kato classes. We consider elliptic curves $E/\mathbb{Q}$ with
$$
\text{ord}_{s=1} L(E, s) = \text{rank}_\mathbb{Z} E(\mathbb{Q}) = 2
$$
and conductor $N \in \{ q, 2q \}$ with $q$ an odd prime. We take a squarefree integer $-\Delta < 0$ such that $K = \mathbb{Q}(\sqrt{-\Delta})$ has class number one, $q$ is inert in $K$, and $L(E^K, 1) \neq 0$, and take a prime $p > 3$ of good ordinary prime for $E$ which splits in $K$ and such that $E[p]$ is irreducible as $G_\mathbb{Q}$-module. For every triple $(E, p, -\Delta)$, letting $f \in S_2(\Gamma_0(N))$ be the newform associated to $E$, we give numerical examples where the associated theta element
$$
\Theta_{E/K}(T) = \Theta_{f/K}(T) \in \mathbb{Z}_p [[T]]
$$
vanishes to order exactly 2 at $T = 0$.

Then, by the work of Bertolini–Darmon [BD95, BD05] on the anticyclotomic Iwasawa main conjecture (see [BD05] Cor. 3)), it follows that $\text{III}(E/K)[p^\infty]$ is finite. Moreover, the residual representation $E[p]$ must ramify at $N^- = q$ by [Rib90, Thm. 1.1] and for each of the examples we checked that $E[p]$ is irreducible, either by [Maz78] when $p \gg 11$ or by checking that $E$ does not admit any rational $m$-isogenies for $m > 3$ according to Cremona’s tables. Thus for every ring class character $\chi$ of $p$-power conductor with $L(E/K, \chi^2, 1) \neq 0$ (as always exist in these examples by virtue of [Vat03 Thm. 1.4], as extended in [CH18 Thm. D]), the examples below provide triples $(E, p, K)$ for which the generalized Kato class
$$
\kappa_{E, K} \in \text{Sel}(\mathbb{Q}, V_p E) \simeq E(\mathbb{Q}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p
$$
is non-vanishing by virtue of Theorem A in the Introduction.

To further explain these examples, we need some more notation. Let $B/\mathbb{Q}$ be the definite quaternion algebra of discriminant $q$. Let $R$ be an Eichler order of level $N/q$ and let $\text{Cl}(R)$ be the class group of $R$. Let $f_E : \text{Cl}(R) \rightarrow \mathbb{Z}$ be the $(p$-adically normalized) Hecke eigenfunction

---

2As extended by Pollack–Weston [PW11] to allow for weaker hypotheses.
associated to $f$ by the Jacquet–Langlands correspondence. Fix an optimal embedding $\mathcal{O}_K \hookrightarrow R$ and an isomorphism $i_p : R \otimes \mathbb{Z}_p \simeq M_2(\mathbb{Z}_p)$ such that $i_p(K)$ lies in the subspace of diagonal matrices. For $a \in \mathbb{Z}_p^\times$ and an integer $n$, put

$$r_n(a) = i_p^{-1}\left( \begin{pmatrix} ap^{-n} \\ 1 \end{pmatrix} \right) \in \hat{B}^\times, \quad \hat{B} := B \otimes \mathbb{Z} \hat{\mathbb{Z}}.$$

Consider the sequence $\{P_n^a\}_{n=0,1,\ldots}$ of right $R$-ideals defined by $P_n^a := (r_n(a)\hat{R}) \cap B$. (The images of these ideals $P_n^a$ in $\text{Cl}(R)$ are usually referred to as Gross points of level $p^n$.) Letting $u = 1 + p$, we define the $n$-th theta element $\Theta_{E/K,n}(T) \in \mathbb{Z}_p[T]$ by

$$\Theta_{E/K,n}(T) := \frac{1}{\alpha_p^{n+1}} \sum_{i=0}^{p^n-1} \sum_{a \in \mathbb{Z}_p} \left( \alpha_p \cdot f_E(P_n^{au}) - f_E(P_n^{au^i}) \right) (1 + T)^i.$$ 

By the definition of theta elements in [BD99 §2.7], if $K$ has class number one, we then have

$$\Theta_{E/K}(T) = \Theta_{E/K,n}(T) \pmod{(1 + T)^{p^n} - 1}.$$ 

Since $(p^n, (1 + T)^{p^n} - 1) \subset (p^n, T^p)$ and $p > 2$, to check the vanishing $\Theta_{E/K}(T)$ to exact order 2 at $T = 0$, it suffices to compute $\Theta_{E/K,n}(T)$ for sufficiently large $n$. The following examples were obtained by implementing the Brandt module package in SAGE.

<table>
<thead>
<tr>
<th>$E$</th>
<th>$p$</th>
<th>$-\Delta$</th>
<th>$\Theta_{E,K,2}(T) \pmod{(p^2, T^p)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>389a1</td>
<td>11</td>
<td>$-2$</td>
<td>$107^2 + 69T^3 + T^4 + 103T^5 + 106T^6 + 66T^7 + 11T^8 + 55T^9 + 110T^{10}$</td>
</tr>
<tr>
<td>433a1</td>
<td>11</td>
<td>$-7$</td>
<td>$88T^2 + 22T^3 + 86T^4 + 7T^5 + 10T^6 + 12T^7 + 29T^8 + 88T^9 + 48T^{10}$</td>
</tr>
<tr>
<td>446c1</td>
<td>7</td>
<td>$-3$</td>
<td>$22T^2 + 2T^3 + 3T^4 + 16T^5 + 11T^6$</td>
</tr>
<tr>
<td>563a1</td>
<td>5</td>
<td>$-1$</td>
<td>$18T^2 + 9T^3 + 5T^4$</td>
</tr>
<tr>
<td>643a1</td>
<td>5</td>
<td>$-1$</td>
<td>$T^2 + 21T^4$</td>
</tr>
<tr>
<td>709a1</td>
<td>11</td>
<td>$-2$</td>
<td>$27T^2 + 114T^3 + 3T^4 + 14T^5 + 36T^6 + 15T^7 + 42T^8 + 44T^9 + 91T^{10}$</td>
</tr>
<tr>
<td>718b1</td>
<td>5</td>
<td>$-19$</td>
<td>$3T^2 + 20T^3 + 12T^4$</td>
</tr>
<tr>
<td>794a1</td>
<td>7</td>
<td>$-3$</td>
<td>$47T^2 + 223T^3 + 8T^4 + 24T^5 + 7T^6$</td>
</tr>
<tr>
<td>997b1</td>
<td>11</td>
<td>$-2$</td>
<td>$71T^2 + 41T^3 + 83T^4 + 19T^5 + 114T^6 + 111T^7 + 101T^8 + 46T^9 + 102T^{10}$</td>
</tr>
<tr>
<td>997c1</td>
<td>11</td>
<td>$-2$</td>
<td>$54T^2 + 38T^3 + 3T^4 + 81T^5 + 82T^6 + 18T^7 + 72T^8 + 95T^9 + 4T^{10}$</td>
</tr>
<tr>
<td>1034a1</td>
<td>5</td>
<td>$-19$</td>
<td>$22T^2 + 4T^3 + 6T^4$</td>
</tr>
<tr>
<td>1171a1</td>
<td>5</td>
<td>$-1$</td>
<td>$6T^2 + 6T^3 + 20T^4$</td>
</tr>
<tr>
<td>1483a1</td>
<td>13</td>
<td>$-1$</td>
<td>$128T^2 + 148T^3 + 127T^4 + 162T^5 + 30T^6 + 149T^7 + 141T^8 + 97T^9 + 49T^{10} + 13T^{11} + 29T^{12}$</td>
</tr>
<tr>
<td>1531a1</td>
<td>5</td>
<td>$-1$</td>
<td>$16T^2 + 7T^3 + 21T^4$</td>
</tr>
<tr>
<td>1613a1</td>
<td>17</td>
<td>$-2$</td>
<td>$128T^2 + 165T^3 + 224T^4 + 287T^5 + 140T^6 + 211T^7 + 147T^8 + 160T^9 + 59T^{10} + 122T^{11} + 195T^{12} + 43T^{13} + 207T^{14} + 214T^{15} + 285T^{16}$</td>
</tr>
<tr>
<td>1627a1</td>
<td>13</td>
<td>$-1$</td>
<td>$101T^2 + 151T^3 + 58T^4 + 104T^5 + 3T^6 + 165T^7 + 128T^8 + 63T^9 + 17T^{10} + 55T^{11} + 166T^{12}$</td>
</tr>
<tr>
<td>1907a1</td>
<td>13</td>
<td>$-1$</td>
<td>$72T^2 + 131T^3 + 32T^4 + 142T^5 + 84T^6 + 104T^7 + 90T^8 + 105T^9 + 38T^{10} + 92T^{11} + 116T^{12}$</td>
</tr>
<tr>
<td>1913a1</td>
<td>7</td>
<td>$-3$</td>
<td>$41T^2 + 16T^3 + 28T^4 + 23T^5 + 14T^6$</td>
</tr>
<tr>
<td>2027a1</td>
<td>13</td>
<td>$-1$</td>
<td>$54T^2 + 128T^3 + 65T^4 + 93T^5 + 83T^6 + 161T^7 + 113T^8 + 133T^9 + 49T^{10} + 151T^{11} + 13T^{12}$</td>
</tr>
<tr>
<td>$E$</td>
<td>$p$</td>
<td>$\Theta_{E/K,3}(T) \mod (p^3, T^p)$</td>
<td></td>
</tr>
<tr>
<td>------------</td>
<td>-----</td>
<td>-----------------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>571b1</td>
<td>5</td>
<td>$-1 \quad 100T^2 + 100T^3 + 15T^4$</td>
<td></td>
</tr>
<tr>
<td>1621a</td>
<td>11</td>
<td>$-2 \quad 1089T^2 + 807T^3 + 986T^5 + 586T^6 + 1098T^7 + 772T^8 + 228T^9 + 1296T^{10}$</td>
<td></td>
</tr>
</tbody>
</table>

**References**


Department of Mathematics, University of California Santa Barbara, CA 93106, USA
Email address: castella@ucsb.edu

Institute of Mathematics, Academia Sinica, Taipei 10617, Taiwan
Email address: mlhsieh@math.sinica.edu.tw