TAMAGAWA NUMBER CONJECTURE FOR CM MODULAR FORMS AND RANKIN–SELBERG CONVOLUTIONS

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In memory of Jan Nekovář

Abstract. Let E/F be an elliptic curve defined over a number field F with complex multiplication by the ring of integers of an imaginary quadratic field K such that the torsion points of E generate over F an abelian extension of K. In this paper we prove the p-part of the Birch and Swinnerton-Dyer formula for E/F in analytic rank 1 for primes $p > 3$ split in K. This was previously known for $F = \mathbb{Q}$ by work of Rubin [Rub91] as a consequence of his proof of the Mazur–Swinnerton-Dyer "main conjecture" for rational CM elliptic curves, but the problem remained wide open for general F.

The approach in this paper, based on a novel application of an idea of Bertolini–Darmon–Prasanna to consider a carefully chosen decomposable Rankin–Selberg convolution of two CM modular forms having the Hecke L-function of interest as one of the factors, circumvents the use of p-adic heights and Bertrand's p-adic transcendence results in previous approaches. It also yields a proof of similar results for CM abelian varieties A/K, and for CM modular forms of higher weight.

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1. Introduction

1.1. **Statement of the main results.** In this paper we prove the following result towards the Birch and Swinnerton-Dyer conjecture for elliptic curves with complex multiplication.

Theorem A. *Let E be an elliptic curve over a number field F with complex multiplication by the ring of integers of an imaginary quadratic field K such that $F(E_{\rm tors})/K$ is abelian. Let $\psi_E : F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}^\times$ be the Hecke character associated to E/F. Suppose*

$$\mathrm{ord}_{s=1}L(\psi_E, s) = 1,$$

and let $p \nmid 6h_K$ be a prime split in K, where $h_K := \#\mathrm{Pic}(\mathcal{O}_K)$ is the class number of K. Assume the conductor $\mathfrak{f}_\psi$ of ψ is prime to p and satisfies $\mathfrak{d}_K \| N_{F/K}(\mathfrak{f}_\psi)$, where $\mathfrak{d}_K := (\sqrt{-D_K})$. Then

$$\mathrm{rank}_{\mathbb{Z}}E(F) = \mathrm{ord}_{s=1}L(E/F, s),$$

Moreover, for all primes $\wp \mid p$ in K we have $\#\text{Ш}(E/F)[\wp^\infty] < \infty$, with

$$\mathrm{ord}_\wp\left(\frac{L^*(E/F, 1)}{\mathrm{Reg}(E)\cdot\Omega(E)}\right) = \mathrm{ord}_\wp\left(\frac{\#\text{Ш}(E/F)[\wp^\infty]\cdot\mathrm{Tam}(E/F)}{\#(E(F)_{\rm tors})^2}\right).$$

where $L^*(E/F, 1)$ is the leading Taylor coefficient of the Hasse–Weil L-function for E/F at $s = 1$.

Remark 1.1.1.

(1) Classical results of Deuring [Deu53] (see also [Sil94, Thm. 10.5]) show that
$$L(E/F, s) = L(\psi_E, s)\cdot L(\overline{\psi}_E, s)$$
if K is contained in F, where $\overline{\psi}_E$ is the complex conjugate of ψ_E, while
$$L(E/F, s) = L(\psi_E, s) = L(\overline{\psi}_E, s)$$
otherwise. Hence the first assertion in Theorem A is that $\mathrm{rank}_{\mathbb{Z}}E(F) = 2$ in the former case, and $\mathrm{rank}_{\mathbb{Z}}E(F) = 1$ in the latter.

(2) The relevance of the condition that the extension $F(E_{\rm tors})/K$ be abelian for the application of Iwasawa theory of K to the arithmetic of elliptic curves with CM by K was first highlighted in work of Arthaud [Art78] generalizing Coates–Wiles [CW77] (see also [Rub81, GS81]).

For $F = \mathbb{Q}$ (which forces K to have class number one), Theorem A was obtained by Rubin [Rub91] as a consequence of his proof of the Iwasawa main conjecture for K, the Gross–Zagier formula [GZ86], and Perrin-Riou's work [PR87a, PR87b] on a p-adic Gross–Zagier formula and Iwasawa theory. The requirement that $F = \mathbb{Q}$ is essential to Rubin's result, as it relies on a link with the main conjecture of Mazur and Swinnerton-Dyer [Maz72, MSD74] for rational elliptic curves. As a consequence, prior to Theorem A the case $F \neq \mathbb{Q}$ remained wide open.

Our proof of Theorem A also gives a new proof of Rubin's result when $F = \mathbb{Q}$, circumventing the use of p-adic heights and of Bertrand's p-adic transcendence results [Ber84] (which remain unknown in higher dimensions). The new method also yields a similar result on the Birch and Swinnerton-Dyer conjecture for higher-dimensional CM abelian varieties A/K (Theorem B), and for higher weight CM forms (see Theorem C and Remark 1.2.2 below).

Theorem B. *Let A/K be an abelian variety with $\mathrm{End}_K(A) \simeq \mathcal{O}_L$ for a CM field L with $[L:K] = \dim(A)$, and let $\lambda : K^\times \backslash \mathbb{A}_K^\times \to \mathbb{C}^\times$ be the associated Hecke character of conductor $\mathfrak{c}$. Suppose*

$$\mathrm{ord}_{s=1} L(\lambda, s) = 1,$$

and let $p \nmid 6h_K$ be a prime split in K. Assume $\mathfrak{c}$ is prime to p and satisfies $\mathfrak{d}_K \| \mathfrak{c}$. Then

$$\mathrm{rank}_{\mathbb{Z}} A(K) = \mathrm{ord}_{s=1} L(A/K, s).$$

Moreover, for all primes $\mathfrak{P} \mid p$ in L we have $\#Ш(A/K)[\mathfrak{P}^\infty] < \infty$, with

$$\mathrm{ord}_{\mathfrak{P}}\left(\frac{L^*(A/K,1)}{\mathrm{Reg}(A)\cdot\Omega(A)}\right) = \mathrm{ord}_{\mathfrak{P}}\left(\frac{\#Ш(A/K)[\mathfrak{P}^\infty]\cdot\mathrm{Tam}(A/K)}{\#(A(K)_{\mathrm{tors}})\cdot\#(A^\vee(K)_{\mathrm{tors}})}\right),$$

where $A^\vee/K$ is the abelian variety dual to A/K.

The periods $\Omega(A)$ and $\Omega(E)$ in the above results are as defined in [FS22, Def. 30], which as explained in [BF24, §1] agree with the periods in the conjecture of Birch Swinnerton-Dyer for abelian varieties over number fields.

1.2. **About the proofs.** The starting point in the proof of our main results is an idea introduced by Bertolini–Darmon–Prasanna [BDP12] in their proof of (a generalization of) Rubin's formula [Rub92] expressing the p-adic logarithm of Heegner points in terms of special values of Katz p-adic L-functions. In Rubin's original proof of the formula, the arithmetic of CM elliptic curves $E/\mathbb{Q}$ is studied using Heegner points for an auxiliary imaginary quadratic field K' satisfying the Heegner hypothesis relative to the conductor of E. In particular, one is forced to take $K' \neq K$, as the L-function

$$L(E/K, s) = L(E, s)^2$$

has always sign $+1$. Letting λ be the Hecke character of K attached to $E/\mathbb{Q}$, the ingenious idea of [BDP12] is to write

$$\lambda = \psi\chi$$

as a product of a suitable Hecke character ψ of infinity type $(-1, 0)$ and a ray class character χ, so that $L(E, s) = L(\lambda, s)$ appears as a factor of the Rankin–Selberg convolution

$$L(g/K, \chi, s) = L(\lambda, s) \cdot L(\psi^\tau \chi, s)$$

for $g = \theta_\psi$, where ψ^τ denotes the composition of ψ with the action of the non-trivial automorphism of $K/\mathbb{Q}$, and use Heegner cycles for the pair (g, χ) as in the general Gross–Zagier formula [YZZ13]. Combined with a factorization and special value formula for the p-adic Rankin L-series for (g, χ) obtained in [BDP13], this leads to a new proof of Rubin's formula.

Towards the application of this idea to our problem, the first part of the paper is devoted to the study of the anticyclotomic Iwasawa main conjecture for "self-dual pairs" (g, χ) of the form

$$(g, \chi) = (\theta_\psi, \chi)$$

with ψ a Hecke a character of K of infinity type $(1-2r, 0)$ for any $r \geq 1$ and χ a finite order Hecke character with central character $\varepsilon_\chi = \varepsilon_g^{-1}$, where ε_g is the nebentypus of g. Expanding on the work of Agboola–Howard [AH06] and Arnold [Arn07], incorporating key ideas and results from Kato [Kat04] and Johnson-Leung–Kings [JLK11], we prove the main conjecture for (g, χ) in this setting, both

- in terms of the p-adic L-function $\mathscr{L}_w(g, \chi)$ of Bertolini–Darmon–Prasanna (Theorem 4.1.1),
- in terms of the $\Lambda_{\mathcal{O}}$-adic Heegner classes $\mathbf{z}_{g,\chi}$ of [CH18] (Corollary 4.6.1).

After inverting p and for $r=1$, the "lower bound" divisibility predicted by these main conjectures was obtained in earlier work of the author with Burungale, Skinner, and Tian in [BCST22]; here we further develop the method to prove the equality predicted by the main conjecture without any ambiguity by powers of p (as is essential for results such as Theorem A). An interesting new insight here is the dependence on a certain root number that seems to not have been observed before (see Theorem 4.5.1 and Remark 4.5.2).

The next step is to deduce from the equality of characteristic ideals in the Iwasawa main conjectures, a formula for the order of the Tate–Shafarevich group attached (after Bloch–Kato [BK90]) to the pair (g,χ). In our rank 1 case, the formula we obtain is in terms of the index of a Heegner class $z_{g,\chi}$ (introduced Theorem 3.2.1) inside the Bloch–Kato Selmer group $\mathrm{Sel}_{\mathrm{BK}}(K,T_{g,\chi})$.

With future arithmetic applications in mind (see Remark 1.2.2), this result also applies in arbitrary weights $2r\geq 2$. Below we let $\mathscr{L}_v(\lambda\mathbf{N}^{-r})$ denote the anticyclotomic Katz p-adic L-function introduced in §2.2.2, which has the trivial character $\mathbb{1}$ outside the range of interpolation.

Theorem C. *Let λ be a Hecke character of infinity type $(1-2r,0)$ for some $r\geq 1$ and conductor $\mathfrak{c}$ prime to p such that $\mathfrak{d}_K\|\mathfrak{c}$. Suppose $\mathscr{L}_v(\lambda\mathbf{N}^{-r})(\mathbb{1})\neq 0$ and $\lambda\mathbf{N}^{-r}$ has root number -1. Let (ψ,χ) be a good pair for λ in the sense of Definition 5.1.1, and let*

$$z_{g,\chi}\in \mathrm{Sel}_{\mathrm{BK}}(K,T_{g,\chi})$$

be the Heegner class associated to the self-dual pair $(g,\chi)=(\theta_\psi,\chi)$. Then $\mathrm{rank}_{\mathscr{O}}\,\mathrm{Sel}_{\mathrm{BK}}(K,T_{g,\chi})=1$, $z_{g,\chi}$ is non-torsion, and

$$\#\big(\mathrm{Sel}_{\mathrm{BK}}(K,T_{g,\chi})/\mathscr{O}\cdot z_{g,\chi}\big)^2=\frac{\#\text{Ш}_{\mathrm{BK}}(W_{g,\chi}/K)}{\#\mathrm{H}^0(K,W_{g,\chi})\cdot\#\mathrm{H}^0(K,W_{g,\chi}^\tau)}\cdot\prod_{w\in\Sigma,v\nmid p}c_w(W_{g,\chi}/K),$$

where $c_w(W_{g,\chi}/K)$ is the Tamagawa number of $W_{g,\chi}$ at w.

Remark 1.2.1. When $r=1$, the assumption $\mathscr{L}_v(\lambda\mathbf{N}^{-r})(\mathbb{1})\neq 0$ and $\lambda\mathbf{N}^{-r}$ has root number -1 can be shown to follow from $\mathrm{ord}_{s=r}L(\lambda,s)=1$; the same implication is expected to hold for $r>1$, but this is not known at present (see Remark 5.2.2 for further details).

Let $\Gamma=\mathrm{Gal}(K_\infty/K)$ denote the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension of K. The proof of Theorem C is based on a new calculation of the Γ-Euler characteristic (in the sense of [Gre99, §4]) of a certain the Selmer group $\mathcal{X}_v(g,\chi)$. In the non-CM case (and for $r=1$), a related computation was obtained by Jetchev–Skinner–Wan [JSW17, §4]. As one of the most novel features in our approach, here we exploit the decomposition

$$\mathcal{X}_v(g,\chi)\simeq\mathcal{X}_v(\lambda\mathbf{N}^{-r})\oplus\mathcal{X}_v(\psi^\tau\chi\mathbf{N}^{-r})\tag{1.1}$$

of Proposition 3.5.1, where $\mathcal{X}_v(\psi^\tau\chi\mathbf{N}^{-r})$ interpolates the Bloch–Kato Selmer group of $\psi^\tau\chi\mathbf{N}^{-r}$ over K_∞/K, while $\mathcal{X}_v(\lambda\mathbf{N}^{-r})$ is seen to agree with the result of *reversing* the local conditions at the primes above p defining the Bloch–Kato Selmer group of $\lambda\mathbf{N}^{-r}$ over the anticyclotomic tower. After computing the Γ-Euler characteristic for each of the summands in (1.1), the expressions we obtain combine quite pleasantly into the formula of Theorem C.

The proof of Theorem A is easily deduced from an application of Theorem B to (the isogeny factors of) $B=\mathrm{Res}_{F/K}(E)$, so it remains to explain how to go from Theorem C to Theorem B, for which we rely on the general Gross–Zagier formula [YZZ13] (in its explicit form in [CST14]). Letting $B_{g,\chi}/K$ be the Serre tensor attached to the pair (g,χ), the deduction relies on the factorization

$$L(B_{g,\chi}/K,s)=\prod_{\sigma:L\hookrightarrow\mathbb{C}}L(g^\sigma/K,\chi^\sigma,s)$$

and the explicit description of the motivic structure attached to Hecke characters in the work of Kato [Kat04] and Burungale–Flach [BF24]. The existence of *strong good pairs* in the sense introduced in Definition 5.1.1 (whose existence is established in Lemma 5.1.3 building on Finis' mod p non-vanishing results [Fin06]) is also used at this point to address a certain period comparison building on work of Hida–Tilouine [HT93, HT94], as completed by Hida [Hid06], on the anticyclotomic Iwasawa main conjecture for CM fields.

Remark 1.2.2. Granted a Gross–Zagier type formula for the generalized Heegner cycles of [BDP13] in the style of S.-W. Zhang [Zha97][1], Theorem C should yield a proof of the p-part of the equivariant Tamagawa number conjecture of Burns–Flach [BF01] for CM motives in analytic rank 1 under standard hypotheses[2]. (See [LV23] for results in this direction in the non-CM case building on [Zha97].) This will be studied in the forthcoming PhD thesis by Mychelle Parker [Par24], which will also include a proof, building on the methods of this paper, of a higher weight analogue of Skinner's p-converse theorem [Ski20] in the CM case.

1.3. **About the hypotheses.** The restriction on the conductor in our results arises from the *Heegner hypothesis* present in some of our sources (such as [CH18] and [JLZ21]), and could be removed with a suitable refinement of the results in [LZZ18]. Our second additional hypothesis that $p \nmid h_K$ would also become superfluous with a better understanding of the relation between modular degrees and congruence numbers studied in [ARS12]. Lastly, removing the hypotheses that $p > 3$ and p be split in K would seem to require new ideas. We hope to come back to this in future work.

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We would like to dedicate this work to the memory of the late Jan Nekovář, whose encouragement was heartfelt and instrumental to us, specially in the early career stages.

2. p-adic L-functions

In this section we describe the p-adic L-functions needed for our arguments. The key result is the factorization of Proposition 2.3.1. The discussion in this section parallels [BCST22, §4], but here we need to pay more attention to integrality properties.

2.1. p-adic L-functions for self-dual pairs.

2.1.1. *Self-dual pairs.* Let $g \in S_{2r}(\Gamma_0(N_g), \varepsilon_g)$ be a newform of even weight $k = 2r \geq 2$, and let $K/\mathbb{Q}$ be an imaginary quadratic field of discriminant $-D_K < 0$ satisfying the *Heegner hypothesis*:

(Heeg) there exists an ideal $\mathfrak{N}_g$ with $\mathcal{O}_K/\mathfrak{N}_g \simeq \mathbb{Z}/N_g\mathbb{Z}$.

We also fix once and for all embedding $\imath_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\imath_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, and assume that

(spl) $(p) = v\overline{v}$ splits in K,

with v the prime of K above p induced by $\imath_p$.

As in [BCST22, §2.1], we say that a Hecke character $\psi = (\psi_w)_w : K^\times\backslash\mathbb{A}_K^\times \to \mathbb{C}^\times$ has *infinity type* $(a, b) \in \mathbb{Z}^2$ if $\psi_\infty(z) = z^a\overline{z}^b$ for all $z \in (K \otimes_\mathbb{Q} \mathbb{R})^\times \simeq \mathbb{C}^\times$ under the identification induced by $\imath_\infty$. In particular, the norm character $\mathbf{N}$ given by $\mathfrak{a} \mapsto \#(\mathcal{O}_K/\mathfrak{a})$ on ideals of $\mathcal{O}_K$, has infinity type $(-1, -1)$. Then the *central character* of ψ is the Dirichlet character ε_ψ defined by

$$\psi|_{\mathbb{A}^\times} = \varepsilon_\psi \cdot |\ |_{\mathbb{A}^\times}^{-a-b}.$$

We say that ψ is *anticyclotomic* if $\psi|_{\mathbb{A}^\times} = 1$; in particular, such ψ has trivial central character and its infinity type is of the form $(n, -n)$.

[1]See forthcoming work of Lilienfeldt–Shnidman for progress in this direction.

[2]That is, injectivity of the p-adic Abel–Jacobi map and non-degeneracy of a (Gillet–Soulé) height pairing.

Definition 2.1.1. Let χ be a finite order character of K. We say that (g,χ) is a *self-dual pair* if

$$\varepsilon_\chi = \varepsilon_g^{-1}.$$

Then the Rankin–Selberg L-function $L(g/K,\chi,s)$ is self-dual, with a functional equation relating its values at s and $2r-s$. The sign $\epsilon(g,\chi)\in\{\pm1\}$ in the functional equation is a product of local signs:

$$\epsilon(g,\chi)=\prod_q \epsilon_q(g,\chi),$$

where q runs over all places of $\mathbb{Q}$.

Let $\hat{K}^\times$ denote group of finite idèles of K. Attached to a character ψ of K of infinity type (a,b) is its *p-adic avatar* $\hat{\psi}: K^\times\backslash\hat{K}^\times\to\mathbb{C}_p^\times$ defined by

$$\hat{\psi}(x)=\imath_p\circ\imath_\infty^{-1}(\psi(x))\sigma(x_p)^a\overline{\sigma}(x_p)^b$$

for all $x\in\hat{K}^\times$, where $\sigma: K\otimes_{\mathbb{Q}}\mathbb{Q}_p\to\mathbb{C}_p$ is the map induced by $\imath_p$, and $\overline{\sigma}:=\sigma\circ\tau$ for the non-trivial automorphism τ of $K/\mathbb{Q}$. Via the (geometrically normalized) reciprocity map $\mathrm{rec}_K: K^\times\backslash\hat{K}^\times\to G_K^{\mathrm{ab}}$, we shall view $\hat{\psi}$ as a Galois character. Since it should not lead to confusion, in the following we shall still use ψ to denote its p-adic avatar $\hat{\psi}$ (except for $\psi=\mathbf{N}$, whose p-adic avatar is $\varepsilon_{\mathrm{cyc}}^{-1}$ for the p-adic cyclotomic character $\varepsilon_{\mathrm{cyc}}: G_K\to\mathbb{Z}_p^\times$).

For $\mathscr{O}$ the ring of integers of a finite extension Φ of $\mathbb{Q}_p$, we say that the pair (g,χ) is *defined over* $\mathscr{O}$ if (under our fixed embeddings $\imath_\infty,\imath_p$) the Fourier coefficients of g and the values of χ are all valued in $\mathscr{O}$.

Definition 2.1.2. For a positive integer c prime to N_g, we let $\Sigma_{\mathrm{cc}}(c,\mathfrak{N}_g,\varepsilon_g)$ denote the set of finite order characters χ such that (g,χ) is a self-dual pair and moreover:

(i) The conductor of χ is

$$\mathfrak{f}_\chi=(c)\mathfrak{N}_{\varepsilon_g},$$

where $\mathfrak{N}_{\varepsilon_g}$ is the unique divisor of $\mathfrak{N}_g$ with norm equal to the conductor of ε_g;

(ii) $\epsilon_q(g,\chi)=+1$ for all finite primes q.

In particular, for $\chi\in\Sigma_{\mathrm{cc}}(c,\mathfrak{N}_g,\varepsilon_g)$ and ξ any anticyclotomic Hecke character of K of infinity type $(n,-n)$ and conductor divisible only by primes that split in K, the L-series $L(g/K,\chi\xi,s)$ is self-dual with center at $s=r$ and sign

$$\epsilon(g,\chi\xi)=\begin{cases}+1 & \text{if } |n|\geq r,\\ -1 & \text{if } |n|<r.\end{cases}$$

The p-adic L-functions in this section interpolate the central L-values $L(g/K,\chi\xi,r)$ for $|n|\geq r$.

2.1.2. *CM periods.* Fix an elliptic curve A_0/F defined over a number field F with complex multiplication by $\mathcal{O}_K$, and let $(\Omega_K,\Omega_p)\in\mathbb{C}^\times\times\mathcal{O}_{\mathbb{C}_p}^\times$ be the complex and p-adic CM periods in [CH18, §2.5] (with A_0 corresponding to the CM elliptic curve denoted A in *loc. cit.*). Put also

$$\Omega:=2\pi i\cdot\Omega_K,$$

and note that this recovers (up to $F^\times$) the complex CM period appearing in [dS87, II.4.2]. For any embedding $\sigma: F\to\mathbb{C}$, we define $\Omega_\sigma\in\mathbb{C}^\times$ by replacing A_0 by A_0^σ in the above definition.

2.1.3. *p-adic interpolation.* For any abelian extension K'/K, let $\Lambda_{\mathscr{O}}(K')$ denote the Iwasawa algebra $\mathscr{O}[\![\mathrm{Gal}(K'/K)]\!] := \varprojlim \mathscr{O}[\mathrm{Gal}(K''/K)]$, where K'' runs over the finite extensions of K contained in K', and the inverse limit is with respect to the natural projection maps. We also put

$$\Lambda_{\mathscr{O}}^{\mathrm{ur}}(K') := \Lambda_{\mathscr{O}}(K')\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p^{\mathrm{ur}},$$

where $\mathbb{Z}_p^{\mathrm{ur}}$ denotes the completion of the ring of integers of the maximal unramified extension of $\mathbb{Q}_p$.

Let H_{p^∞} be the union of the ring class fields of K of p-power conductor; the maximal torsion-free quotient of $\widetilde{\Gamma} := \mathrm{Gal}(H_{p^\infty}/K)$ is by definition the Galois group

$$\Gamma := \mathrm{Gal}(K_\infty/K)$$

of the anticyclotomic $\mathbb{Z}_p$-extension of K. Since anticyclotomic Iwasawa algebras will play a prominent role in the paper, we set

$$\Lambda_{\mathscr{O}} := \Lambda_{\mathscr{O}}(K_\infty), \qquad \Lambda_{\mathscr{O}}^{\mathrm{ur}} := \Lambda_{\mathscr{O}}^{\mathrm{ur}}(K_\infty)$$

for the ease of notation.

Theorem 2.1.3. *Let $(g,\chi) \in S_{2r}(\Gamma_1(N_g)) \times \Sigma_{\mathrm{cc}}(c, \mathfrak{N}_g, \varepsilon_g)$ be a self-dual pair for some positive integer c prime to pN_g defined over $\mathscr{O}$. There exists a "square-root" p-adic L-function*

$$\mathscr{L}_v(g,\chi) \in \Lambda_{\mathscr{O}}^{\mathrm{ur}}(H_{p^\infty})$$

characterized by the property that for every character ξ of $\widetilde{\Gamma}$ crystalline at both v and $\overline{v}$ corresponding to a Hecke character of K of infinity type $(n,-n)$ with $n \geq r$ and $n \equiv 0 \pmod{p-1}$, we have

$$\mathscr{L}_v(g,\chi)^2(\xi) = \frac{\Omega_p^{4n}}{\Omega^{4n}} \cdot \Gamma(r+n)\Gamma(n+1-r) \cdot \left(\frac{2\pi}{\sqrt{D_K}}\right)^{2n-1}$$
$$\times \left(1 - a_p(g)\chi\xi(\overline{v})p^{-r} + \varepsilon_g(p)\chi\xi(\overline{v})^2p^{-1}\right)^2 \cdot L(g/K, \chi\xi, r).$$

Proof. This follows from the results of [CH18, §3] as in [BCST22, Thm. 4.1]. (Note that in the above interpolation property we have replaced the CM period Ω_K from [CH18] by the above Ω, and have omitted terms interpolated by a unit in $\Lambda_{\mathscr{O}}^{\mathrm{ur}}(H_{p^\infty})^\times$.) □

Remark 2.1.4. Reversing the roles of v and $\overline{v}$ in Theorem 2.1.3, we obtain an element $\mathscr{L}_{\overline{v}}(g,\chi) \in \Lambda_{\mathscr{O}}^{\mathrm{ur}}(H_{p^\infty})$ interpolating the central critical values $L(g/K, \chi\xi, r)$ for ξ of infinity type $(-n, n)$ with $n \geq r$ and $n \equiv 0 \pmod{p-1}$. We shall use both $\mathscr{L}_v(g,\chi)$ and $\mathscr{L}_{\overline{v}}(g,\chi)$ in the following.

With a slight abuse of notation, we also denote by $\mathscr{L}_v(g,\chi)$ its image under the natural projection $\Lambda_{\mathscr{O}}^{\mathrm{ur}}(H_{p^\infty}) \to \Lambda_{\mathscr{O}}^{\mathrm{ur}}$, and similarly for $\mathscr{L}_{\overline{v}}(g,\chi)$.

2.2. Katz p-adic L-functions.

The construction of p-adic L-functions in the next result is originally due to Katz [Kat78], but the alternative construction by de Shalit [dS87], whose exposition we follow, will be most convenient for our purposes.

2.2.1. *Two-variable p-adic L-functions.* For an integral ideal $\mathfrak{c}$ of K, let $K(\mathfrak{c}p^\infty)$ be the union of the ray class fields of K conductor $\mathfrak{c}p^n$ for $n \geq 0$, and put

$$G_{\mathfrak{c}} := \mathrm{Gal}(K(\mathfrak{c}p^\infty)/K).$$

For a character ξ of $G_{\mathfrak{c}}$, let ξ^τ denote the composition of ξ with action of the non-trivial automorphism of $K/\mathbb{Q}$, and put $\xi^{-\tau} := (\xi^\tau)^{-1}$.

Theorem 2.2.1. *Let $\mathfrak{c}$ be an integral ideal of K prime to p. There exists an element*

$$\mathscr{L}_{v,\mathfrak{c}} \in \Lambda_{\mathscr{O}}^{\mathrm{ur}}(K(\mathfrak{c}p^\infty))$$

characterized by the property that for every character ξ of $G_{\mathfrak{c}}$ crystalline at both v and $\overline{v}$ corresponding to a Hecke character of infinity type (a,b) with $a > 0$ and $b \leq 0$ we have

$$\mathscr{L}_{v,\mathfrak{c}}(\xi) = \frac{\Omega_p^{a-b}}{\Omega^{a-b}} \cdot \Gamma(a) \cdot \left(\frac{\sqrt{D_K}}{2\pi}\right)^b \cdot (1 - \xi^{-1}(v)p^{-1})(1 - \xi(\overline{v})) \cdot L_{\mathfrak{c}}(\xi, 0),$$

where $L_{\mathfrak{c}}(\xi,s)$ denotes the Hecke L-function of ξ with the Euler factors at the primes $w \mid \mathfrak{c}$ removed. Similarly, there exists an element $\mathscr{L}_{\overline{v},\mathfrak{c}} \in \Lambda_{\mathscr{O}}^{\mathrm{ur}}(K(\mathfrak{c}p^\infty))$ such that for every character ξ of $G_{\mathfrak{c}}$ crystalline at both v and $\overline{v}$ corresponding to a Hecke character of infinity type (b,a) with $a>0$ and $b\le 0$ we have

$$\mathscr{L}_{\overline{v},\mathfrak{c}}(\xi) = \frac{\Omega_p^{a-b}}{\Omega^{a-b}}\cdot\Gamma(a)\cdot\left(\frac{\sqrt{D_K}}{2\pi}\right)^b\cdot(1-\xi^{-1}(\overline{v})p^{-1})(1-\xi(v))\cdot L_{\mathfrak{c}}(\xi,0).$$

Moreover, we have the functional equation

$$\mathscr{L}_{v,\mathfrak{c}}(\xi) = \mathscr{L}_{v,\overline{\mathfrak{c}}}(\xi^{-\tau}\mathbf{N}^{-1}),$$

where the equality is up to a p-adic unit, and similarly for $\mathscr{L}_{\overline{v},\mathfrak{c}}$.

Proof. The construction of the p-adic L-functions with the stated interpolation properties is given in [dS87, Thm. II.4.14]: $\mathscr{L}_{v,\mathfrak{c}}$ (resp. $\mathscr{L}_{\overline{v},\mathfrak{c}}$) corresponds the p-adic measure $\mu(\mathfrak{c}\overline{v}^\infty)$ (resp. $\mu(\mathfrak{c}v^\infty)$) on $G_{\mathfrak{c}}$ constructed in *loc. cit.*. The functional equation is proved in [dS87, Thm. II.6.4], which implies that the interpolation formulae extend from the range $0\le -b<a$ to the range $a>0$ and $b\le 0$ (see [dS87, Cor. II.6.7]). □

2.2.2. *Anticyclotomic projection.* We shall be particularly interested in the anticyclotomic projection of Katz's p-adic L-functions twisted by characters that are self-dual in the following sense.

Definition 2.2.2. We say that a Hecke character ϕ of K is *self-dual* if it satisfies

$$\phi^\tau = \phi^{-1}\mathbf{N}^{-1} \quad \text{and} \quad \varepsilon_\phi = \eta_K,$$

where η_K is the quadratic Dirichlet character corresponding to $K/\mathbb{Q}$.

In particular, if ϕ is self-dual then its p-adic avatar is a conjugate self-dual character of the absolute Galois group $G_K = \mathrm{Gal}(\overline{\mathbb{Q}}/K)$, i.e.

$$\phi^\tau = \varepsilon_{\mathrm{cyc}}\phi^{-1}.$$

Note that for self-dual ϕ, the Hecke L-function $L(\phi,s)$ is self-dual, with a functional equation relating its values at s and $-s$. Moreover, the infinity type of such ϕ is of the form $(1-r,r)$ for some $r\in\mathbb{Z}$ and its conductor is invariant under complex conjugation.

Let F/K be an abelian extension contained in $K(\mathfrak{c}p^\infty)$. For a character ϕ of $\mathrm{Gal}(F/K)$ valued in $\mathscr{O}$ and having $\mathfrak{c}$ as the prime-to-p part of its conductor, we denote by $\mathscr{L}_v(\phi)$ the image of $\mathscr{L}_{v,\mathfrak{c}}$ under the composite map

$$\Lambda_{\mathscr{O}}^{\mathrm{ur}}(K(\mathfrak{c}p^\infty)) \xrightarrow{\mathrm{Tw}_\phi} \Lambda_{\mathscr{O}}^{\mathrm{ur}}(K(\mathfrak{c}p^\infty)) \twoheadrightarrow \Lambda_{\mathscr{O}}^{\mathrm{ur}},$$

where Tw_ϕ is the $\mathscr{O}$-linear isomorphism given by $\gamma\mapsto\phi(\gamma)\gamma$ for $\gamma\in G_{\mathfrak{c}}$ and the second arrow is the natural projection (noting that $K(\mathfrak{c}p^\infty)$ contains H_{p^∞}, and hence also K_∞).

Remark 2.2.3. Although not reflected in the notation for simplicity, when we write $\mathscr{L}_v(\psi)$ (resp. $\mathscr{L}_{\overline{v}}(\psi)$) it will be understood that this is obtained from $\mathscr{L}_{v,\mathfrak{c}}$ (resp. $\mathscr{L}_{\overline{v},\mathfrak{c}}$) are above taking for $\mathfrak{c}$ the prime-to-p part of the conductor of ϕ.

2.3. **Factorization.** Recall that if ψ is a Hecke character of infinity type $(1-2r,0)$ and conductor $\mathfrak{f}_\psi$, then the theta series $g=\theta_\psi$ is an eigenform in $S_{2r}(\Gamma_0(N_g),\varepsilon_g)$ with $N_g = D_K\mathbf{N}(\mathfrak{f}_\psi)$ and $\varepsilon_g=\varepsilon_\psi\eta_K$.

Proposition 2.3.1. *Let ψ be a Hecke character of K of infinity type $(1-2r,0)$ for some $r\ge 1$ and conductor a cyclic ideal $\mathfrak{f}_\psi$ of norm prime to pD_K. Let*

$$(g,\chi) = (\theta_\psi,\chi)\in S_{2r}(\Gamma_0(N_g),\varepsilon_g)\times\Sigma_{\mathrm{cc}}(c,\mathfrak{N}_g,\varepsilon_g)$$

be a self-dual pair for some positive integer c prime to pN_g defined over $\mathscr{O}$. Then for every $w\in\{v,\overline{v}\}$ we have

$$\mathscr{L}_w(g,\chi)^2 = u\cdot\mathscr{L}_w(\psi\chi\mathbf{N}^{-r})\cdot\mathscr{L}_w(\psi^\tau\chi\mathbf{N}^{-r}),$$

where u is a unit in $\Lambda_{\mathscr{O}}^{\mathrm{ur},\times}$.

Proof. We only explain the proof in the case $w = v$, since the argument for $w = \overline{v}$ is virtually the same. By our assumption on $\mathfrak{f}_\psi$, the imaginary quadratic field K satisfies (Heeg) with $\mathfrak{N}_g = \mathfrak{d}_K \cdot \mathfrak{f}_\psi$. Let ξ be a character as in the statement of Theorem 2.1.3, corresponding to a Hecke character of K of infinity type $(n, -n)$ with $n \geq r$, and assume that ξ factors through Γ. Then the central L-value $L(\theta_\psi/K, \chi\xi, r)$ factors as

$$L(\theta_\psi/K, \chi\xi, r) = L(\psi\chi\xi, r) \cdot L(\psi^\tau\chi\xi, r) = L(\psi\chi\mathbf{N}^{-r}\xi, 0) \cdot L(\psi^\tau\chi\mathbf{N}^{-r}\xi, 0).$$

The Hecke characters $\psi\chi\mathbf{N}^{-r}\xi$ and $\psi^\tau\chi\mathbf{N}^{-r}\xi$ have infinity type $(1-r+n, r-n)$ and $(n+r, 1-n-r)$, and one easily checks (see [BDP12, Lem. 3.16]) that they are both self-dual, with prime-to-p conductor $(c)\mathfrak{d}_K$ and $(cM)\mathfrak{d}_K$, respectively, where $\mathfrak{d}_K := (\sqrt{-D_K})$ and $M := \mathbf{N}(\mathfrak{f}_\psi)$; thus they are in the range of interpolation of $\mathscr{L}_{v,(c)\mathfrak{d}_K}$ and $\mathscr{L}_{v,(cM)\mathfrak{d}_K}$, respectively. From the self-duality of $\psi\chi\mathbf{N}^{-r}\xi$ and $\psi^\tau\chi\mathbf{N}^{-r}\xi$ we immediately get the relations

$$(1 - \psi^{-1}\chi^{-1}\xi^{-1}(v)p^{r-1}) = (1 - \psi\chi\xi(\overline{v})p^{-r}), \quad (1 - \psi^{-\tau}\chi^{-1}\xi^{-1}(\overline{v})p^{r-1}) = (1 - \psi^\tau\chi\xi(\overline{v})p^{-r}),$$

and so from the interpolation formula in Theorem 2.2.1 we obtain

$$\begin{aligned}\mathscr{L}_{v,(c)\mathfrak{d}_K}(\psi\chi\mathbf{N}^{-r}\xi) \cdot \mathscr{L}_{v,(c)\mathfrak{d}_K M}(\psi^\tau\chi\mathbf{N}^{-r}\xi) &= \frac{\Omega_p^{2n-2r+1}}{\Omega^{2n-2r+1}} \cdot \frac{\Omega_p^{2n+2r-1}}{\Omega^{2n+2r-1}} \cdot \Gamma(n-r+1) \cdot \Gamma(n+r) \\ &\times \left(\frac{2\pi}{\sqrt{D_K}}\right)^{n-r} \cdot \left(\frac{2\pi}{\sqrt{D_K}}\right)^{n+r-1} \cdot \big(1 - \psi\chi\xi(\overline{v})p^{-r}\big)^2 \cdot \big(1 - \psi^\tau\chi\xi(\overline{v})p^{-r}\big)^2 \\ &\times L(\psi\chi\mathbf{N}^{-r}\xi, 0) \cdot L(\psi^\tau\chi\mathbf{N}^{-r}\xi, 0),\end{aligned} \tag{2.1}$$

using that the prime-to-p conductors of $\psi\chi\mathbf{N}^{-r}\xi$ and $\psi^\tau\chi\mathbf{N}^{-r}\xi$ are $(c)\mathfrak{d}_K$ and $(cM)\mathfrak{d}_K$, respectively, to equivalently write the primitive Hecke L-values. Noting that

$$(1 - \psi\chi\xi(\overline{v})p^{-r}\big) \cdot \big(1 - \psi^\tau\chi\xi(\overline{v})p^{-r}\big) = (1 - a_p(g)\chi\xi(\overline{v})p^{-r} + \varepsilon_g(p)\chi\xi(\overline{v})^2p^{-1}),$$

comparing (2.1) with the interpolation property of Theorem 2.1.3, the result follows. □

3. Selmer groups and main conjectures

Throughout this section, we fix a prime $p > 3$ and integer $N_g \geq 1$ with $p \nmid N_g$, and let

$$(g, \chi) \in S_{2r}(\Gamma_1(N_g)) \times \Sigma_{\mathrm{cc}}(c, \mathfrak{N}_g, \varepsilon_g)$$

be a self-dual pair defined over $\mathscr{O}$ for some positive integer c prime to pN_g and an imaginary quadratic field K satisfying (Heeg) and (spl).

3.1. Selmer groups for self-dual pairs.

Let V_g denote the p-adic Galois representation associated to g by Deligne, and put

$$V_{g,\chi} := V_g(r)|_{G_K} \otimes \chi.$$

Definition 3.1.1. Let $w \in \{v, \overline{v}\}$ be a prime of K above p, and let F/K be a finite extension. For η a prime of F above w, define the local condition $\mathrm{H}^1_w(F_\eta, V_{g,\chi})$ by

$$\mathrm{H}^1_w(F_\eta, V_{g,\chi}) := \begin{cases} \mathrm{H}^1(F_\eta, V_{g,\chi}) & \text{if } \eta \mid w, \\ 0 & \text{if } \eta \mid \overline{w}, \end{cases}$$

and let $\mathrm{H}^1_f(F_\eta, V_{g,\chi})$ denote the Bloch–Kato finite subspace, i.e.

$$\mathrm{H}^1_f(F_\eta, V_{g,\chi}) := \ker\big\{\mathrm{H}^1(F_\eta, V_{g,\chi}) \to \mathrm{H}^1(F_\eta, V_{g,\chi} \otimes \mathbf{B}_{\mathrm{cris}})\big\},$$

where $\mathbf{B}_{\mathrm{cris}}$ is Fontaine's ring of crystalline periods. For a finite prime $\eta \nmid p$ of F, let

$$\mathrm{H}^1_{\mathrm{ur}}(F_\eta, V_{g,\chi}) := \ker\big\{\mathrm{H}^1(F_\eta, V_{g,\chi}) \to \mathrm{H}^1(I_\eta, V_{g,\chi})\big\}$$

be the unramified subspace.

- The *w-Selmer group* of $V_{g,\chi}$ is
$$\mathrm{Sel}_w(F,V_{g,\chi}) := \ker\Big\{\mathrm{H}^1(F,V_{g,\chi}) \to \prod_{\eta\mid p}\frac{\mathrm{H}^1(F_\eta,V_{g,\chi})}{\mathrm{H}^1_w(F_\eta,V_{g,\chi})}\times\prod_{\eta\nmid p}\frac{\mathrm{H}^1(F_\eta,V_{g,\chi})}{\mathrm{H}^1_{\mathrm{ur}}(F_\eta,V_{g,\chi})}\Big\}.$$
- The *Bloch–Kato Selmer group* of $V_{g,\chi}$ is
$$\mathrm{Sel}_{\mathrm{BK}}(F,V_{g,\chi}) := \ker\Big\{\mathrm{H}^1(F,V_{g,\chi}) \to \prod_{\eta\mid p}\frac{\mathrm{H}^1(F_\eta,V_{g,\chi})}{\mathrm{H}^1_f(F_\eta,V_{g,\chi})}\times\prod_{w\nmid p}\frac{\mathrm{H}^1(F_\eta,V_{g,\chi})}{\mathrm{H}^1_{\mathrm{ur}}(F_\eta,V_{g,\chi})}\Big\}.$$
(Note that since p is odd, for $\eta\mid\infty$ the groups $\mathrm{H}^1(F_\eta,V_{g,\chi})$ all vanish.)

Let $T_{g,\chi}\subset V_{g,\chi}$ be a G_K-stable $\mathscr{O}$-lattice, and define $W_{g,\chi}$ by the exact sequence
$$0\to T_{g,\chi}\to V_{g,\chi}\to W_{g,\chi}\to 0. \tag{3.1}$$
Then we define the Selmer groups $\mathrm{Sel}_w(F,T_{g,\chi}),\mathrm{Sel}_w(F,W_{g,\chi}),\mathrm{Sel}_{\mathrm{BK}}(F,T_{g,\chi}),\mathrm{Sel}_{\mathrm{BK}}(F,W_{g,\chi})$ as above, with the corresponding local conditions obtained from those of $V_{g,\chi}$ by propagating via (3.1).

Finally, for $\star\in\{v,\overline{v},\mathrm{BK}\}$, we define the Λ-adic Selmer groups
$$\check{\mathcal{S}}_\star(g,\chi) := \varprojlim_F \mathrm{Sel}_\star(F,T_{g,\chi}),\qquad \mathcal{S}_\star(g,\chi) := \varinjlim_F \mathrm{Sel}_\star(F,W_{g,\chi}),$$
where F runs over the finite extensions of K contained in K_∞, and the limits are with respect to the corestriction and restriction maps, respectively, and put
$$\mathcal{X}_\star(g,\chi) := \mathrm{Hom}_{\mathrm{cts}}(\mathcal{S}_\star(g,\chi),\mathbb{Q}_p/\mathbb{Z}_p)$$
for the Pontryagin dual of $\mathcal{S}_\star(g,\chi)$.

3.2. **Generalized Heegner cycles.** We recall the construction of $\Lambda_{\mathscr{O}}$-adic classes $\mathbf{z}_{g,\chi}$ interpolating the generalized Heegner cycles of Bertolini–Darmon–Prasanna [BDP13] over the anticyclotomic tower K_∞/K, as well as their link with the p-adic L-functions $\mathscr{L}_w(g,\chi)$.

A main result of [BDP13] is the following formula for the value of $\mathscr{L}_w(g,\chi)$ at the trivial character $\xi=\mathbb{1}$ of Γ, which lies outside the range of p-adic interpolation in Theorem 2.1.3. For $w\in\{v,\overline{v}\}$ a prime of K above p, denote by
$$\log_{V_{g,\chi}}:\mathrm{H}^1_f(K_w,V_{g,\chi})\to\frac{\mathbf{D}_{\mathrm{dR}}(V_{g,\chi})}{\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V_{g,\chi})}\simeq\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V^*_{g,\chi}(1))^\vee$$
the Bloch–Kato logarithm map, and recall the CM elliptic curve A_0/F used for the definition of the CM periods in §2.1.2.

Theorem 3.2.1 (Bertolini–Darmon–Prasanna)**.** *There exists a class $z_{g,\chi}\in\mathrm{Sel}_{\mathrm{BK}}(K,V_{g,\chi})$ such that*
$$\mathscr{L}_w(g,\chi)(\mathbb{1})=\frac{c^{-r}}{\Gamma(r)}\cdot\big(1-a_p(g)\chi(\overline{w})p^{-r}+\varepsilon_g(p)\chi(\overline{w})^2p^{-1}\big)\cdot\Big\langle\log_{V_{g,\chi}}(\mathrm{loc}_{\overline{w}}(z_{g,\chi})),\omega_g'\wedge\omega_{A_0}^{r-1}\eta_{A_0}^{r-1}\Big\rangle,$$
where $\omega_g'\in\mathbf{D}_{\mathrm{dR}}(V_g^*)$ is a differential attached to g as in [KLZ20]*, and ω_{A_0} and η_{A_0} are differentials attached to A_0 as in* [BDP13, §1.4]*.*

Proof. This is [BDP13, Thm. 5.13], as reformulated in [JLZ21, Thm. 7.2.4]. □

By interpolating p-power conductor variants of the classes $z_{g,\chi}$ into a $\Lambda_{\mathscr{O}}$-adic class $\mathbf{z}_{g,\chi}\in\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)$, a generalization of Theorem 3.2.1 allows one to recover $\mathscr{L}_v(g,\chi)$ as the image of $\mathbf{z}_{g,\chi}$ under a generalized Coleman power series map. This was first done in a joint work of the author with M.-L. Hsieh [CH18] for $\chi=\mathbb{1}$, and later in [JLZ21] in the level of generality required for this paper.

To state the result, given a class $\mathbf{z}\in\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)$, we let $\mathbf{z}(\mathbb{1})$ denote the image of $\mathbf{z}$ under the natural projection
$$\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)\to\mathrm{Sel}_{\mathrm{BK}}(K,V_{g,\chi}).$$
Let α be the p-adic unit root of $x^2-a_p(g)x+\varepsilon_g(p)p^{2r-1}$, and let $\varpi\in\mathscr{O}$ be a uniformizer.

Theorem 3.2.2. *Let $w \in \{v, \overline{v}\}$ be a prime of K above p. There exits a class $\mathbf{z}_{g,\chi} \in \check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)$ with*

$$\mathbf{z}_{g,\chi}(\mathbb{1}) = \mathcal{E}_p(g,\chi) \cdot z_{g,\chi},$$

where $\mathcal{E}_p(g,\chi) = \big(1 - \frac{\chi(v)p^{r-1}}{\alpha}\big)\big(1 - \frac{\chi(\overline{v})p^{r-1}}{\alpha}\big)$, and an injective $\Lambda_{\mathscr{O}}^{\mathrm{ur}}$-module homomorphism

$$\mathrm{Col}_{\overline{w}} : \varprojlim_{F \subset K_\infty} \prod_{\eta | \overline{w}} \mathrm{H}^1_f(F_\eta, T_{g,\chi}) \hat{\otimes}_{\mathscr{O}} \Lambda_{\mathscr{O}}^{\mathrm{ur}} \to \Lambda_{\mathscr{O}}^{\mathrm{ur}}$$

with finite cokernel for which we have the "explicit reciprocity law"

$$\mathrm{Col}_{\overline{w}}\big(\mathrm{loc}_{\overline{w}}(\mathbf{z}_{g,\chi})\big) = \mathscr{L}_w(g,\chi).$$

Proof. The construction of $\mathbf{z}_{g,\chi}$ is given in [CH18, §5.2] in the case $\varepsilon_g = 1$ and in [JLZ21, Thm. 5.4.1][3] in general. Similarly, the construction of $\mathrm{Col}_{\overline{w}}$ and the proof of the explicit reciprocity law is given in [CH18, §5.3] in the case $\varepsilon_g = 1$, and in [JLZ21, §8] in general. □

3.3. **Main conjectures for self-dual pairs.** We now state the anticyclotomic Iwasawa main conjectures that will be studied in this paper.

Conjecture 3.3.1 (Iwasawa–Greenberg main conjecture)**.** Let $w \in \{v, \overline{v}\}$ be a prime of K above p, and assume that $\mathscr{L}_w(g,\chi) \neq 0$. Then:

(i) $\mathrm{rank}_{\Lambda_{\mathscr{O}}}\big(\check{\mathcal{S}}_w(g,\chi)\big) = \mathrm{rank}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_w(g,\chi)\big) = 0$;

(ii) We have

$$\mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_w(g,\chi)\big) = \big(\mathscr{L}_w(g,\chi)^2\big)$$

as ideals in $\Lambda_{\mathscr{O}}^{\mathrm{ur}}$.

Remark 3.3.2. It follows from Perrin-Riou's results [PR89] that the characteristic ideal of $\mathcal{X}_w(g,\chi)$ is independent of the choice of G_K-stable $\mathscr{O}$-lattice $T_{g,\chi}$ in $V_{g,\chi}$ used in the definition of $\mathcal{X}_w(g,\chi)$ (see [KO20, Prop. 2.9]). Note that this is consistent with Conjecture 3.3.1(ii), since the periods used in the construction of $\mathscr{L}_w(g,\chi)$ depend only on K.

The following is a natural higher weight extension of the Heegner point main conjecture of [BT20] and [BCST22] (see also [LV19] for an analogous higher weight extension of the original Heegner point main conjecture by Perrin-Riou [PR87a]).

Conjecture 3.3.3 (Heegner cycle main conjecture)**.** The following hold:

(i) $\mathbf{z}_{g,\chi}$ is not $\Lambda_{\mathscr{O}}$-torsion;

(ii) $\mathrm{rank}_{\Lambda_{\mathscr{O}}}\big(\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)\big) = \mathrm{rank}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_{\mathrm{BK}}(g,\chi)\big) = 1$;

(iii) We have

$$\mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_{\mathrm{BK}}(g,\chi)_{\mathrm{tors}}\big) = \mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)/\Lambda_{\mathscr{O}} \cdot \mathbf{z}_{g,\chi}\big)^2,$$

where the subscript tors denotes the maximal $\Lambda_{\mathscr{O}}$-torsion submodule.

Remark 3.3.4. It follows from Nekovář's results [Nek06] that there is a $\Lambda_{\mathscr{O}}$-module pseudo-isomorphism

$$\mathcal{X}_{\mathrm{BK}}(g,\chi)_{\mathrm{tors}} \sim M \oplus M$$

for a torsion $\Lambda_{\mathscr{O}}$-module M with $\mathrm{char}_{\Lambda_{\mathscr{O}}}(M)^\iota = \mathrm{char}_{\Lambda_{\mathscr{O}}}(M)$, where ι denotes the involution on $\Lambda_{\mathscr{O}}$ given by $\gamma \mapsto \gamma^{-1}$ for $\gamma \in \Gamma$ (see also [How04, p. 1464]). Thus we see that the equality of characteristic ideals in Conjecture 3.3.3 can alternatively be written as

$$\mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_{\mathrm{BK}}(g,\chi)_{\mathrm{tors}}\big) = \mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)/\Lambda_{\mathscr{O}} \cdot \mathbf{z}_{g,\chi}\big) \cdot \mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)/\Lambda_{\mathscr{O}} \cdot \mathbf{z}_{g,\chi}\big)^\iota.$$

[3] Note that this theorem achieves more, also interpolating $z_{g,\chi}$ along a "weight" variable for g, vastly generalizing the results of [How07, Cas13, Cas20, Ota20] in the p-ordinary case; cf. [Dis22, BL21]. Here we only need the interpolation result in the anticyclotomic direction.

3.4. **Selmer groups for Hecke characters.** In this subsection, the imaginary quadratic field K is only required to satisfy (spl) for our fixed prime $p > 3$.

Let ξ be a Hecke character of K of infinity type $(a, b) \in \mathbb{Z}^2$ with p-adic avatar valued in the ring of integers $\mathscr{O}$ of a finite extension $\Phi/\mathbb{Q}_p$. Put

$$V_\xi = \Phi(\xi),$$

where $\Phi(\xi)$ denotes the one-dimensional Φ-vector space on which G_K act via ξ.

Definition 3.4.1. Let $w \in \{v, \overline{v}\}$ be a prime of K above p, and let F/K be a finite extension. For η a prime of F above p, put

$$\mathrm{H}^1_f(F_\eta, V_\xi) := \ker\left\{\mathrm{H}^1(F_\eta, V_\xi) \to \mathrm{H}^1(F_\eta, V_\xi \otimes \mathbf{B}_{\mathrm{cris}})\right\}$$

and

$$\mathrm{H}^1_w(F_\eta, V_\xi) := \begin{cases} \mathrm{H}^1(F_\eta, V_\xi) & \text{if } \eta \mid w, \\ 0 & \text{if } \eta \mid \overline{w}, \end{cases}$$

and for η a finite prime of F not dividing p, let $\mathrm{H}^1_{\mathrm{ur}}(F_\eta, V_\xi) = \ker\{\mathrm{H}^1(F_\eta, V_\xi) \to \mathrm{H}^1(I_\eta, V_\xi)\}$ be the unramified subspace.

- The *Bloch–Kato* Selmer group of V_ξ is

$$\mathrm{Sel}_{\mathrm{BK}}(F, V_\xi) := \ker\left\{\mathrm{H}^1(F, V_\xi) \to \prod_{\eta \mid p} \frac{\mathrm{H}^1(F_\eta, V_\xi)}{\mathrm{H}^1_f(F_\eta, V_\xi)} \times \prod_{\eta \nmid p} \frac{\mathrm{H}^1(F_\eta, V_\xi)}{\mathrm{H}^1_{\mathrm{ur}}(F_\eta, V_\xi)}\right\}.$$

- The *w-Selmer group* of V_ξ is

$$\mathrm{Sel}_w(F, V_\xi) := \ker\left\{\mathrm{H}^1(F, V_\xi) \to \prod_{\eta \mid p} \frac{\mathrm{H}^1(F_\eta, V_\xi)}{\mathrm{H}^1_w(F_\eta, V_\xi)} \times \prod_{\eta \nmid p} \frac{\mathrm{H}^1(F_\eta, V_\xi)}{\mathrm{H}^1_{\mathrm{ur}}(F_\eta, V_\xi)}\right\}.$$

Let T_ξ denote the free rank 1 $\mathscr{O}$-module on which G_K acts via ξ, and put $W_\xi = V_\xi/T_\xi$. Similarly as above, for $\star \in \{\mathrm{BK}, v, \overline{v}\}$, we define $\mathrm{Sel}_\star(F, T_\xi)$ and $\mathrm{Sel}_\star(F, W_\xi)$ by propagation via $0 \to T_\xi \to V_\xi \to W_\xi \to 0$, define the $\Lambda_{\mathscr{O}}$-adic Selmer groups

$$\check{\mathcal{S}}_\star(\xi) := \varprojlim_{F \subset K_\infty} \mathrm{Sel}_\star(F, T_\xi), \qquad \mathcal{S}_\star(\xi) := \varinjlim_{F \subset K_\infty} \mathrm{Sel}_\star(F, W_\xi),$$

with F running over the finite extensions of K contained in K_∞, and let

$$\mathcal{X}_\star(\xi) := \mathrm{Hom}_{\mathrm{cts}}(\mathcal{S}_\star(\xi), \mathbb{Q}_p/\mathbb{Z}_p)$$

be the Pontryagin dual of $\mathcal{S}_\star(\xi)$.

Lemma 3.4.2. *Suppose ξ has infinity type (a, b). Let F/K be a finite extension and let η be a prime of F above p. Then*

$$\mathrm{H}^1_f(F_\eta, V_\xi) = \begin{cases} \mathrm{H}^1(F_\eta, V_\xi) & \text{if } \eta \mid v \text{ and } a > 0, \text{ or } \eta \mid \overline{v} \text{ and } b > 0, \\ 0 & \text{else.} \end{cases}$$

In particular,

$$\mathcal{X}_{\mathrm{BK}}(\xi) = \begin{cases} \mathcal{X}_v(\xi) & \text{if } a > 0 \text{ and } b \leq 0, \\ \mathcal{X}_{\overline{v}}(\xi) & \text{if } b > 0 \text{ and } a \leq 0. \end{cases}$$

Proof. With the convention that the p-adic cyclotomic character $\varepsilon_{\mathrm{cyc}} : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ has Hodge–Tate weight $+1$, our convention on infinity types implies that the p-adic avatar $\xi : G_K \to \mathscr{O}$ has Hodge–Tate weight a (resp. b) at v (resp. $\overline{v}$). In view of [BK90, Thm. 4.1(ii)], this implies the result. $\square$

3.5. **Decompositions.** When g is the theta series of a Hecke character ψ of K, then $V_g \simeq \mathrm{Ind}_K^{\mathbb{Q}} V_\psi$ and so

$$V_{g,\chi} \simeq V_{\psi\chi}(r) \oplus V_{\psi^\tau\chi}(r). \tag{3.2}$$

We fix an isomorphism as above, and let $T_{g,\chi} \subset V_{g,\chi}$ be the G_K-stable $\mathscr{O}$-lattice with $T_{g,\chi} \simeq T_{\psi\chi}(r) \oplus T_{\psi^\tau\chi}(r)$ under that isomorphism, so that

$$W_{g,\chi} := V_{g,\chi}/T_{g,\chi} \simeq W_{\psi\chi}(r) \oplus W_{\psi^\tau\chi}(r)$$

as G_K-modules.

Proposition 3.5.1. *Suppose $g = \theta_\psi \in S_{2r}(\Gamma_1(N_g))$ is the theta series of a Hecke character ψ of K, and χ is a finite order character with $\chi \in \Sigma_{\rm cc}(c, \mathfrak{N}_g, \varepsilon_g)$. Then there are $\Lambda_{\mathscr{O}}$-module isomorphisms*

$$\check{\mathcal{S}}_v(g,\chi) \simeq \check{\mathcal{S}}_v(\psi\chi\mathbf{N}^{-r}) \oplus \check{\mathcal{S}}_v(\psi^\tau\chi\mathbf{N}^{-r}),$$
$$\check{\mathcal{S}}_{\rm BK}(g,\chi) \simeq \check{\mathcal{S}}_{\overline{v}}(\psi\chi\mathbf{N}^{-r}) \oplus \check{\mathcal{S}}_v(\psi^\tau\chi\mathbf{N}^{-r}),$$

and similarly

$$\mathcal{X}_v(g,\chi) \simeq \mathcal{X}_v(\psi\chi\mathbf{N}^{-r}) \oplus \mathcal{X}_v(\psi^\tau\chi\mathbf{N}^{-r}),$$
$$\mathcal{X}_{\rm BK}(g,\chi) \simeq \mathcal{X}_{\overline{v}}(\psi\chi\mathbf{N}^{-r}) \oplus \mathcal{X}_v(\psi^\tau\chi\mathbf{N}^{-r}).$$

Proof. The first and third isomorphisms are immediate from (3.2) and the definitions. On the other hand, note that the character $\psi\chi\varepsilon_{\rm cyc}^r$ has Hodge–Tate weight $1-r$ and r above v and $\overline{v}$, respectively, while $\psi^\tau\chi\varepsilon_{\rm cyc}^r$ has Hodge–Tate weight r and $1-r$ above v and $\overline{v}$, respectively. In view of Lemma 3.4.2, the second and fourth isomorphisms thus follow the decomposition

$$\mathrm{H}^1_f(F_\eta, V_{g,\chi}) = \mathrm{H}^1_f(F_\eta, V_{\psi\chi}(r)) \oplus \mathrm{H}^1_f(F_\eta, V_{\psi^\tau\chi}(r))$$

induced by (3.2) for any $\eta \mid p$. $\square$

Remark 3.5.2. In the situation of Proposition 3.5.1, the local conditions defining the Selmer groups $\check{\mathcal{S}}_v(\psi\chi\mathbf{N}^{-r})$ and $\mathcal{X}_v(\psi\chi\mathbf{N}^{-r})$ at the primes above p are *reversed* with respect to those defining the corresponding Bloch–Kato Selmer groups $\check{\mathcal{S}}_{\rm BK}(\psi\chi\mathbf{N}^{-r}) = \check{\mathcal{S}}_{\overline{v}}(\psi\chi\mathbf{N}^{-r})$ and $\mathcal{X}_{\rm BK}(\psi\chi\mathbf{N}^{-r}) = \mathcal{X}_{\overline{v}}(\psi\chi\mathbf{N}^{-r})$ (see Lemma 3.4.2).

3.6. **Equivalence between the main conjectures.** In the case where $V_{g,\chi}|_{G_K}$ is irreducible, one can show that $\mathscr{L}_w(g,\chi) \neq 0$ for both values of $w \in \{v, \overline{v}\}$ (see [CH18, Thm. 3.9]), and Conjecture 3.3.3 can be shown to be equivalent to either version (v or $\overline{v}$) of Conjecture 3.3.1. In the weight 2 case, the argument for this appears in several places in the literature (see e.g. [Wan21, §3] or [BCK21, §5]).

In Theorem 4.5.1 below, we shall see that when $g = \theta_\psi$ has CM by K (and so $V_{g,\chi}|_{G_K}$ is reducible), the vanishing or not of $\mathscr{L}_w(g,\chi)$ *depends* on the value of the root number

$$w(\psi\chi\mathbf{N}^{-r}) \in \{\pm 1\}$$

giving the sign in the functional equation relating $L(\psi\chi, s)$ and $L(\psi\chi, 2r-s)$. In fact, in this case we shall see that $\mathscr{L}_w(g,\chi)$ vanishes for one value of $w(\psi\chi\mathbf{N}^{-r})$, and it is nonzero for the other (see Remark 4.5.2). Thus in this section we carefully explain the relation between Conjecture 3.3.3 and Conjecture 3.3.1 in general.

Lemma 3.6.1. *For any prime w of K above p, the quotient*

$$\mathcal{H}_w := \varprojlim_{F \subset K_\infty} \prod_{\eta | w} \frac{\mathrm{H}^1(F_\eta, T_{g,\chi})}{\mathrm{H}^1_f(F_\eta, T_{g,\chi})}$$

is $\Lambda_{\mathscr{O}}$-torsion-free.

Proof. From Lemma 3.4.2 we immediately see that

$$\mathcal{H}_v \simeq \varprojlim_{F\subset K_\infty} \prod_{\eta|v} \mathrm{H}^1(F_\eta, T_{\psi\chi}(r)), \qquad \mathcal{H}_{\overline{v}} \simeq \varprojlim_{F\subset K_\infty} \prod_{\eta|\overline{v}} \mathrm{H}^1(F_\eta, T_{\psi^\tau\chi}(r)).$$

The result thus follows from [PR92, Prop. 2.1.6] as in [Arn07, Lem. 2.8]. (Note that [Arn07, Lem. 2.5] also applies in our case, since neither of the characters $\psi\chi\mathbf{N}^{-r}$ and $\psi^\tau\chi\mathbf{N}^{-r}$ has infinity type of the form (a,b) with $a=-b$.) ∎

Proposition 3.6.2. *Suppose $\mathbf{z}_{g,\chi}$ is non-torsion, and the localization map*

$$\mathrm{loc}_{\overline{v}} : \check{\mathcal{S}}_{\mathrm{BK}}(g,\chi) \to \varprojlim_{F\subset K_\infty} \prod_{\eta|\overline{v}} \mathrm{H}^1_f(F_\eta, T_{g,\chi}) \tag{3.3}$$

is nonzero. Then the following are equivalent:

(i^+) $\mathrm{rank}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_v(g,\chi)\big) = \mathrm{rank}_{\Lambda_{\mathscr{O}}}\big(\check{\mathcal{S}}_v(g,\chi)\big) = 0$;

(i^-) $\mathrm{rank}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_{\mathrm{BK}}(g,\chi)\big) = \mathrm{rank}_{\Lambda_{\mathscr{O}}}\big(\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)\big) = 1$,

and in that case, the following are equivalent:

(ii^+) $\mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_v(g,\chi)\big) \subset \big(\mathscr{L}_v(g,\chi)^2\big)$ *in* $\Lambda_{\mathscr{O}}^{\mathrm{ur}}$;

(ii^-) $\mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_{\mathrm{BK}}(g,\chi)_{\mathrm{tors}}\big) \subset \mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)/\Lambda_{\mathscr{O}}\cdot\mathbf{z}_{g,\chi}\big)^2$ *in* $\Lambda_{\mathscr{O}}$,

and the same holds true for the opposite divisibilities. In particular, Conjecture 3.3.1 for $w=v$ and Conjecture 3.3.3 are equivalent.

Proof. The argument is similar to that in [BCST22, Thm. 5.2]. For $\circ,\bullet \in \{\mathrm{str}, f, \mathrm{rel}\}$, let $\check{\mathcal{S}}_{\circ,\bullet}(g,\chi)$ and $\mathcal{X}_{\circ,\bullet}(g,\chi)$ denote the Selmer groups defined as in §3.1 by with the local conditions $\circ$ and $\bullet$ at the primes above v and $\overline{v}$, respectively. Thus, for instance, $\mathcal{X}_{\mathrm{rel},\mathrm{str}}(g,\chi)$ and $\mathcal{X}_{f,f}(g,\chi)$ are the previously defined $\mathcal{X}_v(g,\chi)$ and $\mathcal{X}_{\mathrm{BK}}(g,\chi)$, respectively.

By global duality, $\mathrm{coker}(\mathrm{loc}_{\overline{v}})$ is the same[4] as the kernel of the projection $\mathcal{X}_{\mathrm{rel},f}(g,\chi) \to \mathcal{X}_{\mathrm{BK}}(g,\chi)$. Similarly, by global duality the cokernel of the localization map

$$\mathrm{loc}'_{\overline{v}} : \check{\mathcal{S}}_{\mathrm{rel},f}(g,\chi) \to \varprojlim_{F\subset K_\infty} \prod_{\eta|\overline{v}} \mathrm{H}^1_f(F_\eta, T_{g,\chi})$$

is identified with the kernel of the projection $\mathcal{X}_v(g,\chi) \to \mathcal{X}_{f,\mathrm{str}}(g,\chi)$. Since the same argument as in [Cas17, Lem. 2.3] shows that $\mathrm{rank}_{\Lambda_{\mathscr{O}}}(\mathcal{X}_{\mathrm{rel},f}(g,\chi)) = 1 + \mathrm{rank}_{\Lambda_{\mathscr{O}}}(\mathcal{X}_{f,\mathrm{str}}(g,\chi))$, under the assumption on $\mathrm{loc}_{\overline{v}}$ in the statement we thus conclude that

$$\mathrm{rank}_{\Lambda_{\mathscr{O}}}(\mathcal{X}_{\mathrm{BK}}(g,\chi)) = 1 + \mathrm{rank}_{\Lambda_{\mathscr{O}}}(\mathcal{X}_v(g,\chi)),$$

whence the equivalence $(i^+) \iff (i^-)$.

Now assume that (i^+) (and therefore also (i^-)) holds. As above, from global duality we have the short exact sequences

$$\begin{aligned} &0 \to \mathrm{coker}(\mathrm{loc}_{\overline{v}}) \to \mathcal{X}_{\mathrm{rel},f}(g,\chi) \to \mathcal{X}_{\mathrm{BK}}(g,\chi) \to 0, \\ &0 \to \mathrm{coker}(\mathrm{loc}'_{\overline{v}}) \to \mathcal{X}_v(g,\chi) \to \mathcal{X}_{f,\mathrm{str}}(g,\chi) \to 0. \end{aligned} \tag{3.4}$$

Since the target of $\mathrm{loc}'_{\overline{v}}$ has $\Lambda_{\mathscr{O}}$-rank 1, and $\mathrm{rank}_{\Lambda_{\mathscr{O}}}(\check{\mathcal{S}}_v(g,\chi)) = 0$, the quotient $\check{\mathcal{S}}_{\mathrm{rel},f}(g,\chi)/\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)$ is $\Lambda_{\mathscr{O}}$-torsion; since this quotient injects via loc_v into $\mathcal{H}_v$, which is $\Lambda_{\mathscr{O}}$-torsion-free by Lemma 3.6.1, we conclude that $\check{\mathcal{S}}_{\mathrm{rel},f}(g,\chi) = \check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)$, so in particular $\mathrm{loc}_{\overline{v}} = \mathrm{loc}'_{\overline{v}}$. From (3.4) we thus obtain

$$\begin{aligned} \mathrm{char}_{\Lambda_{\mathscr{O}}}(\mathcal{X}_v(g,\chi)) &= \mathrm{char}_{\Lambda_{\mathscr{O}}}(\mathcal{X}_{f,\mathrm{str}}(g,\chi)) \cdot \mathrm{char}_{\Lambda_{\mathscr{O}}}(\mathrm{coker}(\mathrm{loc}_{\overline{v}})) \\ &= \mathrm{char}_{\Lambda_{\mathscr{O}}}(\mathcal{X}_{\mathrm{rel},f}(g,\chi)_{\mathrm{tors}}) \cdot \mathrm{char}_{\Lambda_{\mathscr{O}}}(\mathrm{coker}(\mathrm{loc}_{\overline{v}})) \\ &= \mathrm{char}_{\Lambda_{\mathscr{O}}}(\mathcal{X}_{\mathrm{BK}}(g,\chi)_{\mathrm{tors}}) \cdot \mathrm{char}_{\Lambda_{\mathscr{O}}}(\mathrm{coker}(\mathrm{loc}_{\overline{v}}))^2, \end{aligned} \tag{3.5}$$

[4] Using the fact that the representation $V_{g,\chi}$ is conjugate self-dual, so the Selmer group dual to $\check{\mathcal{S}}_{f,\mathrm{str}}(g,\chi) = \ker(\mathrm{loc}_{\overline{v}})$ is identified with $\mathcal{X}_{\mathrm{rel},f}(g,\chi)$.

using a variant of [Cas17, Lem. 2.3(3)] for the second equality. On the other hand, since $\varprojlim_F \mathrm{H}^1(F, T_{g,\chi})$ is $\Lambda_{\mathscr{O}}$-torsion–free, the map $\mathrm{loc}_{\overline{v}}$ defines the short exact sequence

$$0 \to \check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)/\Lambda_{\mathscr{O}} \cdot \mathbf{z}_{g,\chi} \to \varprojlim_{F \subset K_\infty} \oplus_{\eta | \overline{v}} \mathrm{H}^1_f(F_\eta, T_{g,\chi})/\Lambda_{\mathscr{O}} \cdot \mathrm{loc}_{\overline{v}}(\mathbf{z}_{g,\chi}) \to \mathrm{coker}(\mathrm{loc}_{\overline{v}}) \to 0,$$

which together with Theorem 3.2.2 yields the equality

$$\mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)/\Lambda_{\mathscr{O}} \cdot \mathbf{z}_{g,\chi}\big) \cdot \mathrm{char}_{\Lambda_{\mathscr{O}}}(\mathrm{coker}(\mathrm{loc}_{\overline{v}})) = \big(\mathscr{L}_v(g,\chi)\big) \tag{3.6}$$

in $\Lambda_{\mathscr{O}}^{\mathrm{ur}}$. Combining (3.5) and (3.6) we deduce

$$\mathrm{char}_{\Lambda_{\mathscr{O}}}(\mathcal{X}_v(g,\chi)) \cdot \mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)/\Lambda_{\mathscr{O}} \cdot \mathbf{z}_{g,\chi}\big)^2 = \mathrm{char}_{\Lambda_{\mathscr{O}}}(\mathcal{X}_v(g,\chi)_{\mathrm{tors}}) \cdot \big(\mathscr{L}_v(g,\chi)^2\big),$$

which readily yields the equivalence $(ii^+) \iff (ii^-)$. ☐

Remark 3.6.3. Theorem 3.2.2 also relates the image of $\mathbf{z}_{g,\chi}$ under the localization map

$$\mathrm{loc}_v : \check{\mathcal{S}}_{\mathrm{BK}}(g,\chi) \to \varprojlim_{F \subset K_\infty} \prod_{\eta | v} \mathrm{H}^1_f(F_\eta, T_{g,\chi})$$

to the p-adic L-function $\mathscr{L}_{\overline{v}}(g,\chi)$ of Theorem 2.1.3. The same argument as in Proposition 3.6.2 shows that, assuming the non-vanishing of both $\mathbf{z}_{g,\chi}$ and $\mathrm{loc}_v(\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi))$, the following are equivalent:

(i) Conjecture 3.3.1 for $w = \overline{v}$;
(ii) Conjecture 3.3.3.

4. Proof of the Iwasawa main conjectures

In this section we prove Conjecture 3.3.1 and Conjecture 3.3.3 for self-dual pairs (g,χ) in the case where $g = \theta_\psi$ is the theta series of a Hecke character ψ. In the course of the proof we shall establish the non-triviality of $\mathbf{z}_{g,\chi}$ in this case (which in weights $2r > 2$ appears to be new).

4.1. **Statement.**

The main result of this section is the following, where K is an imaginary quadratic field satisfying (spl) for a prime $p > 3$.

For a self-dual character ϕ, we denote by $w(\phi) \in \{\pm 1\}$ the sign in the functional equation

$$L(\phi, s) = w(\phi) L(\phi, -s).$$

Theorem 4.1.1. *Let ψ be a Hecke character of infinity type $(1-2r, 0)$ for some $r \geq 1$ and conductor a cyclic ideal $\mathfrak{f}_\psi$ of norm prime to pD_K. Set*

$$g = \theta_\psi, \quad N_g = D_K \cdot \mathbf{N}(\mathfrak{f}_\psi), \quad \mathfrak{N}_g = \mathfrak{d}_K \cdot \mathfrak{f}_\psi.$$

Let c be a positive integer prime to pN_g, let χ be a finite order character in $\Sigma_{\mathrm{cc}}(c, \mathfrak{N}_g, \varepsilon_g)$, and suppose (g,χ) is defined over $\mathscr{O}$.

(i) *If $w(\psi\chi\mathbf{N}^{-r}) = -1$, then $\check{\mathcal{S}}_v(g,\chi) = 0$ and $\mathcal{X}_v(g,\chi)$ is $\Lambda_{\mathscr{O}}$-torsion, with*

$$\mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_v(g,\chi)\big) = \mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathscr{L}_v(g,\chi)^2\big)$$

in $\Lambda_{\mathscr{O}}^{\mathrm{ur}}$.

(ii) *If $w(\psi\chi\mathbf{N}^{-r}) = +1$, the same results hold with v replaced by $\overline{v}$.*

Hence Conjecture 3.3.1 holds for (g,χ).

Remark 4.1.2. The above assumption on $\mathfrak{f}_\psi$ implies that $\mathcal{O}_K/\mathfrak{N}_g \simeq \mathbb{Z}/N_g\mathbb{Z}$, i.e. K satisfies (Heeg) relative to N_g. Moreover, it is easy to see that in the setting of Theorem 4.1.1, the character $\psi\chi\mathbf{N}^{-r}$ (and hence also $\psi^\tau\chi\mathbf{N}^{-r}$) is self-dual (see [BDP12, Lem. 3.16]).

By Proposition 2.3.1 and Proposition 3.5.1, the proof of Theorem 4.1.1 is reduced to the study of the relation between the $\Lambda_{\mathscr{O}}$-adic Selmer groups attached to the self-dual characters $\psi\chi\mathbf{N}^{-r}$ and $\psi^{\tau}\chi\mathbf{N}^{-r}$ and the p-adic L-functions $\mathscr{L}_w(\psi\chi\mathbf{N}^{-r})$ and $\mathscr{L}_w(\psi^{\tau}\chi\mathbf{N}^{-r})$, respectively. The main difficulty arises from the fact that, as noted in Remark 3.5.2, the Selmer group $\mathcal{X}_v(\psi\chi\mathbf{N}^{-r})$ is different from the Bloch–Kato Selmer group over K_∞/K, and so a relation between the characteristic ideal of $\mathcal{X}_v(\psi\chi\mathbf{N}^{-r})$ and the p-adic l-function $\mathscr{L}_v(\psi\chi\mathbf{N}^{-r})$ is not immediate from the main conjecture in this setting.

The proof of Theorem 4.1.1 is concluded in §4.6, where we also deduce from it a similar result on Conjecture 3.3.3.

4.2. **Explicit reciprocity law.** For an ideal $\mathfrak{c}\subset\mathcal{O}_K$ prime to p and a non-trivial ideal $\mathfrak{a}$ prime to $6\mathfrak{c}p$, let $c_{\mathfrak{a}}(K(\mathfrak{c}p^k))\in\mathrm{H}^1(K(\mathfrak{c}p^k),\mathbb{Z}_p(1))$ be the elliptic unit denoted $\vartheta_{\mathfrak{a}}(\mathfrak{c}p^k)$ in [AH06, §2.3] and ${}_{\mathfrak{a}}z_{\mathfrak{c}p^k}$ in [Kat04, §15.5]; these are norm-compatible at k varies.

As a piece of notation, for an infinite abelian extension K'/K and a $\mathbb{Z}_p$-module T with a continuous linear G_K-action, put

$$\mathrm{H}^1_{\mathrm{Iw}}(K',T):=\varprojlim_{K''\subset K'}\mathrm{H}^1(K'',T),$$

with K'' running over the finite extensions of K contained in K', and limit with respect to corestriction. For a character $\phi:\mathrm{Gal}(K(\mathfrak{c}p^\infty)/K)\to\mathscr{O}^\times$, we put $\phi^*=\varepsilon_{\mathrm{cyc}}\phi^{-1}$ and let $c_{\mathfrak{a}}(K(\mathfrak{c}p^\infty))^{\phi^{-1}}$ denote the image of $\varprojlim_k c_{\mathfrak{a}}(K(\mathfrak{c}p^k))$ under the twisting homomorphism

$$\mathrm{H}^1_{\mathrm{Iw}}(K(\mathfrak{c}p^\infty),\mathbb{Z}_p(1))\xrightarrow{\otimes\phi^{-1}}\mathrm{H}^1_{\mathrm{Iw}}(K(\mathfrak{c}p^\infty),T_{\phi^*}).$$

For any subextension L of $K(\mathfrak{c}p^\infty)$ (not necessarily finite over K), we define $c_{\mathfrak{a}}(L)^{\phi^{-1}}$ to be the image of $c_{\mathfrak{a}}(K(\mathfrak{c}p^\infty))^{\phi^{-1}}$ under the corestriction map $\mathrm{H}^1_{\mathrm{Iw}}(K(\mathfrak{c}p^\infty),T_{\phi^*})\to\mathrm{H}^1_{\mathrm{Iw}}(L,T_{\phi^*})$.

The next result is a refinement of Yager's work [Yag82] (specialized to the anticyclotomic setting) building on Kato's explicit reciprocity law [Kat04].

Theorem 4.2.1. *Let $w\in\{v,\overline{v}\}$ be a prime of K above p, and let ϕ be a self-dual Hecke character of K with values in $\mathscr{O}$. Then there is an injective $\Lambda_{\mathscr{O}}^{\mathrm{ur}}$-module homomorphism*

$$\mathrm{Col}_w:\varprojlim_{F\subset K_\infty}\prod_{\eta|w}\mathrm{H}^1(F_\eta,T_{\phi^\tau})\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p^{\mathrm{ur}}\to\Lambda_{\mathscr{O}}^{\mathrm{ur}}$$

with finite cokernel, and a "twisted elliptic unit" $\mathbf{c}_{\phi^\tau}\in\mathrm{H}^1_{\mathrm{Iw}}(K_\infty,T_{\phi^\tau})$, such that

$$\mathrm{Col}_w\big(\mathrm{loc}_w(\mathbf{c}_{\phi^\tau})\big)=\mathscr{L}_w(\phi).$$

Proof. We first explain the construction of the map Col_w for $w=v$. For any field $F\subset\overline{\mathbb{Q}}$ over K, let F_v denote the completion of F at the prime above p induced by $\imath_p$. Then $H_{p^\infty,v}$ contains $K_{\infty,v}$, and it follows by local Class Field Theory that $H_{p^\infty,v}$ is obtained by adjoining to H_v the torsion points of a height 1 Lubin–Tate formal group relative to the extension H_v/K_v (see [Shn16, Prop. 39]). Thus, as explained in [CH22, §3], from the Perrin-Riou big exponential map for H_{p^∞}/H_v (see [PR94, Kob23]) we have a $\mathbb{Z}_p[\![\Gamma_v]\!]$-linear map

$$\mathrm{Col}_v:\varprojlim_{F\subset K_\infty}\mathrm{H}^1(F_v,T_{\phi^\tau})\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p^{\mathrm{ur}}\to\mathscr{O}[\![\Gamma_v]\!]\hat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p^{\mathrm{ur}}$$

interpolating the Bloch–Kato logarithm and dual exponential maps for varying specializations (see [CH22, Thm. 3.4]) which gives the map Col_v in the statement after tensoring with $\mathbb{Z}_p[\![\Gamma]\!]$ over $\mathbb{Z}_p[\![\Gamma_v]\!]$. The maps $\mathrm{Col}_{\overline{v}}$ is constructed in the same manner, replacing $\imath_p$ by $\imath_p\circ\tau$, where $\tau\in G_{\mathbb{Q}}$ is the complex conjugation induced by $\imath_\infty$; the claim that both maps Col_w are injective with finite cokernel follows form the theory of Coleman power series as in [Kat04, §17.10].

On the other hand, letting $\mathfrak{c}$ be the prime-to-p part of the conductor of ϕ, we can find $\mathfrak{a}$ coprime to $6\mathfrak{c}p$ with $\sigma_{\mathfrak{a}} - \phi^\tau(\sigma_{\mathfrak{a}})$ invertible in $\Lambda_{\mathscr{O}}^{\mathrm{ur}}$ (cf. [dS87, p. 77]), and setting

$$\mathbf{c}_{\phi^\tau} := (\sigma_{\mathfrak{a}} - \phi^\tau(\sigma_{\mathfrak{a}}))^{-1} \cdot c_{\mathfrak{a}}(K_\infty)^{\phi^{-1}}$$

(which is independent of $\mathfrak{a}$ by [Kat04, (15.4.4] and belongs to $\mathrm{H}^1_{\mathrm{Iw}}(K_\infty, T_{\phi^*}) = \mathrm{H}^1_{\mathrm{Iw}}(K_\infty, T_{\phi^\tau})$ by the self-duality of ϕ), the last assertion follows directly from the interpolations properties of $\mathscr{L}_w(\phi)$ and Col_w together with Kato's explicit reciprocity law [Kat04, Prop. 15.9]. □

Remark 4.2.2. Specializing Yager's result [Yag82] to the anticyclotomic line as in [AH06, Prop. 2.3.4] and [Arn07, Prop. 2.6] yields an analogue of Theorem 4.2.1 under the assumption that $p \nmid [K(\mathfrak{c}) : K]$ and $\phi|_{G_{K_w}} \not\equiv 1 \pmod{\varpi}$ (i.e. ϕ is "non-anomalous" at w). Indeed, following the argument in *loc. cit.*, by local Tate duality one can show that under these additional hypotheses the corestriction map

$$\Bigg(\varprojlim_{F \subset K(\mathfrak{c}p^\infty)} \prod_{\eta|w} \mathrm{H}^1(F_\eta, T_{\phi^*})\Bigg) \otimes_{\Lambda_{\mathscr{O}}(K(\mathfrak{c}p^\infty))} \Lambda_{\mathscr{O}} \to \varprojlim_{F \subset K_\infty} \prod_{\eta|w} \mathrm{H}^1(F_\eta, T_{\phi^*})$$

is an isomorphism.

4.3. **Main conjectures for characters.** The 2-variable main conjecture for K was proved by Rubin [Rub91] under mild hypotheses on the prime p, and by Johnson-Leung–Kings [JLK11] in general.

The problem of deducing from the 2-variable main conjecture a proof of the anticyclotomic main conjecture for self-dual Hecke characters ϕ (which is particularly subtle in the case of root number $w(\phi) = -1$), was first studied in detail by Agboola–Howard [AH06] in the case of CM elliptic curves $E/\mathbb{Q}$, and by Arnold [Arn07] for general CM forms.

In this section, we extend their methods to deduce the results we need on the relation between the Selmer groups $\mathcal{X}_v(\phi)$ and the p-adic L-functions $\mathscr{L}_v(\phi)$.

4.3.1. *The case of* $\lambda = \psi\chi$.

Theorem 4.3.1. *Let λ be a Hecke character of infinity type $(1-2r, 0)$ for some $r \geq 1$ with $\varepsilon_\lambda = \eta_K$, conductor $\mathfrak{c}$ prime to p. Suppose*

$$w(\lambda\mathbf{N}^{-r}) = -1.$$

Then $\check{\mathcal{S}}_v(\lambda\mathbf{N}^{-r}) = 0$ and $\mathcal{X}_v(\lambda\mathbf{N}^{-r})$ is $\Lambda_{\mathscr{O}}$-torsion, with

$$\mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_v(\lambda\mathbf{N}^{-r})\big) = \big(\mathscr{L}_v(\lambda\mathbf{N}^{-r})\big)$$

as ideals in Λ^{ur}.

Remark 4.3.2. Since $\mathcal{X}_v(\lambda\mathbf{N}^{-r})$ is not the same as $\mathcal{X}_{\mathrm{BK}}(\lambda\mathbf{N}^{-r})$ (see Remark 3.5.2; in fact, we shall see that their $\Lambda_{\mathscr{O}}$-ranks are always different), the result of Theorem 4.3.1 is not covered by the main results of [AH06] and [Arn07], which only concern the Bloch–Kato Selmer groups.

By anticyclotomic descent, the proof of Theorem 4.3.1 will be obtained from a twisted variant of the results by Rubin [Rub91] and Johnson-Leung–Kings [JLK11] on the 2-variable Iwasawa main conjecture for K. The starting point is the following key consequence of Greenberg's nonvanishing results [Gre85].

Proposition 4.3.3. *Let $w(\lambda\mathbf{N}^{-r}) \in \{\pm 1\}$ be the root number of $\lambda\mathbf{N}^{-r}$. Then*

$$\mathscr{L}_v(\lambda\mathbf{N}^{-r}) \neq 0 \quad \Longleftrightarrow \quad w(\lambda\mathbf{N}^{-r}) = -1.$$

Proof. We adapt the argument in the proof of [Arn07, Prop. 2.3]. Put $\phi_0 = \overline{\lambda}/\lambda$, and take $m > 0$ (to be possibly enlarged later) large enough so that $\phi := \phi_0^m$ factors through Γ and has trivial conductor. Then for $n \gg 0$, the character $\lambda\mathbf{N}^{-r}\phi^n$ is in the range of interpolation of $\mathscr{L}_{v,\mathfrak{c}}$ (indeed, its infinity type $(1-r+mn(2r-1), r+mn(1-2r)))$ and so from Theorem 2.2.1 and the functional equation for Hecke L-functions[5] we immediately obtain

$$\mathscr{L}_v(\lambda\mathbf{N}^{-r})(\phi^n) \doteq L(\lambda^{2mn-1}, c),$$

where $c = mn(2r-1) - r + 1$ is the center of the functional equation. Applying Weil's formula for root numbers as stated in [AH06, Prop. 2.1.6], arguing as in [Arn07, p. 51] we see that (after possibly replacing m by $2m$) $w(\lambda^{2mn-1}) = -w(\lambda\mathbf{N}^{-r})$, and the result follows from [Gre85, Thm. 1]. $\square$

Proposition 4.3.4. *If $w(\lambda\mathbf{N}^{-r}) = -1$, then $\mathcal{X}_v(\lambda\mathbf{N}^{-r})$ is $\Lambda_{\mathscr{O}}$-torsion.*

Proof. By Theorem 4.2.1 (with $\phi = \lambda\mathbf{N}^{-r}$) and Proposition 4.3.3, if $w(\lambda\mathbf{N}^{-r}) = -1$ then the map

$$\mathrm{loc}_v : \check{\mathcal{S}}_{\mathrm{rel}}(\lambda^\tau\mathbf{N}^{-r}) \to \varprojlim_{F\subset K_\infty} \prod_{\eta|v} \mathrm{H}^1(F_\eta, T_{\lambda^\tau}(r))$$

is nonzero[6]. On the other hand, by [Arn07, Thm. 2.14] (an application of the Euler system machinery, [Rub00, Thm. 2.3.3]), the $\Lambda_{\mathscr{O}}$-module $\mathcal{X}_{\mathrm{str}}(\lambda\mathbf{N}^{-r})$ is torsion. By the global duality exact sequence

$$\check{\mathcal{S}}_{\mathrm{rel}}(\lambda^\tau\mathbf{N}^{-r}) \to \varprojlim_{F\subset K_\infty} \prod_{\eta|v} \mathrm{H}^1(F_\eta, T_{\lambda^\tau}(r)) \to \mathcal{X}_v(\lambda\mathbf{N}^{-r}) \to \mathcal{X}_{\mathrm{str}}(\lambda\mathbf{N}^{-r}) \to 0,$$

this yields the result. $\square$

Together with Proposition 4.3.4, the next result completes the proof of Theorem 4.3.1.

Proposition 4.3.5. *Suppose $w(\lambda\mathbf{N}^{-r}) = -1$. Then*

$$\mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_v(\lambda\mathbf{N}^{-r})\big) = \big(\mathscr{L}_v(\lambda\mathbf{N}^{-r})\big)$$

as ideals in $\Lambda_{\mathscr{O}}^{\mathrm{ur}}$.

Proof. Denote by $\mathcal{C}_{\lambda^\tau\mathbf{N}^{-r}}(K_\infty)$ the $\Lambda_{\mathscr{O}}$-submodule of $\check{\mathcal{S}}_{\mathrm{rel}}(\lambda^\tau\mathbf{N}^{-r})$ generated by $\mathbf{c}_{\lambda^\tau\mathbf{N}^{-r}}$. By Theorem 4.2.1 (with $\phi = \lambda\mathbf{N}^{-r}$) and Proposition 4.3.3, $\mathcal{C}_{\lambda^\tau\mathbf{N}^{-r}}(K_\infty)$ is not $\Lambda_{\mathscr{O}}$-torsion, and by Theorem 2.14 in [Arn07] we know that $\check{\mathcal{S}}_{\mathrm{rel}}(\lambda^\tau\mathbf{N}^{-r})$ is torsion-free of $\Lambda_{\mathscr{O}}$-rank 1, $\mathcal{X}_{\mathrm{str}}(\lambda\mathbf{N}^{-r})$ is $\Lambda_{\mathscr{O}}$-torsion, and we have the divisibility

$$\mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_{\mathrm{str}}(\lambda\mathbf{N}^{-r})\big) \supset \mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\check{\mathcal{S}}_{\mathrm{rel}}(\lambda^\tau\mathbf{N}^{-r})/\mathcal{C}_{\lambda^\tau\mathbf{N}^{-r}}(K_\infty)\big).$$

Note that this divisibility is independent of the value of $w(\lambda\mathbf{N}^{-r})$, and in *loc. cit.* the result is stated for $T_\lambda(r)$ and $W_{\lambda^\tau}(r)$ in our notations (corresponding to T and W^* in *loc. cit.*), rather than for our $T_{\lambda^\tau}(r)$ and $W_\lambda(r)$ as above.

On the other hand, if $w(\lambda\mathbf{N}^{-r}) = -1$, in view of Proposition 4.3.4 the arguments in [Arn07, §3][7] leading to the proof of Proposition 3.8 in *loc. cit.* apply *verbatim* with $T_\lambda(r)$ and $W_{\lambda^\tau}(r)$ replaced by $T_{\lambda^\tau}(r)$ and $W_\lambda(r)$, hence yielding a proof of the converse divisibility, and so

$$\mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_{\mathrm{str}}(\lambda\mathbf{N}^{-r})\big) = \mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\check{\mathcal{S}}_{\mathrm{rel}}(\lambda^\tau\mathbf{N}^{-r})/\mathcal{C}_{\lambda^\tau\mathbf{N}^{-r}}(K_\infty)\big). \tag{4.1}$$

Now, from Poitou–Tate duality we have the exact sequence

$$0 \to \check{\mathcal{S}}_{\bar{v}}(\lambda^\tau\mathbf{N}^{-r}) \to \check{\mathcal{S}}_{\mathrm{rel}}(\lambda^\tau\mathbf{N}^{-r}) \xrightarrow{\mathrm{loc}_v} \varprojlim_{F\subset K_\infty} \prod_{\eta|v} \mathrm{H}^1(F_\eta, T_{\lambda^\tau}(r)) \to \mathcal{X}_v(\lambda\mathbf{N}^{-r}) \to \mathcal{X}_{\mathrm{str}}(\lambda\mathbf{N}^{-r}) \to 0, \tag{4.2}$$

[5] See e.g. [dS87, p. 37], whose conventions on infinity type are *opposite* to ours.

[6] Note that the implicit inclusion $\mathbf{c}_{\lambda^\tau\mathbf{N}^{-r}} \in \check{\mathcal{S}}_{\mathrm{rel}}(\lambda^\tau\mathbf{N}^{-r})$ follows from [AH06, Lem. 2.4.2].

[7] Using [JLK11, Thm. 5.2], which proves an extension of Rubin's result on the 2-variable Iwasawa main conjecture for K used in [Arn07, Thm. 3.2] under weaker hypotheses on p.

which from the above it implies that $\check{\mathcal{S}}_{\overline{v}}(\lambda^\tau\mathbf{N}^{-r})$ is $\Lambda_{\mathscr{O}}$-torsion, and so being also $\Lambda_{\mathscr{O}}$-torsion-free ,it vanishes. Thus from (4.2) we obtain the exact sequence

$$\begin{aligned}0 \to \check{\mathcal{S}}_{\mathrm{rel}}(\lambda^\tau\mathbf{N}^{-r})/\mathcal{C}_{\lambda^\tau\mathbf{N}^{-r}}(K_\infty) \xrightarrow{\mathrm{loc}_v} \varprojlim_{F\subset K_\infty}\prod_{\eta\mid v}\mathrm{H}^1(F_\eta,T_{\lambda^\tau}(r))/\mathrm{loc}_v(\mathcal{C}_{\lambda^\tau\mathbf{N}^{-r}}(K_\infty))\\ \to \mathcal{X}_v(\lambda\mathbf{N}^{-r}) \to \mathcal{X}_{\mathrm{str}}(\lambda\mathbf{N}^{-r}) \to 0.\end{aligned}$$

Since by Theorem 4.2.1 the map Col_v defines a $\Lambda_{\mathscr{O}}^{\mathrm{ur}}$-module pseudo-isomorphism

$$\left(\varprojlim_{F\subset K_\infty}\prod_{\eta\mid v}\mathrm{H}^1(F_\eta,T_{\lambda^\tau}(r))/\mathrm{loc}_v(\mathcal{C}_{\lambda^\tau\mathbf{N}^{-r}}(K_\infty))\right)\hat{\otimes}_{\mathscr{O}}\mathbb{Z}_p^{\mathrm{ur}} \to \Lambda_{\mathscr{O}}^{\mathrm{ur}}/(\mathscr{L}_v(\lambda\mathbf{N}^{-r}))$$

together with (4.1) this concludes the proof. □

4.3.2. *The case of $\psi^\tau\chi$.*

Theorem 4.3.6. *Let ψ be a Hecke character of infinity type $(1-2r,0)$ for some $r\geq 1$ and let χ be a finite order character such that $\psi^\tau\chi\mathbf{N}^{-r}$ is self-dual with root number*

$$w(\psi^\tau\chi\mathbf{N}^{-r}) = +1$$

and conductor prime to p. Then $\check{\mathcal{S}}_v(\psi^\tau\chi\mathbf{N}^{-r})=0$ and $\mathcal{X}_v(\psi^\tau\chi\mathbf{N}^{-r})$ is $\Lambda_{\mathscr{O}}$-torsion, with

$$\mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_v(\psi^\tau\chi\mathbf{N}^{-r})\big) = \big(\mathscr{L}_v(\psi^\tau\chi\mathbf{N}^{-r})\big).$$

as ideals in $\Lambda_{\mathscr{O}}^{\mathrm{ur}}$.

Proof. We being by noting that $L(\psi^\tau\chi,s)=L(\psi\chi^\tau,s)$, and by our assumption the self-dual character $\psi\chi^\tau\mathbf{N}^{-r}$ has root number $w(\psi\chi^\tau\mathbf{N}^{-r})=+1$. Since $\mathcal{X}_{\mathrm{BK}}(\psi\chi^\tau\mathbf{N}^{-r})=\mathcal{X}_{\overline{v}}(\psi\chi^\tau\mathbf{N}^{-r})$ by Lemma 3.4.2 (indeed, the infinity type of $\psi\chi^\tau\mathbf{N}^{-r}$ is $(1-r,r)$), by the same argument as in the proof of [AH06, Thm. 2.4.17(1)] and [Arn07, Thm. 3.9], but replacing the use of [AH06, Prop. 2.3.4] and [Arn07, Prop. 2.6] by an appeal to Theorem 4.2.1 above, we deduce that $\check{\mathcal{S}}_{\overline{v}}(\psi^\tau\chi\mathbf{N}^{-r})=0$ and $\mathcal{X}_{\overline{v}}(\psi\chi^\tau\mathbf{N}^{-r})$ is $\Lambda_{\mathscr{O}}$-torsion, with

$$\mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_{\overline{v}}(\psi\chi^\tau\mathbf{N}^{-r})\big) = \big(\mathscr{L}_{\overline{v}}(\psi\chi^\tau\mathbf{N}^{-r})\big)$$

as ideals in $\Lambda_{\mathscr{O}}^{\mathrm{ur}}$.

Let ι denote the involution of $\Lambda_{\mathscr{O}}$ given by $\gamma\mapsto\gamma^{-1}$ for $\gamma\in\Gamma$. Since by [Arn07, Prop. 4.1], and as a direct consequence of the interpolation property of Theorem 2.2.1 (see e.g. [BCG+20, Lem. 3.3.2(a)]), we have the equalities

$$\begin{aligned}\mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_{\overline{v}}(\psi\chi^\tau\mathbf{N}^{-r})\big) &= \mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_v(\psi^\tau\chi\mathbf{N}^{-r})^\iota\big),\\ \big(\mathscr{L}_{\overline{v}}(\psi\chi^\tau\mathbf{N}^{-r})\big) &= \big(\mathscr{L}_v(\psi^\tau\chi\mathbf{N}^{-r})^\iota\big),\end{aligned}$$

the result follows. □

4.4. p**-parity conjecture.** By Nekovář's methods, we deduce from some of the preceding results a proof of the p-parity conjecture for CM forms of higher weight (see [Guo93] for earlier results in this context). This will be an important ingredient in the proof a higher weight CM p-converse theorem (in the style of [Ski20] and [BCST22]) in forthcoming work [CP24].

Corollary 4.4.1. *Let λ be a Hecke character of infinity type $(1-2r,0)$ for some $r\geq 1$ with $\varepsilon_\lambda=\eta_K$ and conductor $\mathfrak{c}$ coprime to p and such that $\mathfrak{d}_K\|\mathfrak{c}$. Then*

$$\mathrm{ord}_{s=r}L(\lambda,s)\equiv\dim_\Phi\mathrm{Sel}_{\mathrm{BK}}(K,V_\lambda(r))\pmod 2,$$

and hence the p-parity conjecture holds for λ.

Proof. We argue similarly as in the proof of [CH18, Thm. 6.4]. By Nekovář's general result [Nek07, Cor. (5.3.2)] (see also [Nek09]), it suffices to show that if $w(\lambda\mathbf{N}^{-r}) = -1$ (resp. $w(\lambda\mathbf{N}^{-r}) = +1$), there exists a finite order character ϕ of Γ such that $\dim_\Phi \mathrm{Sel}_{\mathrm{BK}}(K, V_{\lambda\phi}(r)) = 1$ (resp. $\mathrm{Sel}_{\mathrm{BK}}(K, V_{\lambda\phi}(r)) = 0$).

Suppose first that $w(\lambda\mathbf{N}^{-r}) = -1$. In view of Proposition 4.3.3, we can take a finite order character ϕ of Γ such that $\mathscr{L}_v(\lambda\mathbf{N}^{-r})(\phi) \neq 0$. By Theorem 4.3.1 and a variant of Mazur's control theorem for $\mathcal{X}_v(\lambda\phi\mathbf{N}^{-r})$ (see [Cas23, Prop. 6.2.1]), it follows that $\mathrm{Sel}_v(K, V_{\lambda\phi}(r)) = 0$, and by Theorem 4.2.1 it follows that the localization map

$$\mathrm{loc}_v : \mathrm{Sel}_v(K, V_{\lambda^\tau\phi^\tau}(r)) \to \mathrm{H}^1(K_v, V_{\lambda^\tau\phi^\tau}(r)) \tag{4.3}$$

is nonzero[8]. From the Poitou–Tate exact sequence

$$\mathrm{Sel}_v(K, V_{\lambda^\tau\phi^\tau}(r)) \to \mathrm{H}^1(K_v, V_{\lambda^\tau\phi^\tau}(r)) \to \mathrm{Sel}_{\mathrm{rel}}(K, V_{\lambda\phi}(r))^* \to \mathrm{Sel}_{\overline{v}}(K, V_{\lambda\phi}(r))^* \to 0,$$

it follows that

$$\mathrm{Sel}_{\mathrm{rel}}(K, V_{\lambda\phi}(r)) = \mathrm{Sel}_{\overline{v}}(K, V_{\lambda\phi}(r)) = \mathrm{Sel}_{\mathrm{BK}}(K, V_{\lambda\phi}(r)), \tag{4.4}$$

using Lemma 3.4.2 for the last equality. Since the nonvanishing of (4.3) gives $\mathrm{loc}_{\overline{v}}(\mathrm{Sel}_{\overline{v}}(K, V_{\lambda\phi}(r))) \neq 0$, from the tautological exact sequence

$$\mathrm{Sel}_v(K, V_{\lambda\phi}(r)) \to \mathrm{Sel}_{\mathrm{rel}}(K, V_{\lambda\phi}(r)) \to \mathrm{H}^1(K_{\overline{v}}, V_{\lambda\phi}(r))$$

and (4.4) we obtain $\dim_\Phi \mathrm{Sel}_{\mathrm{BK}}(K, V_{\lambda\phi}(r)) = \dim_\Phi \mathrm{Sel}_v(K, V_{\lambda\phi}(r)) + 1 = 1$, as desired.

The case $w(\lambda\mathbf{N}^{-r}) = +1$ is easier: By [Arn07, Prop. 2.3], we can take a finite order character ϕ of Γ such that $\mathscr{L}_{\overline{v}}(\lambda\mathbf{N}^{-r})(\phi) \neq 0$, and then by Theorem 3.9 in *op. cit.* it follows that $\mathrm{Sel}_{\overline{v}}(K, V_{\lambda\phi}(r)) = 0$, so by Lemma 3.4.2 the Bloch–Kato Selmer group $\mathrm{Sel}_{\mathrm{BK}}(K, V_{\lambda\phi}(r))$ vanishes as desired. □

4.5. **Non-triviality of $\mathbf{z}_{g,\chi}$.** For g of weight $k = 2$, the non-triviality of $\mathbf{z}_{g,\chi}$ follows from the work of Cornut–Vatsal [CV05, CV07]. For g of even weight $k > 2$, assuming that the residual representation attached to $V_g|_{G_K}$ is absolutely *irreducible*, the non-triviality of $\mathbf{z}_{g,\chi}$ follows from the combination of Theorem 3.9 and Theorem 5.7 in [CH18], and also from [Bur20, Thm. 4.3]; in both cases, the result is deduced from Hida's methods [Hid10, Hid04], showing the non-vanishing of $\mathscr{L}_v(g,\chi)$, and a form of Theorem 3.2.2.

Here we are interested in the case where $g = \theta_\psi$ has CM by K, so in particular $V_g|_{G_K}$ is reducible. In the weight 2 case, an alternative proof of Cornut–Vatsal's nonvanishing result was given in [BD20] building on the $\Lambda_{\mathscr{O}}$-adic Gross–Zagier formula of [Dis17] and the nonvanishing result of [Bur15]. Here we prove the non-triviality of $\mathbf{z}_{g,\chi}$ for $g = \theta_\psi$ of arbitrary even weight.

Theorem 4.5.1. *Let ψ be a Hecke character of infinity type $(1-2r, 0)$ for some $r \geq 1$ and conductor a cyclic ideal $\mathfrak{f}_\psi$ of norm prime to pD_K. Let*

$$(g,\chi) = (\theta_\psi, \chi) \in S_{2r}(\Gamma_1(N_g)) \times \Sigma_{\mathrm{cc}}(c, \mathfrak{N}_g, \varepsilon_g)$$

be a self-dual pair for some positive integer c prime to pN_g. Then:

(i) *If $w(\psi\chi\mathbf{N}^{-r}) = -1$, then $\mathscr{L}_v(g,\chi) \neq 0$ and $\mathrm{loc}_{\overline{v}}(\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)) \neq 0$.*
(ii) *If $w(\psi\chi\mathbf{N}^{-r}) = +1$, then $\mathscr{L}_{\overline{v}}(g,\chi) \neq 0$ and $\mathrm{loc}_v(\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)) \neq 0$.*

In particular, $\mathbf{z}_{g,\chi} \neq 0$ regardless of the sign of $w(\psi\chi\mathbf{N}^{-r})$.

Proof. Since $\mathscr{L}_v(g,\chi)$ is a unit multiple of $\mathscr{L}_v(\psi\chi\mathbf{N}^{-r})\mathscr{L}_v(\psi^\tau\chi\mathbf{N}^{-r})$ by Proposition 2.3.1, in the case $w(\psi\chi\mathbf{N}^{-r}) = -1$ the nonvanishing of $\mathscr{L}_v(g,\chi)$ follows from Proposition 4.3.3 and Theorem 4.3.6; the nonvanishing of $\mathrm{loc}_{\overline{v}}(\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi))$ then follows from Theorem 3.2.2.

[8]Indeed, letting $\mathbf{c}_{\lambda^\tau\mathbf{N}^{-r}}(\phi^\tau)$ denote the specialization of $\mathbf{c}_{\lambda^\tau\mathbf{N}^{-r}}$ at ϕ^τ, the inclusion $\mathbf{c}_{\lambda^\tau\mathbf{N}^{-r}}(\phi^\tau) \in \mathrm{Sel}_v(K, V_{\lambda^\tau\phi^\tau}(r))$ follows from Theorem 4.2.1, the interpolation property in Theorem 2.2.1, and the fact that $L(\lambda, r) = 0$.

In the case $w(\psi\chi\mathbf{N}^{-r}) = +1$, the same argument as in Proposition 2.3.1 shows that $\mathscr{L}_{\overline{v}}(g,\chi)$ is a unit multiple of $\mathscr{L}_{\overline{v}}(\psi\chi\mathbf{N}^{-r})\mathscr{L}_{\overline{v}}(\psi^\tau\chi\mathbf{N}^{-r})$. Directly from Theorem 2.2.1 (see [BCG$^+$20, Lem. 3.3.2(a)]), we have the relations

$$\big(\mathscr{L}_{\overline{v}}(\psi\chi\mathbf{N}^{-r})\big) = \big(\mathscr{L}_v(\psi^\tau\chi^\tau\mathbf{N}^{-r})^\iota\big), \quad \big(\mathscr{L}_{\overline{v}}(\psi^\tau\chi\mathbf{N}^{-r})\big) = \big(\mathscr{L}_v(\psi\chi^\tau\mathbf{N}^{-r})^\iota\big).$$

Note that the character $\psi^\tau\chi^\tau\mathbf{N}^{-r}$ lies inside the range of p-adic interpolation for $\mathscr{L}_{v,\mathfrak{c}}$, while $\psi\chi^\tau\mathbf{N}^{-r}$ lies outside this range. As in Theorem 4.3.6 and Proposition 4.3.3, we see that

$$\begin{aligned} w(\psi^\tau\chi^\tau\mathbf{N}^{-r}) = +1 \quad &\Longrightarrow \quad \mathscr{L}_v(\psi^\tau\chi^\tau\mathbf{N}^{-r}) \neq 0, \\ w(\psi\chi^\tau\mathbf{N}^{-r}) = -1 \quad &\Longrightarrow \quad \mathscr{L}_v(\psi\chi^\tau\mathbf{N}^{-r}) \neq 0, \end{aligned}$$

respectively, which together with Theorem 3.2.2 yields the result. □

Remark 4.5.2. It is interesting to note that the preceding results show in fact the equivalences

$$\begin{aligned} \mathrm{loc}_{\overline{v}}(\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)) \neq 0 \quad &\Longleftrightarrow \quad \mathscr{L}_v(g,\chi) \neq 0 \quad \Longleftrightarrow \quad w(\psi\chi\mathbf{N}^{-r}) = -1, \\ \mathrm{loc}_{v}(\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)) \neq 0 \quad &\Longleftrightarrow \quad \mathscr{L}_{\overline{v}}(g,\chi) \neq 0 \quad \Longleftrightarrow \quad w(\psi\chi\mathbf{N}^{-r}) = +1. \end{aligned}$$

Indeed, the right pair of equivalences and the backward direction for the left pair directly follow from the proof of Theorem 4.5.1. For the left pair of implications $\Longrightarrow$, note that by [AH06, Thm. 2.4.17] (as extended in [Arn07, Thm. 3.9]) we have the implications

$$\begin{aligned} w(\psi\chi\mathbf{N}^{-r}) = +1 \quad &\Longrightarrow \quad \check{\mathcal{S}}_{\mathrm{BK}}(\psi\chi\mathbf{N}^{-r}) = 0, \\ w(\psi\chi\mathbf{N}^{-r}) = -1 \quad &\Longrightarrow \quad \check{\mathcal{S}}_{\mathrm{BK}}(\psi^\tau\chi\mathbf{N}^{-r}) = 0, \end{aligned}$$

using that $w(\psi^\tau\chi\mathbf{N}^{-r}) = -w(\psi\chi\mathbf{N}^{-r})$ for the second implication. Since Proposition 3.5.1 gives the equalities

$$\mathrm{loc}_{\overline{v}}(\check{\mathcal{S}}_{\mathrm{BK}}(\psi\chi\mathbf{N}^{-r})) = \mathrm{loc}_{\overline{v}}(\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)), \quad \mathrm{loc}_{v}(\check{\mathcal{S}}_{\mathrm{BK}}(\psi^\tau\chi\mathbf{N}^{-r})) = \mathrm{loc}_{v}(\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)),$$

this yields the claim.

4.6. Proof of Theorem 4.1.1 and Heegner cycle main conjecture.

Proof of Theorem 4.1.1. Since $(g,\chi) = (\theta_\psi,\chi)$ is a self-dual pair, the L-function $L(g/K,\chi,s)$ is self-dual with sign -1 and center at $s = r$, and the decomposition (3.2) gives

$$L(g/K,\chi,s) = L(\psi\chi,s)\cdot L(\psi^\tau\chi,s).$$

The L-functions in the right-hand side of this factorization are self-dual with opposite signs,

$$w(\psi\chi\mathbf{N}^{-r}) = -w(\psi^\tau\chi\mathbf{N}^{-r}).$$

Suppose first that $w(\psi\chi\mathbf{N}^{-r}) = -1$. By Proposition 3.5.1, the assertions that $\check{\mathcal{S}}_v(g,\chi)$ vanishes and $\mathcal{X}_v(g,\chi)$ is $\Lambda_{\mathscr{O}}$-torsion follow from Theorem 4.3.1 and Theorem 4.3.6. Together with the factorization in Proposition 2.3.1, the same theorems give the equalities

$$\begin{aligned} \mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_v(g,\chi)\big) &= \mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_v(\psi\chi\mathbf{N}^{-r})\big)\cdot\mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_v(\psi^\tau\chi\mathbf{N}^{-r})\big) \\ &= \big(\mathscr{L}_v(\psi\chi\mathbf{N}^{-r})\cdot\mathscr{L}_v(\psi^\tau\chi\mathbf{N}^{-r})\big) \\ &= \big(\mathscr{L}_v(g,\chi)^2\big) \end{aligned} \tag{4.5}$$

in $\Lambda_{\mathscr{O}}^{\mathrm{ur}}$, yielding the result in this case.

In the case $w(\psi\chi\mathbf{N}^{-r}) = +1$, we apply the result of Theorem 4.3.1 and Theorem 4.3.6 to $\psi\chi^\tau\mathbf{N}^{-r}$ (which has $w(\psi\chi^\tau\mathbf{N}^{-r}) = -1$) and $\psi^\tau\chi^\tau\mathbf{N}^{-r}$ (which has $w(\psi^\tau\chi^\tau\mathbf{N}^{-r}) = +1$), respectively, to obtain

$$\check{\mathcal{S}}_v(\psi\chi^\tau\mathbf{N}^{-r}) = \check{\mathcal{S}}_v(\psi^\tau\chi^\tau\mathbf{N}^{-r}) = 0$$

and that $\mathcal{X}_v(\psi\chi^\tau\mathbf{N}^{-r})$ and $\mathcal{X}_v(\psi^\tau\chi^\tau\mathbf{N}^{-r})$ are both $\Lambda_{\mathscr{O}}$-torsion, with

$$\mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_v(\psi\chi^\tau\mathbf{N}^{-r})\big) = \big(\mathscr{L}_v(\psi\chi^\tau\mathbf{N}^{-r})\big), \quad \mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_v(\psi^\tau\chi^\tau\mathbf{N}^{-r})\big) = \big(\mathscr{L}_v(\psi^\tau\chi^\tau\mathbf{N}^{-r})\big)$$

as ideals in $\Lambda_{\mathscr{O}}^{\mathrm{ur}}$. As in the proof of Theorem 4.3.6, by the action of complex conjugation this gives $\check{\mathcal{S}}_v(\psi^\tau\chi\mathbf{N}^{-r}) = \check{\mathcal{S}}_v(\psi\chi\mathbf{N}^{-r}) = 0$ and that $\mathcal{X}_{\overline{v}}(\psi^\tau\chi\mathbf{N}^{-r})$ and $\mathcal{X}_{\overline{v}}(\psi\chi\mathbf{N}^{-r})$ are both $\Lambda_{\mathscr{O}}$-torsion, with

$$\mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_{\overline{v}}(\psi^\tau\chi\mathbf{N}^{-r})\big) = \big(\mathscr{L}_{\overline{v}}(\psi^\tau\chi\mathbf{N}^{-r})\big), \quad \mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_{\overline{v}}(\psi\chi\mathbf{N}^{-r})\big) = \big(\mathscr{L}_{\overline{v}}(\psi\chi\mathbf{N}^{-r})\big),$$

which by the $\overline{v}$-versions of Proposition 2.3.1 and Proposition 3.5.1 yields the result as above. □

By the nonvanishing results established in the course of proving Theorem 4.1.1, we can deduce the following result on Conjecture 3.3.3.

Corollary 4.6.1. *Let (g,χ) be a self-dual pair as in Theorem 4.1.1. Then $\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)$ and $\mathcal{X}_{\mathrm{BK}}(g,\chi)$ both have $\Lambda_{\mathscr{O}}$-rank one, and*

$$\mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\mathcal{X}_{\mathrm{BK}}(g,\chi)_{\mathrm{tors}}\big) = \mathrm{char}_{\Lambda_{\mathscr{O}}}\big(\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)/\Lambda_{\mathscr{O}}\cdot\mathbf{z}_{g,\chi}\big)^2.$$

In other words, Conjecture 3.3.3 holds for (g,χ).

Proof. By Theorem 4.5.1, $\mathbf{z}_{g,\chi}$ is not $\Lambda_{\mathscr{O}}$-torsion, and $\mathrm{loc}_{\overline{v}}(\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)) \neq 0$ (resp. $\mathrm{loc}_v(\check{\mathcal{S}}_{\mathrm{BK}}(g,\chi)) \neq 0$) when $w(\psi\chi\mathbf{N}^{-r}) = -1$ (resp. $w(\psi\chi\mathbf{N}^{-r}) = +1$). Hence by Proposition 3.6.2 (see also Remark 3.6.3) the result follows from Theorem 4.1.1. □

5. Formula for the Bloch–Kato $\text{Ш}_{\mathrm{BK}}(W_{g,\chi}/K)$

Let K be an imaginary quadratic field satisfying (spl) for a prime $p > 3$. Let λ be a Hecke character of K of infinity type $(1-2r, 0)$ for some $r \geq 1$ and central character

$$\varepsilon_\lambda = \eta_K. \tag{sd}$$

In particular, $\lambda\mathbf{N}^{-r}$ is self-dual in the sense of Definition 2.2.2, and the conductor $\mathfrak{c}$ of λ is divisible by $\mathfrak{d}_K := (\sqrt{-D_K})$. Throughout this section we assume that

$$p \nmid \mathfrak{c} \quad \text{and} \quad \mathfrak{d}_K \| \mathfrak{c}. \tag{div}$$

By self-duality, $\mathfrak{c}$ is invariant under complex conjugation, and so by (div) can write

$$\mathfrak{c} = (c)\mathfrak{d}_K \tag{5.1}$$

for a unique positive integer c prime to pD_K.

5.1. Strong good pairs.

The following definition is a natural higher weight extension (and strengthening) of the notion of "good pair" for λ introduced in [BDP12].

Definition 5.1.1. Suppose the self-dual character $\lambda\mathbf{N}^{-r}$ has root number $w(\lambda\mathbf{N}^{-r}) = -1$. We say that a pair of Hecke characters (ψ,χ) is a *good pair* for λ if it satisfies

- (S1) ψ has infinity type $(1-2r, 0)$ and conductor a cyclic ideal $\mathfrak{f}_\psi$ of norm prime to pD_K.
- (S2) χ is a finite order character such that
$$(g,\chi) = (\theta_\psi,\chi) \in S_{2r}(\Gamma_1(N_g)) \times \Sigma_{\mathrm{cc}}(c,\mathfrak{N}_g,\varepsilon_g),$$
where $\mathfrak{N}_g := \mathfrak{d}_K\cdot\mathfrak{f}_\psi$ and $N_g = D_K\mathbf{N}(\mathfrak{f}_\psi)$.
- (S3) $\lambda = \psi\chi$.
- (S4) $L(\psi^\tau\chi, r) \neq 0$.

If $r = 1$, we say that (ψ,χ) is a *strong good pair* for λ if in addition

- (S5) $\psi^- := \psi^\tau/\psi$ has order prime to p, and $p \nmid \mathfrak{f}_{\psi^-}$.
- (S6) Every prime $\mathfrak{l} \mid \mathfrak{f}_{\psi^-}$ splits in K.
- (S7) $\psi^-|_{G_v} \neq \mathbb{1}$, where $G_v \subset G_K$ is a decomposition group at v.
- (S8) ψ^- has order ≥ 3.

(S9) The value

$$L^{\mathrm{alg}}(\psi^-,0) := \left(\frac{2\pi}{\sqrt{D_K}}\right)\cdot(1-\psi^-(\overline{v}))(1-\psi^-(\overline{v})p^{-1})\cdot\frac{L(\psi^-,0)}{\Omega}$$

is a p-adic unit.

Remark 5.1.2. Note that by Theorem 2.2.1 the normalized L-value $L^{\mathrm{alg}}(\psi^-,0)$ is the same as the p-adic L-value $\mathscr{L}_{v,\mathfrak{c}}(\psi^-)$ for $\mathfrak{c}=\mathfrak{f}_{\psi^-}$; having infinity type $(1,-1)$, ψ^- lies within its range of p-adic interpolation of $\mathscr{L}_{\mathfrak{c},v}$. In particular, by the functional equation of Theorem 2.2.1, if (ψ,χ) is a strong good pair for λ, then the p-adic L-function $\mathscr{L}_v(\psi^-\mathbf{N}^{-1})\in\Lambda_{\mathscr{O}}^{\mathrm{ur}}$ is invertible.

For $r=1$, the existence of good pairs for λ is shown in [BDP12, Prop. 3.28] building on Greenberg [Gre85] and Rohrlich [Roh84] nonvanishing results. Adapting their argument, and building on Finis' mod p non-vanishing results [Fin06], we can prove the following.

Lemma 5.1.3. *Suppose $\lambda\mathbf{N}^{-r}$ has root number -1. Then there exist good pairs for λ. Moreover, if $r=1$, then there exist strong good pairs for λ.*

Proof. Take ψ a Hecke character of K of infinity type $(1-2r,0)$ and conductor a cyclic ideal $\mathfrak{f}_\psi$ of norm coprime to $p\mathfrak{c}$, and put

$$\chi := \psi^{-1}\lambda.$$

By (5.1), ψ satisfies (S1), and by construction, the pair (ψ,χ) satisfies (S3). By [BDP12, Rem. 3.20], it follows that χ satisfies (S2), so it remains to verify the much subtler condition (S4).

We begin by noting (as used in the proof of Theorem 4.1.1) the sign in the functional equation for $L(\psi^\tau\chi,s)$ relating its values at s and $2r-s$ has sign $w(\psi^\tau\chi\mathbf{N}^{-r})=+1$. Indeed, by the inclusion $\chi\in\Sigma_{\mathrm{cc}}(c,\mathfrak{N}_g,\varepsilon_g)$, the sign in the functional equation for $L(g/K,\chi,s)=L(\theta_\psi/K,\chi,s)$ is $\epsilon(g,\chi)=-1$, and so the claim follow from the factorization

$$L(\theta_\psi/K,\chi,s)=L(\lambda,s)\cdot L(\psi^\tau\chi,s) \tag{5.2}$$

and the assumption $w(\lambda\mathbf{N}^{-r})=-1$. Let $\ell=\lambda\overline{\lambda}$ be a prime split in K, with $\ell\neq p$ and prime to the conductors of ψ and χ. Given a pair (ψ,χ) satisfying conditions (S1)–(S3), for any finite order character α of conductor dividing λ^∞, the pair $(\psi',\chi'):=(\psi\alpha,\chi\alpha^{-1})$ satisfies the same conditions (with $\mathfrak{f}_\psi$ replaced by $\mathfrak{f}_\psi\lambda^m$ for some $m\geq 0$), while (5.2) becomes

$$L(\theta_{\psi'}/K,\chi',s)=L(\lambda,s)\cdot L(\psi^\tau\chi\alpha^-,s),$$

where $\alpha^-:=\alpha^\tau/\alpha$. Since α^- is anticyclotomic of ℓ-power conductor, by Theorem 1.1 of [Fin06] all but finitely many of the values $L(\psi^\tau\chi\alpha^-,r)$ are nonzero (even mod ϖ as α varies[9]), this shows the first assertion.

Moreover, when $r=1$ noting that

$$L((\psi')^-,0)=L(\psi^-\alpha^-,0),$$

the same [Fin06, Thm. 1.1] ensures we can find α as above so that further $L^{\mathrm{alg}}(\psi^-\alpha^-,0)$ is a p-adic unit and conditions (S5)–(S8) hold for $(\psi')^-$, whence the result. ☐

Remark 5.1.4. In the proof of Lemma 5.1.3, the use of the main result of [Fin06] could alternatively be replaced by an appeal to Hsieh's mod p nonvanishing result [Hsi12, Thm. A].

[9] Reversing the roles of ℓ and p. Note that $\psi^\tau\chi\alpha^-\mathbf{N}^{-r}$ has infinity type $(r,1-r)$, and our conventions are *opposite* to those in [Fin06].

5.2. **Proving Selmer rank 1.** Let $f \in S_{2r}(\Gamma_0(N_f))$ be an eigenform of even weight $2r \geq 2$ and level N_f prime to p. Granted the injectivity of a certain p-adic Abel–Jacobi map, the works of Nekovář [Nek92] and S.-W. Zhang [Zha97] (extending to weights $2r > 2$ landmark results of Kolyvagin [Kol88] and Gross–Zagier [GZ86]) yield a proof of the implication

$$\operatorname{ord}_{s=r} L(f,s) = 1 \quad \Longrightarrow \quad \dim_\Phi \operatorname{Sel}_{\mathrm{BK}}(\mathbb{Q}, V_f(r)) = 1. \tag{5.3}$$

Key to the proof of this result are Heegner cycles attached to an auxiliary imaginary quadratic field K' satisfying the Heegner hypothesis (Heeg) relative to N_f.

Here we are interested in the case where $f = \theta_\lambda$ is the theta series of a Hecke character λ of infinity type $(1-2r, 0)$, so $\varepsilon_\lambda = \eta_K$. In particular, the conductor $\mathfrak{c}$ of λ is divisible by $\mathfrak{d}_K$, and so K does *not* satisfy the Heegner hypothesis (Heeg) relative to $N_f = D_K \mathbf{N}(\mathfrak{c})$.

For later use, here we prove the following analogue of (5.3) for $f = \theta_\lambda$ with CM by K using only arithmetic over K.

Theorem 5.2.1. *Let λ be a Hecke character of K of infinity type $(1-2r, 0)$ for some $r \geq 1$, conductor $\mathfrak{c}$ prime to p, and central character $\varepsilon_\lambda = \eta_K$. Then*

$$\mathscr{L}_v(\lambda \mathbf{N}^{-r})(0) \neq 0 \quad \Longrightarrow \quad \dim_\Phi \operatorname{Sel}_{\mathrm{BK}}(K, V_\lambda(r)) = 1.$$

Equivalently, if $\mathscr{L}_v(\lambda \mathbf{N}^{-r})(0)$ then $\operatorname{Sel}_{\mathrm{BK}}(\mathbb{Q}, V_f(r))$ is 1-dimensional, where $f = \theta_\lambda$.

Remark 5.2.2. When $r = 1$ and $\mathfrak{d}_K \| \mathfrak{c}$, the assumption $\mathscr{L}_v(\lambda \mathbf{N}^{-r})(0) \neq 0$ follows from $\operatorname{ord}_{s=1} L(\lambda, s) = 1$. Indeed, letting (ψ, χ) be a good pair for λ (as they exist by Lemma 5.1.3; note that $w(\lambda \mathbf{N}^{-1}) = -1$ by the assumption that $\operatorname{ord}_{s=1} L(\lambda, s) = 1$), putting $g = \theta_\psi$ and letting $z_{g,\chi} \in \operatorname{Sel}_{\mathrm{BK}}(K, T_{g,\chi})$ be the Heegner class of Theorem 3.2.1, which in this case arises as the image under the Kummer map

$$B_{g,\chi}(K) \otimes_{\mathcal{O}_L} \mathscr{O} \to \operatorname{Sel}_{\mathrm{BK}}(K, T_{g,\chi}) \tag{5.4}$$

of a Heegner cycle on a certain CM abelian variety $B_{g,\chi}/K$, we have

$$\begin{aligned}
&\text{(by (5.2))} && \operatorname{ord}_{s=1} L(\lambda, s) = 1 && \Longrightarrow \quad \operatorname{ord}_{s=1} L(g/K, \chi, s) = 1 \\
&\text{(by [YZZ13])} && && \Longrightarrow \quad z_{g,\chi} \neq 0 \\
&\text{(by Theorem 3.2.1)} && && \Longrightarrow \quad \mathscr{L}_v(g, \chi)(0) \neq 0 \\
&\text{(by Proposition 2.3.1)} && && \Longrightarrow \quad \mathscr{L}_v(\lambda \mathbf{N}^{-1})(0) \neq 0.
\end{aligned}$$

Note that the second implication relies on the injectivity (5.4), and the third on the non-triviality of the localization map

$$\operatorname{loc}_{\overline{v}} : \operatorname{Sel}_{\mathrm{BK}}(K, T_{g,\chi}) \to \mathrm{H}^1_f(K_{\overline{v}}, T_{g,\chi}),$$

both of which are expected—but not known—to continue to hold in weight $2r > 2$, with (5.4) replaced by the p-adic étale Abel–Jacobi map on generalized Kuga–Sato varieties studied in [BDP13].

Proof of Theorem 5.2.1. Let $\mathbf{c}_{\lambda \mathbf{N}^{-r}} \in \mathrm{H}^1_{\mathrm{Iw}}(K_\infty, T_\lambda(r))$ be the $\Lambda_{\mathscr{O}}$-adic class (of twisted elliptic units) of Theorem 4.2.1, and denote by $\mathbf{c}_{\lambda \mathbf{N}^{-r}}(\mathbb{1})$ its image under the projection

$$\mathrm{H}^1_{\mathrm{Iw}}(K_\infty, T_\lambda(r)) \to \mathrm{H}^1(K, T_\lambda(r)).$$

As a consequence of Theorem 4.2.1, we have

$$\mathscr{L}_v(\lambda \mathbf{N}^{-r})(0) \neq 0 \quad \Longrightarrow \quad \operatorname{loc}_{\overline{v}}(\mathbf{c}_{\lambda \mathbf{N}^{-r}}(\mathbb{1})) \neq 0.$$

We claim that $\mathbf{c}_{\lambda \mathbf{N}^{-r}}(\mathbb{1})$ lands in $\operatorname{Sel}_{\mathrm{BK}}(K, T_\lambda(r))$. Indeed, since $\lambda \mathbf{N}^{-r}$ has infinity type $(1-r, r)$, by Lemma 3.4.2 the claim amounts to the assertion that

$$\operatorname{loc}_v(\mathbf{c}_{\lambda \mathbf{N}^{-r}}(\mathbb{1})) = 0, \tag{5.5}$$

which by Theorem 4.2.1 for $w = \overline{v}$ amounts to the assertion that $\mathscr{L}_v(\lambda^\tau \mathbf{N}^{-r})(0) = 0$. Since $\lambda^\tau \mathbf{N}^{-r}$ is within the range of interpolation of $\mathscr{L}_{v,\mathfrak{c}}$ (as its infinity type is $(r, 1-r)$), by Theorem 2.2.1 we have

$$\mathscr{L}_v(\lambda^\tau \mathbf{N}^{-r})(0) = 0 \quad \Longleftrightarrow \quad L(\lambda^\tau, r) = L(\lambda, r) = 0,$$

whence the claim. In particular, this shows that the map

$$\mathrm{loc}_{\overline{v}} : \mathrm{Sel}_{\mathrm{BK}}(K, T_\lambda(r)) \to \mathrm{H}^1_f(K_{\overline{v}}, T_\lambda(r)) = \mathrm{H}^1(K_{\overline{v}}, T_\lambda(r)) \tag{5.6}$$

is nonzero, using Lemma 3.4.2 for the last equality.

It follows from the discussion in §3.3 and Theorem 6.4.1 in [Rub00] that the class $\mathbf{c}_{\lambda\mathbf{N}^{-r}}(\mathbb{1})$ extends to an anticyclotomic Euler system for $T_\lambda(r)$; since the above shows that this Euler system is non-trivial, by [Rub00, Thm. 2.2.3] (see also [*loc. cit.*, §9.3]) it follows that $\mathrm{rank}_{\mathscr{O}} \mathrm{Sel}_{\mathrm{rel}}(K, T_\lambda(r)) = 1$. From the global duality exact sequence

$$\mathrm{Sel}_{\mathrm{BK}}(K, T_{\lambda^\tau}(r)) \xrightarrow{\mathrm{loc}_v} \mathrm{H}^1(K_v, T_{\lambda^\tau}(r)) \to \mathrm{Sel}_{\mathrm{rel}}(K, W_\lambda(r))^\vee \to \mathrm{Sel}_{\mathrm{BK}}(K, W_\lambda(r))^\vee \to 0,$$

noting that the non-vanishing of (5.6) implies the non-vanishing of the map loc_v in this sequence, we conclude that $\mathrm{corank}_{\mathscr{O}} \mathrm{Sel}_{\mathrm{BK}}(K, W_\lambda(r)) = \mathrm{corank}_{\mathscr{O}} \mathrm{Sel}_{\mathrm{rel}}(K, W_\lambda(r)) = 1$, whence the result. ☐

5.3. **Proof of Theorem C.** We begin by noting that by the assumption that $\lambda\mathbf{N}^{-r}$ has root number -1, Lemma 5.1.3 ensures the existence of good pairs for λ. We fix once and for all a good pair (ψ, χ) for λ, and put $g = \theta_\psi$. Exploiting the decomposition

$$\mathcal{X}_v(g, \chi) \simeq \mathcal{X}_v(\lambda\mathbf{N}^{-r}) \oplus \mathcal{X}_v(\psi^\tau\chi\mathbf{N}^{-r})$$

from Proposition 3.5.1, the formula for $\#\text{Ш}_{\mathrm{BK}}(W_{g,\chi}/K)$ of Theorem C will be obtained by computing the Γ-Euler characteristic for the constituent Selmer groups in this decomposition and combining the resulting formulae. Note that some of the intermediate results will be obtained in greater generality than needed for the proof of Theorem C.

For a finite extension Φ of $\mathbb{Q}_p$ with ring of integers $\mathscr{O}$, and a Φ-linear G_K-representation V with a fixed G_K-stable $\mathscr{O}$-lattice $T \subset V$ with $W := V/T$, recall that the Bloch–Kato Tate–Shafarevic group $\text{Ш}_{\mathrm{BK}}(W/K)$ is defined by

$$\text{Ш}_{\mathrm{BK}}(W/K) := \mathrm{Sel}_{\mathrm{BK}}(K, W)/\mathrm{Sel}_{\mathrm{BK}}(K, W)_{\mathrm{div}},$$

where $\mathrm{Sel}_{\mathrm{BK}}(K, W)_{\mathrm{div}}$ denotes the maximal divisible submodule of $\mathrm{Sel}_{\mathrm{BK}}(K, W)$, and for a finite prime $w \nmid p$ of K, the Tamagawa number of W at w is defined by

$$c_w(W/K) := \#\big(\mathrm{H}^1_{\mathrm{ur}}(K_w, W)/\mathrm{H}^1_f(K_w, W)\big),$$

where $\mathrm{H}^1_{\mathrm{ur}}(K_w, W) := \ker\{\mathrm{H}^1(K_w, W) \to \mathrm{H}^1(I_w, W)\}$ is the unramified submodule, and $\mathrm{H}^1_f(K_w, W)$ is the image of the unramified subspace $\mathrm{H}^1_{\mathrm{ur}}(K, V)$ under the natural map $\mathrm{H}^1(K_w, V) \to \mathrm{H}^1(K_w, W)$ (which is thus contained in $\mathrm{H}^1_{\mathrm{ur}}(K_w, W)$).

5.3.1. *Formula for $\psi^\tau\chi$.*

Proposition 5.3.1. *Let (ψ, χ) be as in Theorem 4.3.6, and let $\mathscr{F}_v(\psi^\tau\chi\mathbf{N}^{-r}) \in \Lambda_{\mathscr{O}}$ be a generator of the characteristic ideal of $\mathcal{X}_v(\psi^\tau\chi\mathbf{N}^{-r})$. If $\mathrm{Sel}_{\mathrm{BK}}(K, W_{\psi^\tau\chi}(r))$ is finite, then $\mathscr{F}_v(\psi^\tau\chi\mathbf{N}^{-r})(0) \neq 0$ with*

$$\mathscr{F}_v(\psi^\tau\chi\mathbf{N}^{-r})(0) \ \sim_p \ \#\mathrm{H}^0(K_{\overline{v}}, W_{\psi^\tau\chi}(r))^2 \cdot \frac{\#\text{Ш}_{\mathrm{BK}}(W_{\psi^\tau\chi}(r)/K)}{\#\mathrm{H}^0(K, W_{\psi^\tau\chi}(r))^2} \cdot \prod_{w \in \Sigma, w \nmid p} c_w(W_{\psi^\tau\chi}(r)/K).$$

Proof. By Lemma 3.4.2, $\mathcal{X}_v(\psi^\tau\chi\mathbf{N}^{-r})$ interpolates Selmer groups $\mathrm{Sel}_{\mathrm{BK}}(F, W_{\psi^\tau\chi}(r))$ as F varies over the finite extensions of K contained in K_∞. Therefore, by the control theorem of [Arn07, Prop. 4.3] the natural restriction map

$$\mathrm{Sel}_{\mathrm{BK}}(K, W_{\psi^\tau\chi}(r)) = \mathrm{Sel}_v(K, W_{\psi^\tau\chi}(r)) \to \mathcal{S}_v(\psi^\tau\chi\mathbf{N}^{-r})^\Gamma$$

has finite kernel and cokernel, and so the finiteness of $\mathrm{Sel}_{\mathrm{BK}}(K, W_{\psi^\tau\chi}(r))$ gives $\#\mathcal{S}_v(\psi^\tau\chi\mathbf{N}^{-r})^\Gamma < \infty$. Letting $\mathscr{F}_v(\psi^\tau\chi\mathbf{N}^{-r}) \in \Lambda_{\mathscr{O}}$ be any generator of $\mathrm{char}_{\Lambda_{\mathscr{O}}}(\mathcal{X}_v(\psi^\tau\chi\mathbf{N}^{-r}))$, it is then easily checked that the Γ-coinvariants $\mathcal{S}_v(\psi^\tau\chi\mathbf{N}^{-r})_\Gamma$ are also finite, that $\mathscr{F}_v(\psi^\tau\chi\mathbf{N}^{-r})(0)\neq 0$, and that

$$\mathscr{F}_v(\psi^\tau\chi\mathbf{N}^{-r})(0)\ \sim_p\ \frac{\#\mathcal{S}_v(\psi^\tau\chi\mathbf{N}^{-r})^\Gamma}{\#\mathcal{S}_v(\psi^\tau\chi\mathbf{N}^{-r})_\Gamma} \tag{5.7}$$

(see [Gre99, Lem. 4.2] for example). For the proof of the formula in the Proposition, we shall adapt the methods of [Gre99, §4] and [JSW17, §3].

First we set up some notations. Fix Σ a finite set of primes of K containing the archimedean prime, the primes above p, the primes dividing the conductor of ψ^τ or χ and their complex conjugates, and let K^Σ denote the maximal extension of K unramified outside Σ. Put

$$\mathcal{P}^\Sigma(K):=\frac{\mathrm{H}^1(K_v,W_{\psi^\tau\chi}(r))}{\mathrm{H}^1(K_v,W_{\psi^\tau\chi}(r))_{\mathrm{div}}}\times\mathrm{H}^1(K_{\overline{v}},W_{\psi^\tau\chi}(r))\times\prod_{w\in\Sigma,w\nmid p}\mathrm{H}^1(K_w,W_{\psi^\tau\chi}(r)).$$

Similarly, consider the $\Lambda_{\mathscr{O}}$-module $\mathbf{W}_{\psi^\tau\chi}(r):=W_{\psi^\tau\chi}(r)\otimes_{\mathscr{O}}\Lambda_{\mathscr{O}}^\vee$ with diagonal G_K-action, letting G_K act on the second factor through the inverse of the tautological character $G_K\twoheadrightarrow\Gamma\hookrightarrow\Lambda_{\mathscr{O}}^\times$, and put

$$\mathcal{P}^\Sigma(K_\infty):=\{0\}\times\mathrm{H}^1(K_{\overline{v}},\mathbf{W}_{\psi^\tau\chi}(r))\times\prod_{w\in\Sigma,w\nmid p}\mathrm{H}^1(K_w,\mathbf{W}_{\psi^\tau\chi}(r)),$$

thinking of $\{0\}$ as a subspace of $\mathrm{H}^1(K_v,\mathbf{W}_{\psi^\tau\chi}(r))$. Then from the definitions and Shapiro's lemma we have exact sequences

$$0\to\mathrm{Sel}_{\mathrm{BK}}(K,W_{\psi^\tau\chi}(r))\to\mathrm{H}^1(K^\Sigma/K,W_{\psi^\tau\chi}(r))\to\mathcal{P}^\Sigma(K), \tag{5.8}$$

$$0\to\mathcal{S}_v(\psi^\tau\chi\mathbf{N}^{-r})\to\mathrm{H}^1(K^\Sigma/K,\mathbf{W}_{\psi^\tau\chi}(r))\to\mathcal{P}^\Sigma(K_\infty). \tag{5.9}$$

Denote by $\mathcal{G}^\Sigma(K)\subset\mathcal{P}^\Sigma(K)$ and $\mathcal{G}^\Sigma(K_\infty)\subset\mathcal{P}^\Sigma(K_\infty)$ the image of the right maps in the above exact sequence, whose Γ-invariants then fit into the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Sel}_{\mathrm{BK}}(K,W_{\psi^\tau\chi}(r)) \arrow[r]\arrow[d,"r^*"] & \mathrm{H}^1(K^\Sigma/K,W_{\psi^\tau\chi}(r)) \arrow[r]\arrow[d,"s^*"] & \mathcal{G}^\Sigma(K) \arrow[r]\arrow[d,"t^*"] & 0\\
0 \arrow[r] & \mathcal{S}_v(\psi^\tau\chi\mathbf{N}^{-r})^\Gamma \arrow[r] & \mathrm{H}^1(K^\Sigma/K,\mathbf{W}_{\psi^\tau\chi}(r))^\Gamma \arrow[r] & \mathcal{G}^\Sigma(K_\infty)^\Gamma.
\end{tikzcd} \tag{5.10}$$

The surjectivity of s^* follows from the fact that Γ has p-cohomological dimension 1, while the kernel of s^* is given by

$$\mathrm{H}^1(\Gamma,\mathrm{H}^0(K,\mathbf{W}_{\psi^\tau\chi}(r)))=\mathrm{H}^1(\Gamma,\mathrm{H}^0(K_\infty,W_{\psi^\tau\chi}(r))),$$

whose order is the same as $\mathrm{H}^0(K,W_{\psi^\tau\chi}(r))$ (using that $\#\mathrm{H}^1(K_\infty,W_{\psi^\tau\chi}(r))<\infty$ by [Arn07, Lem. 2.5]). Hence from the Snake Lemma applied to (5.10) we immediately get the relation

$$\#\mathcal{S}_v(\psi^\tau\chi\mathbf{N}^{-r})^\Gamma=\#\mathrm{Sel}_{\mathrm{BK}}(K,W_{\psi^\tau\chi}(r))\cdot\frac{\#\ker(t^*)}{\#\mathrm{H}^0(K,W_{\psi^\tau\chi}(r))}. \tag{5.11}$$

To compute the order of $\ker(t^*)$, we first the order of the kernel of the map

$$\tau^*=(\tau_w^*)_{w\in\Sigma,w\nmid p}:\mathcal{P}^\Sigma(K)\to\mathcal{P}^\Sigma(K_\infty)^\Gamma.$$

For $w=v$ we find

$$\begin{aligned}\ker(\tau_v^*)=\frac{\mathrm{H}^1(K_v,W_{\psi^\tau\chi}(r))}{\mathrm{H}^1(K_v,W_{\psi^\tau\chi}(r))_{\mathrm{div}}}&\simeq\mathrm{H}^1(K_v,T_{\psi\chi^\tau}(r))_{\mathrm{tors}}\\
&=\ker\big(\mathrm{H}^1(K_v,T_{\psi\chi^\tau}(r))\to\mathrm{H}^1(K_v,V_{\psi\chi^\tau}(r))\big)\\
&=\mathrm{coker}\big(\mathrm{H}^0(K_v,V_{\psi\chi^\tau}(r))\to\mathrm{H}^0(K_v,W_{\psi\chi^\tau}(r))\big)\\
&=\mathrm{H}^0(K_v,W_{\psi\chi^\tau}(r)),\end{aligned} \tag{5.12}$$

using local Tate duality, the cohomology long exact sequence associated to $0 \to T_{\psi\chi^\tau}(r) \to V_{\psi\chi^\tau}(r) \to W_{\psi\chi^\tau}(r) \to 0$, and the vanishing of $\mathrm{H}^0(K_v, V_{\psi\chi^\tau}(r))$ (as follows from [Arn07, Lem. 2.5]). Similarly, we find

$$\#\ker(\tau_{\overline{v}}^*) = \#\mathrm{H}^1(\Gamma, \mathrm{H}^0(K_{\overline{v}}, \mathbf{W}_{\psi^\tau\chi}(r))) = \#\mathrm{H}^0(K_{\overline{v}}, W_{\psi^\tau\chi}(r)), \tag{5.13}$$

using the finiteness of $\mathrm{H}^0(K_{\overline{v}}, \mathbf{W}_{\psi^\tau\chi}(r))$ (a consequence of [Arn07, Lem. 2.5]) for the last equality. On the other hand, consider a prime $w \in \Sigma$ with $w \nmid p$. If $W_{\psi^\tau\chi}$ is unramified at w, then $\ker(\tau_w^*) = 0$ by the same argument as in [JSW17, p. 389]; if $W_{\psi^\tau\chi}$ is ramified at w, then the group $\mathrm{H}^0(K_w, \mathbf{W}_{\psi^\tau\chi}(r))$ is finite (see [Arn07, Lem. 3.4]), and as in (5.13) we find $\#\ker(\tau_w^*) = \#\mathrm{H}^0(K_w, W_{\psi^\tau\chi}(r))$. Together with [BK90, Lem. 5.10] and the discussion in [*op. cit.*, p. 373], this shows

$$\#\ker(\tau_w^*) = c_w(W_{\psi^\tau\chi}(r)/K) \tag{5.14}$$

for all $w \in \Sigma$ with $w \nmid p$.

Next, the relation between $\#\ker(t^*)$ and $\#\ker(\tau^*)$ can be found similarly as in [Gre99, Lem. 4.7]. Indeed, by [PW11, Prop. A.2] (extending and generalizing [GV00, Prop. 2.1] to the anticyclotomic setting), the $\Lambda_{\mathscr{O}}$-torsionness of $\mathcal{X}_v(\psi^\tau\chi\mathbf{N}^{-r})$ (as shown in Theorem 4.3.6) implies surjectivity of the right arrow in (5.9), i.e. $\mathcal{G}^\Sigma(K_\infty) = \mathcal{P}^\Sigma(K_\infty)$. On the other hand, from Poitou–Tate duality we have the exact sequence

$$0 \to \mathrm{Sel}_{\mathrm{BK}}(K, W_{\psi^\tau\chi}(r)) \to \mathrm{H}^1(K, W_{\psi^\tau\chi}(r)) \to \mathcal{P}^\Sigma(K) \to \mathrm{Sel}_{\overline{v}}(K, T_{\psi\chi^\tau}(r))^\vee \to \mathrm{H}^2(K, W_{\psi^\tau\chi}(r))^\vee \to 0,$$

using that $\mathrm{Hom}(W_{\psi^\tau\chi}(r), \mu_{p^\infty}) \simeq T_{\psi\chi^\tau}(r)$ by the self-duality of $\psi^\tau\chi\mathbf{N}^{-r}$. Since $\#\mathrm{Sel}_{\mathrm{BK}}(K, W_{\psi^\tau\chi}(r)) < \infty$ by assumption, the generalization of Cassels' theorem in [Gre99, Prop. 4.13] gives[10]

$$\mathcal{P}^\Sigma(K)/\mathcal{G}^\Sigma(K) \simeq \mathrm{H}^0(K, W_{\psi\chi^\tau}(r))^\vee,$$

and with this isomorphism, the argument in [Gre99, Prop. 4.7] gives

$$\#\ker(t^*) = \#\ker(\tau^*) \cdot \frac{\#\mathcal{S}_v(\psi^\tau\chi\mathbf{N}^{-r})_\Gamma}{\#\mathrm{H}^0(K, W_{\psi\chi^\tau}(r))}.$$

Substituting (5.12), (5.13), and (5.14) into this, and the resulting formula for $\ker(t^*)$ into (5.11), we obtain

$$\#\mathcal{S}_v(\psi^\tau\chi\mathbf{N}^{-r})^\Gamma = \#\text{Ш}_{\mathrm{BK}}(K, W_{\psi^\tau\chi}(r)) \cdot \#\mathrm{H}^0(K_{\overline{v}}, W_{\psi^\tau\chi}(r))^2 \cdot \frac{\mathcal{S}_v(\psi^\tau\chi\mathbf{N}^{-r})_\Gamma}{\#\mathrm{H}^0(K, W_{\psi^\tau\chi}(r))^2} \cdot \prod_{w\in\Sigma, w\nmid p} c_w(W_{\psi^\tau\chi}(r)/K),$$

using $\mathrm{H}^0(K_v, W_{\psi\chi^\tau}(r)) \simeq \mathrm{H}^0(K_{\overline{v}}, W_{\psi^\tau\chi}(r))$ and $\mathrm{H}^0(K, W_{\psi\chi^\tau}(r)) \simeq \mathrm{H}^0(K, W_{\psi^\tau\chi}(r))$ by the action of complex conjugation and the equality $\text{Ш}_{\mathrm{BK}}(K, W_{\psi^\tau\chi}(r)) = \mathrm{Sel}_{\mathrm{BK}}(K, W_{\psi^\tau\chi}(r))$ that follows from the finiteness of the latter. Together with (5.7) this concludes the proof. ☐

5.3.2. *Interlude: p-part of TNC in rank* 0. Combined with Theorem 4.3.6, the preceding proposition yields a result on the Tamagawa number conjecture of Bloch–Kato [BK90] that will play an important role in the proof of Theorem B.

Recall that a Hecke character $\xi : K\backslash\mathbb{A}_K^\times \to \mathbb{C}^\times$ of infinity type $(a, b) \in \mathbb{Z}^2$ in the sense of §2.1 can alternatively be viewed as homomorphism

$$\xi : \mathbb{A}_K^\times/U \to \mathbb{C}^\times$$

trivial on an open subgroup U of $\mathbb{A}_K^\times$ (that can be described explicitly in terms of the conductor of ξ) satisfying $\xi(\alpha) = \alpha^{-a}(\alpha^\tau)^{-b}$ for all $\alpha \in K^\times$. Under our fixed embedding $\imath_\infty$, ξ then takes values in a number field L, and for every embedding $\sigma \in \mathrm{Hom}(L, \mathbb{C})$ we obtain the character

$$\xi_\sigma : \mathbb{A}_K^\times/U \xrightarrow{\xi} L^\times \xrightarrow{\sigma} \mathbb{C}^\times$$

[10] Taking $M = W_{\psi^\tau\chi}(r)$ in the notations of *loc. cit.*. Note also that the hypothesis in *loc. cit.* that $\mathrm{H}^0(K_w, M)$ is finite for some finite $w \in \Sigma$ holds in our case taking $w = v$ thanks to [Arn07, Lem. 2.5].

by composition. Combining their associated Hecke L-functions, can consider the $L_{\mathbb{C}} := L\otimes_{\mathbb{Q}}\mathbb{C}$-valued L-function

$$(L(\xi_\sigma, s))_{\sigma:L\hookrightarrow\mathbb{C}},$$

with $L(\xi, s)$ corresponding to $\sigma : L \subset \overline{\mathbb{Q}} \xrightarrow{\imath_\infty} \mathbb{C}$.

Theorem 5.3.2. *Let (ψ, χ) be as in Theorem 4.3.6, with values in a number field L, and suppose that $L(\psi^\tau\chi, r) \neq 0$. Then*

$$\left(\frac{L((\psi^\tau\chi)_\sigma, r)}{\Omega_\sigma}\right) \in L^\times \subset L_{\mathbb{C}}^\times,$$

and for all primes $\mathfrak{P}$ of L above p we have

$$\mathrm{ord}_{\mathfrak{P}}\left(\frac{L(\psi^\tau\chi, r)}{\Omega}\right) = \mathrm{ord}_{\mathfrak{P}}\left(\frac{\#\text{Ш}_{\mathrm{BK}}(W_{\psi^\tau\chi}(r)/K)}{\#\mathrm{H}^0(K, W_{\psi^\tau\chi}(r))^2}\right) + \sum_{w\in\Sigma, w\nmid p} \mathrm{ord}_{\mathfrak{P}}(c_w(W_{\psi^\tau\chi}(r)/K)).$$

In other words, the p-part of the Tamagawa number conjecture holds for $\psi^\tau\chi$.

Proof. The first assertion is proved in [GS81] and (more generally) in [Bla86] (cf. [BDP12, Thm. 2.12]), and the second is immediate from the combination of Theorem 2.2.1, Theorem 4.3.6, and Proposition 5.3.1. $\square$

5.3.3. *Formula for λ.*

Proposition 5.3.3. *Let λ be as in Theorem 4.3.1, and suppose $\mathscr{L}_v(\lambda\mathbf{N}^{-r})(0) \neq 0$. Then the map*

$$\mathrm{loc}_{\overline{v}} : \mathrm{Sel}_{\mathrm{BK}}(K, T_\lambda(r)) \to \mathrm{H}^1_f(K_{\overline{v}}, T_\lambda(r)) \tag{5.15}$$

is nonzero, and letting $\mathscr{F}_v(\lambda\mathbf{N}^{-r}) \in \Lambda_{\mathscr{O}}$ be any generator of the characteristic ideal of $\mathcal{X}_v(\lambda\mathbf{N}^{-r})$ we have

$$\mathscr{F}_v(\lambda\mathbf{N}^{-r})(0) \ \sim_p\ \#\mathrm{H}^0(K_{\overline{v}}, W_\lambda(r))^2 \cdot \frac{\#\text{Ш}_{\mathrm{BK}}(W_\lambda(r)/K)}{\#\mathrm{H}^0(K, W_\lambda(r))^2} \cdot \#\mathrm{coker}(\mathrm{loc}_{\overline{v}/\mathrm{tors}})^2 \cdot \prod_{w\in\Sigma, w\nmid p} c_w(W_\lambda(r)/K),$$

where $\mathrm{loc}_{\overline{v}/\mathrm{tors}}$ is the composition of $\mathrm{loc}_{\overline{v}}$ with $\mathrm{H}^1_f(K_{\overline{v}}, T_\lambda(r)) \to \mathrm{H}^1_f(K_{\overline{v}}, T_\lambda(r))/\mathrm{H}^1_f(K_{\overline{v}}, T_\lambda(r))_{\mathrm{tors}}$.

Proof. By Theorem 4.3.1 we know that $\mathcal{X}_v(\lambda\mathbf{N}^{-r})$ is $\Lambda_{\mathscr{O}}$-torsion and the nonvanishing of $\mathscr{L}_v(\lambda\mathbf{N}^{-r})(0)$ implies that $\mathscr{F}_v(\lambda\mathbf{N}^{-r})(0) \neq 0$. Since by [Cas23, Prop. 6.2.1] the natural restriction map

$$\mathrm{Sel}_v(K, W_\lambda(r)) \to \mathcal{S}_v(\lambda\mathbf{N}^{-r})^\Gamma$$

has finite kernel and cokernel, it follows that $\mathrm{Sel}_v(K, W_\lambda(r))$ if finite. The same argument as in Proposition 5.3.1 then gives

$$\mathscr{F}_v(\lambda\mathbf{N}^{-r})(0) \ \sim_p\ \#\mathrm{H}^0(K_{\overline{v}}, W_\lambda(r))^2 \cdot \frac{\#\mathrm{Sel}_v(K, W_\lambda(r))}{\#\mathrm{H}^0(K, W_\lambda(r))^2} \cdot \prod_{w\in\Sigma, w\nmid p} c_w(W_\lambda(r)/K). \tag{5.16}$$

Thus it remains to show the nonvanishing of (5.15) and to relate the orders of $\mathrm{Sel}_v(K, W_\lambda(r))$ and $\text{Ш}_{\mathrm{BK}}(K, W_\lambda(r))$.

By Theorem 4.2.1, letting $\mathbf{c}_{\lambda\mathbf{N}^{-r}}(\mathbb{1}) \in \mathrm{H}^1(K, T_\lambda(r))$ denote the image of $\mathbf{c}_{\lambda\mathbf{N}^{-r}}$ under the projection $\mathrm{H}^1_{\mathrm{Iw}}(K_\infty, T_\lambda(r)) \to \mathrm{H}^1(K, T_\lambda(r))$, the nonvanishing of $\mathscr{L}_v(\lambda\mathbf{N}^{-r})(0)$ implies that $\mathrm{loc}_{\overline{v}}(\mathbf{c}_{\lambda\mathbf{N}^{-r}}(\mathbb{1})) \neq 0$. Thus to show the nonvanishing of (5.15), it suffices to show the inclusion $\mathbf{c}_{\lambda\mathbf{N}^{-r}}(\mathbb{1}) \in \mathrm{Sel}_{\mathrm{BK}}(K, T_\lambda(r))$; but this was shown in the proof of Theorem 5.2.1 under the present hypotheses. Hence $\mathrm{loc}_{\overline{v}/\mathrm{tors}}$ has finite cokernel, and from the global duality argument in the proof of [JSW17, Prop. 3.2.1][11] we find

$$\#\mathrm{Sel}_v(K, W_\lambda(r)) = \#\text{Ш}_{\mathrm{BK}}(W_\lambda(r)/K) \cdot \#\mathrm{coker}(\mathrm{loc}_{\overline{v}/\mathrm{tors}})^2,$$

which together with (5.16) yields the result. $\square$

[11] Note that the irreducibility assumption ($\mathrm{irred}_{\mathcal{K}}$) in *loc. cit.* is only used to deduce that $W^* \simeq T^\tau$, which is automatic in our case.

5.3.4. *A consequence of the explicit reciprocity law.* Building on the explicit reciprocity law for $\mathbf{z}_{g,\chi}$, we can obtain a useful expression for the order of the cokernel of the map $\mathrm{loc}_{\overline{v}/\mathrm{tors}}$ in Proposition 5.3.3.

Proposition 5.3.4. *Let λ be as in Theorem 4.3.1 with $\mathfrak{d}_K \| \mathfrak{c}$, and let (ψ, χ) be a good pair for λ. If $\mathscr{L}_v(\lambda\mathbf{N}^{-r})(0) \neq 0$ then the following hold:*

1. $\mathscr{L}_v(g,\chi)(0) \neq 0$.
2. *$z_{g,\chi}$ is non-torsion.*
3. $\mathrm{rank}_{\mathscr{O}} \mathrm{Sel}_{\mathrm{BK}}(K, T_{g,\chi}) = 1$.
4. *The map $\mathrm{loc}_{\overline{v}/\mathrm{tors}}$ of Proposition 5.3.3 satisfies*

$$\#\mathrm{coker}(\mathrm{loc}_{\overline{v}/\mathrm{tors}}) = \frac{\#\big(\mathscr{O}/\mathscr{L}_v(g,\chi)(0)\big)}{\#\big(\mathrm{Sel}_{\mathrm{BK}}(K, T_{g,\chi})/\mathscr{O}\cdot z_{g,\chi}\big)} \cdot \frac{1}{\#\mathrm{H}^0(K_{\overline{v}}, W_\lambda(r))} \cdot \frac{1}{\#\mathrm{H}^0(K_{\overline{v}}, W_{\psi^\tau\chi}(r))}.$$

Proof. Part (1) follows from the factorization of Proposition 2.3.1, the fact that $\mathscr{L}_v(\psi^\tau\chi\mathbf{N}^{-r})(0)$ is a nonzero multiple of $L(\psi^\tau\chi, r)$ by Theorem 2.2.1, and the Definition 5.1.1 of good pair; part (2) then follows from Theorem 3.2.1. Part (3) follows from Theorem 5.2.1, so it remains to prove (4).

Note that by Lemma 3.4.2 we have

$$\mathrm{H}^1_f(K_{\overline{v}}, T_\lambda(r)) = \mathrm{H}^1(K_{\overline{v}}, T_\lambda(r)), \qquad \mathrm{H}^1_f(K_{\overline{v}}, T_{\psi^\tau\chi}(r)) = \mathrm{H}^1(K_{\overline{v}}, T_{\psi^\tau\chi}(r))_{\mathrm{tors}},$$

and so from the decomposition $T_{g,\chi} \simeq T_\lambda(r) \oplus T_{\psi^\tau\chi}(r)$ we see that the map $\mathrm{loc}_{\overline{v}/\mathrm{tors}}$ of Proposition 5.3.3 is the same as the composite

$$\mathrm{loc}'_{\overline{v}/\mathrm{tors}} : \mathrm{Sel}_{\mathrm{BK}}(K, T_{g,\chi}) \xrightarrow{\mathrm{loc}_{\overline{v}}} \mathrm{H}^1_f(K_{\overline{v}}, T_{g,\chi}) \to \mathrm{H}^1_f(K_{\overline{v}}, T_{g,\chi})_{/\mathrm{tors}},$$

where $\mathrm{H}^1_f(K_{\overline{v}}, T_{g,\chi})_{/\mathrm{tors}} := \mathrm{H}^1_f(K_{\overline{v}}, T_{g,\chi})/\mathrm{H}^1_f(K_{\overline{v}}, T_{g,\chi})_{\mathrm{tors}}$. As a consequence of Theorem 3.2.2, the argument in Lemma 1.2.3 in [BCGS23] (using Remark 1.2.4 in *loc. cit.*) shows that

$$\#\mathrm{coker}(\mathrm{loc}'_{\overline{v}/\mathrm{tors}}) = \frac{\#\big(\mathscr{O}/\mathscr{L}_v(g,\chi)(0)\big)}{\#\big(\mathrm{Sel}_{\mathrm{BK}}(K, T_{g,\chi})/\mathscr{O}\cdot z_{g,\chi}\big)} \cdot \frac{1}{\#\mathrm{H}^0(K_{\overline{v}}, W_{g,\chi})}.$$

Since $\#\mathrm{H}^0(K_{\overline{v}}, W_{g,\chi}) = \#\mathrm{H}^0(K_{\overline{v}}, W_\lambda(r)) \cdot \#\mathrm{H}^0(K_{\overline{v}}, W_{\psi^\tau\chi}(r))$, the proof of part (4) follows. □

Corollary 5.3.5. *Let λ be as in Theorem 4.3.1 with $\mathfrak{d}_K \| \mathfrak{c}$ and $\mathscr{L}_v(\lambda\mathbf{N}^{-r})(0) \neq 0$. Then for any generator $\mathscr{F}_v(\lambda\mathbf{N}^{-r}) \in \Lambda_{\mathscr{O}}$ of $\mathrm{char}_{\Lambda_{\mathscr{O}}}(\mathcal{X}_v(\lambda\mathbf{N}^{-r}))$ we have*

$$\begin{aligned}\mathscr{F}_v(\lambda\mathbf{N}^{-r})(0) \ \sim_p\ & \frac{\#\text{Ш}_{\mathrm{BK}}(W_\lambda(r)/K)}{\#\mathrm{H}^0(K, W_\lambda(r))^2} \cdot \frac{\#\big(\mathscr{O}/\mathscr{L}_v(g,\chi)(0)\big)^2}{\#\big(\mathrm{Sel}_{\mathrm{BK}}(K, T_{g,\chi})/\mathscr{O}\cdot z_{g,\chi}\big)^2} \\ & \times \frac{1}{\#\mathrm{H}^0(K_{\overline{v}}, W_{\psi^\tau\chi}(r))^2} \cdot \prod_{w\in\Sigma, w\nmid p} c_w(W_\lambda(r)/K).\end{aligned}$$

Proof. This is the combination of Proposition 5.3.3 and Proposition 5.3.4. Note that the contribution from $\#\mathrm{H}^0(K_{\overline{v}}, W_\lambda(r))$ in the two formulae cancel each other. □

Proof of Theorem C. Multiplying the formulas (up to a unit) for $\mathscr{F}_v(\psi^\tau\chi\mathbf{N}^{-r})(0)$ and $\mathscr{F}_v(\lambda\mathbf{N}^{-r})(0)$ in Proposition 5.3.1 and Corollary 5.3.5, and using Proposition 2.3.1 and Proposition 3.5.1, we obtain

$$\#\big(\mathscr{O}/\mathscr{F}_v(g,\chi)(0)\big) = \frac{\#\text{Ш}_{\mathrm{BK}}(W_{g,\chi}/K)}{\#\mathrm{H}^0(K, W_{g,\chi})^2} \cdot \frac{\#\big(\mathscr{O}/\mathscr{L}_v(g,\chi)(0)\big)^2}{\#\big(\mathrm{Sel}_{\mathrm{BK}}(K, T_{g,\chi})/\mathscr{O}\cdot z_{g,\chi}\big)^2} \cdot \prod_{w\in\Sigma, w\nmid p} c_w(W_{g,\chi}/K).$$

Since Theorem 4.1.1 gives

$$\#\big(\mathscr{O}/\mathscr{F}_v(g,\chi)(0)\big) = \#\big(\mathscr{O}/\mathscr{L}_v(g,\chi)(0)\big)^2,$$

and this is nonzero by part (1) of Proposition 5.3.3, this concludes the proof of Theorem C. □

6. The p-part of the Birch Swinnerton-Dyer formula

In this section we deduce our main results on the p-part of the Birch and Swinnerton-Dyer formula in analytic rank 1. After a period comparison (responsible for our assumption that $p \nmid h_K$), the result is obtained by specializing Theorem C to the weight 2 case and combining it with the general Gross–Zagier formula [YZZ13, CST14].

6.1. Periods and heights of Heegner points.

Fix a prime p, and let $g \in S_2(\Gamma_1(N_g))$ be a newform of weight 2 and level N_g with $p \nmid N_g$.

Definition 6.1.1. The *canonical period* of g is

$$\Omega_g^{\rm can} := \frac{8\pi^2 \langle g, g\rangle_\Gamma}{\eta_g},$$

where $\langle g, g\rangle_\Gamma = \int_{\Gamma_1(N_g)\backslash\mathfrak{H}} g(z)\overline{g(z)}\frac{dxdy}{y^2}$ is the Petersson norm of g, and η_g is congruence number of g relative to $S_2(\Gamma_1(N_g))$ (see [Hid81, §7]).

We are interested in the case where $g = \theta_\psi$ is the theta series of a Hecke character ψ of an imaginary quadratic field K satisfying (spl). Suppose L is a number field containing the Fourier coefficients of g, and let Φ be the completion of L at a fixed prime above p, with ring of integers $\mathscr{O}$.

From now on, we also suppose $p > 3$.

Proposition 6.1.2. *Let ψ be a Hecke character of K of infinity type $(-1, 0)$, put $g = \theta_\psi$, and suppose $\psi^- := \psi^\tau/\psi$ satisfies conditions (S5)–(S9) of Definition 5.1.1. Then*

$$\Omega_g^{\rm can} = \Omega^2$$

up to a unit in $\mathcal{O}_L^\times$.

Proof. Let $\boldsymbol{g}$ be the Hida family passing through the ordinary p-stabilization

$$g_\alpha := g(q) - \varepsilon_g(p)p\psi(\overline{v})^{-1}g(q^p),$$

and let $\eta_{\boldsymbol{g}}$ be its associated congruence power series, normalized so that its specialization at the trivial character gives the congruence number η_{g_α} of g_α. Under conditions (S5)–(S8) of Definition 5.1.1, it follows from the proof of the anticyclotomic Iwasawa main conjecture for CM fields by Hida–Tilouine [HT93, HT94] and Hida [Hid06] that

$$\eta_{\boldsymbol{g}} = h_K \cdot \mathscr{L}_v(\psi^-\mathbf{N}^{-1}) \tag{6.1}$$

up to a p-adic unit, where $h_K = \#\mathrm{Pic}(\mathcal{O}_K)$ is the class number of K. (Note that in (6.1) we have used the functional equation of Theorem 2.2.1 to write $\mathscr{L}_v(\psi^-\mathbf{N}^{-1})$ in place of $\mathscr{L}_v(\psi^-)$.)

Having infinity type $(2, 0)$, the character $\psi^-\mathbf{N}^{-1}$ is in the range of interpolation of the Katz v-adic L-function of Theorem 2.2.1, and from there we obtain

$$h_K \cdot \frac{\mathscr{L}_v(\psi^-\mathbf{N}^{-1})(\mathbb{1})}{\Omega_p^2} = \mathcal{E}_p(\psi^-) \cdot h_K \cdot \frac{L(\psi^-, 1)}{\Omega^2}, \tag{6.2}$$

where $\mathcal{E}_p(\psi^-) := (1-\psi^-(v))(1-\psi^-(v)p^{-1})$. By Hida's formula (see [HT93, Thm. 7.1]) and Dirichlet's class number formula, we have

$$\langle g, g\rangle_\Gamma = \frac{D_K^2}{2^3\pi^2} \cdot \frac{h_K}{w_K\sqrt{D_K}} \cdot L(\psi^-, 1),$$

where $w_K = \#\mathcal{O}_K^\times$, and so together with (6.1), equality (6.2) can be rewritten as the equality

$$\frac{\eta_{g_\alpha}}{\Omega_p^2} \cdot u = \mathcal{E}_p(\psi^-) \cdot \frac{8\pi^2\langle g, g\rangle_\Gamma}{\Omega^2} \tag{6.3}$$

for some $u \in \mathcal{O}_{\mathbb{C}_p}^\times$. Since the right-hand side of (6.2) is in $L^\times$ as a consequence of the proof of Deligne's conjecture for Hecke characters (see [GS81] and [Bla86]), it follows that $u/\Omega_p^2 \in \mathcal{O}_L^\times$, and hence (6.3) together with the well-known relation $\eta_{g_\alpha} = \mathcal{E}_p(\psi^-)\eta_g$ (following from [KLZ17, Prop. 7.3.1]) yields the result. $\square$

Using Proposition 6.1.2, we can now rewrite the general Gross–Zagier formula for modular curves in a form that will be convenient for us.

Let $B_{g,\chi}/K$ be the abelian variety (up to K-isogeny) associated to (g,χ). Letting B_ψ/K be a CM abelian variety in the isogeny class of such abelian varieties associated to ψ by Casselman's theorem (see [Shi71, Thm. 6]), it can be described as the Serre tensor $B_{g,\chi} := B_\psi \otimes \chi$, and satisfies

$$L(B_{g,\chi}/K, s) = \prod_{\sigma : L \hookrightarrow \mathbb{C}} L(g^\sigma/K, \chi^\sigma, s).$$

After possibly changing $B_{g,\chi}$ it within its K-isogeny class, we have an embedding $\mathcal{O}_L \hookrightarrow \mathrm{End}_K(B_{g,\chi})$.

The Néron–Tate height pairing gives a $\mathbb{Q}$-bilinear non-degenerate pairing

$$\langle \cdot, \cdot \rangle_{\mathrm{NT}} : B_{g,\chi}(K)_{\mathbb{Q}} \times B_{g,\chi}^\vee(K)_{\mathbb{Q}} \to \mathbb{R},$$

where we put $(\cdot)_{\mathbb{Q}} = (\cdot) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $B_{g,\chi}^\vee/K$ is the dual abelian variety. As explained in [YZZ13, §1.2.4], $\langle \cdot, \cdot \rangle_{\mathrm{NT}}$ induces an L-linear pairing

$$\langle \cdot, \cdot \rangle_L : B_{g,\chi}(K)_{\mathbb{Q}} \otimes_L B_{g,\chi}^\vee(K)_{\mathbb{Q}} \to L_{\mathbb{R}} := L \otimes_{\mathbb{Q}} \mathbb{R}. \tag{6.4}$$

In the next result we view $L(B_{g,\chi}/K, s)$ as valued in $L_{\mathbb{C}} := L \otimes_{\mathbb{Q}} \mathbb{C}$.

Theorem 6.1.3. *Let ψ be as in Proposition 6.1.2, put $g = \theta_\psi$, and let χ be a finite order character such that $(g,\chi) \in S_2(\Gamma_0(N_g), \varepsilon_g) \times \Sigma_{\mathrm{cc}}(c, \mathfrak{N}_g, \varepsilon_g)$ for some positive integer c prime to N_g. If $p \nmid h_K$, there exist Heegner points $y_{g,\chi} \in B_{g,\chi}(K)$ and $y'_{g,\chi} \in B_{g,\chi}^\vee(K)$ such that*

$$\frac{L'(B_{g,\chi}/K, 1)}{\Omega_g^{\mathrm{can}}} = \langle y_{g,\chi}, y'_{g,\chi} \rangle_L,$$

where the equality is up to a p-adic unit.

Proof. The general Gross–Zagier formula of [YZZ13], as made explicit in [CST14, Thm. 1.5], reads

$$L'(B_{g,\chi}/K, 1) = \frac{8\pi^2 \langle g, g \rangle_\Gamma}{u_c^2 \cdot \sqrt{D_K} \cdot c} \cdot \frac{\langle y_{g,\chi}, y'_{g,\chi} \rangle_L}{\deg(\pi_g)},$$

where $u_c := \frac{1}{2}\#\mathcal{O}_c^\times$ and $\pi_g : X_1(N_g) \to A_g$ is an optimal quotient. Hence it suffices to show that the congruence number η_g and $\deg(\pi_g)$ differ by a p-adic unit. As shown in the proof of Proposition 6.1.2, we have the equality up to a p-adic unit

$$\mathcal{E}_p(\psi^-) \cdot \eta_g = h_K \cdot \mathscr{L}_v(\psi^- \mathbf{N}^{-1})(\mathbb{1}).$$

In particular, if $p \nmid h_K$, by (S9) in Definition 5.1.1 (which ψ^- is assumed to satisfy) and Remark 5.1.2, this shows that η_g is a p-adic unit. Since [ARS12, Thm. 3.6(a)] shows that $\deg(\pi_g)$ divides η_g, the result follows. $\square$

Remark 6.1.4. Let $\mathfrak{P}$ be a prime of L above p and denote by $\mathscr{O}$ the ring of integers of the completion of L at $\mathfrak{P}$. Then $T_{g,\chi} \simeq \varprojlim_m B_{g,\chi}[\mathfrak{P}^m]$ as G_K-modules, and the Heegner point $y_{g,\chi}$ of Theorem 6.1.3 can be taken so that its image under the Kummer map

$$B_{g,\chi} \otimes_{\mathcal{O}_L} \mathscr{O} \to \varprojlim_m \mathrm{Sel}_{\mathfrak{P}^m}(B_{g,\chi}/K) \simeq \mathrm{Sel}_{\mathrm{BK}}(K, T_{g,\chi})$$

agrees with the Heegner class $z_{g,\chi}$ of Theorem 3.2.1. and $y'_{g,\chi}$ so that its Kummer image agrees with image of $z_{g,\chi}$ under the isomorphism $\mathrm{Sel}_{\mathrm{BK}}(K, T_{g,\chi}) \simeq \mathrm{Sel}_{\mathrm{BK}}(K, T_{g,\chi}^\tau)$ given by the action of complex conjugation.

6.2. **Theorem B implies Theorem A.** As is well-known (see e.g. [GS81, Thm. 4.1]), the assumption that the field extension $F(E_{\rm tors})/K$ is abelian implies that the Weil restriction $B := {\rm Res}_{F/K}(E)$ is an abelian variety with complex multiplication by an order in a product of CM fields

$$L = L_1 \times \cdots \times L_r$$

containing K with $[L:K] = \sum_{i=1}^r [L_i:K] = \dim(B)$. Moreover, for each i there is an abelian variety B_i/K with CM by an order in L_i which combine to an isogeny

$$B \to \prod_{i=1}^r B_i$$

defined over K. By invariance of the Birch Swinnerton-Dyer conjecture under isogenies and restriction of scalars, Theorem A thus follows from Theorem B applied to each of the isogeny factors B_i.

6.3. **Proof of Theorem B.**

Proof of Theorem B(i). Let $\mathcal{L}$ be a number field containing the values of ψ and χ (so $\mathcal{L}$ contains the field L of values of $\lambda = \psi\chi$). Since ${\rm ord}_{s=1}L(\lambda,s) = 1$, $\lambda\mathbf{N}^{-1}$ has root number $w(\lambda\mathbf{N}^{-1}) = -1$, and by Lemma 5.1.3 we can fix a good pair (ψ,χ) for λ, and let $B_{g,\chi}/K$ be the associated abelian variety. By Theorem 6.1.3, the nonvanishing of $L'(g/K,\chi,1)$ also gives $y_{g,\chi} \notin B_{g,\chi}(K)_{\rm tors}$, and so

$${\rm rank}_{\mathcal{O}_\mathcal{L}} B_{g,\chi}(K) \geq 1.$$

On the other hand, let $\mathfrak{P}$ be a prime of $\mathcal{L}$ above p, and denote by $\mathscr{O}_\mathfrak{P}$ and $T_{\lambda,\mathfrak{P}}(1)$ the completion of $\mathcal{O}_\mathcal{L}$ at $\mathfrak{P}$ and the associated $\mathscr{O}_\mathfrak{P}$-module of rank 1 with G_K-action via the $\mathfrak{P}$-adic avatar of λ, and define $T_{\psi^\tau\chi,\mathfrak{P}}(1)$, $T_{g,\chi,\mathfrak{P}}$, $W_{\psi^\tau\chi,\mathfrak{P}}(1)$, etc. similarly. By Theorem 5.2.1 and Remark 5.2.2, the assumption that ${\rm ord}_{s=1}L(\lambda,s) = 1$ implies that ${\rm rank}_{\mathscr{O}_\mathfrak{P}}{\rm Sel}_{\rm BK}(K,T_{\lambda,\mathfrak{P}}) = 1$; since by Theorem 4.3.6 the nonvanishing of $L(\psi^\tau\chi,1)$ gives $\#{\rm Sel}_{\rm BK}(K,T_{\psi^\tau\chi,\mathfrak{P}}(1)) < \infty$, by the decomposition

$${\rm Sel}_{\rm BK}(K,T_{g,\chi,\mathfrak{P}}) \simeq {\rm Sel}_{\rm BK}(K,T_{\lambda,\mathfrak{P}}(1)) \oplus {\rm Sel}_{\rm BK}(K,T_{\psi^\tau\chi,\mathfrak{P}}(1))$$

it follows that ${\rm rank}_{\mathscr{O}_\mathfrak{P}}{\rm Sel}_{\rm BK}(K,T_{g,\chi,\mathfrak{P}}) = 1$, and hence

$${\rm rank}_{\mathcal{O}_\mathcal{L}} B_{g,\chi}(K) \leq 1.$$

Thus ${\rm rank}_{\mathcal{O}_\mathcal{L}} B_{g,\chi}(K) = 1$ and $\#\text{Ш}(B_{g,\chi}/K)[\mathfrak{P}^\infty] < \infty$; in particular, ${\rm rank}_\mathbb{Z} B_{g,\chi}(K) = [\mathcal{L}:\mathbb{Q}]$. Since up to isogeny the CM abelian variety A is the CM abelian variety B_λ attached to K by Casselman's theorem, there is an isogeny defined over K

$$i_\lambda : B_{g,\chi} \to A \otimes_{\mathcal{O}_L} \mathcal{O}_\mathcal{L}$$

compatible with the action of $\mathcal{O}_\mathcal{L}$ by endomorphisms on both sides (cf. [BDP12, Lem. 2.9]), so the above shows that

$${\rm rank}_\mathbb{Z} A(K) = [L:\mathbb{Q}].$$

Since $L(A/K,s) = \prod_{\sigma:L\hookrightarrow\mathbb{C}} L(\lambda^\sigma,s)$, this gives

$${\rm ord}_{s=1}L(A/K,s) = {\rm rank}_\mathbb{Z} A(K).$$

Moreover, since we have also shown that

$${\rm rank}_{\mathscr{O}_\mathfrak{P}}{\rm Sel}_{\rm BK}(K,T_{\lambda,\mathfrak{P}}(1)) = {\rm rank}_{\mathscr{O}_\mathfrak{P}}{\rm Sel}_{\rm BK}(K,T_{g,\chi,\mathfrak{P}}) = 1,$$

the conclusion $\#\text{Ш}(A/K)[\mathfrak{P}^\infty] < \infty$ also follows. □

Proof of Theorem B(ii). Put $\mathcal{L}_\mathbb{C} = \mathcal{L} \otimes_\mathbb{Q} \mathbb{C} \simeq \prod_\sigma \mathbb{C}$, where σ runs over all field embeddings $\mathcal{L} \hookrightarrow \mathbb{C}$, and write the $\mathcal{L}$-linear pairing $\langle\cdot,\cdot\rangle_\mathcal{L}$ in (6.4) as $(\langle\cdot,\cdot\rangle_{\mathcal{L},\sigma})_\sigma$ according to this decomposition, and write the associated regulator ${\rm Reg}(B_{g,\chi})$ as $({\rm Reg}^\sigma(B_{g,\chi}))_\sigma$. Similar remarks apply to the L-linear pairing

$$\langle\cdot,\cdot\rangle_L : A(K)_\mathbb{Q} \otimes_L A^\vee(K)_\mathbb{Q} \to L_\mathbb{R}.$$

Then by Theorem 6.1.3 we have

$$\left(\frac{L'(g^\sigma/K,\chi^\sigma,1)}{\Omega^{\mathrm{can}}_{g^\sigma}\cdot\langle y_{g,\chi},y'_{g,\chi}\rangle_{\mathcal{L},\sigma}}\right)_\sigma \in \mathcal{L}^\times \subset \mathcal{L}^\times_{\mathbb{C}}. \tag{6.5}$$

Recall the K-isogeny $i_\lambda : B_{g,\chi}\to A\otimes_{\mathcal{O}_L}\mathcal{O}_{\mathcal{L}}$. Put

$$y_\lambda := i_\lambda(y_{g,\chi}) \in A(K),$$

and define $y'_\lambda \in A^\vee(K)$ by $i^\vee_\lambda(y'_\lambda) = y'_{g,\chi}$. Then from the factorization $L(g/K,\chi,s) = L(\lambda,s)L(\psi^\tau\chi,s)$, Proposition 6.1.2, and the projection formula for heights (see [MT83, (3.4.3)]), we have

$$(6.5) = \left(\frac{L'(\lambda^\sigma,1)}{\Omega_\sigma\cdot\langle y_\lambda,y'_\lambda\rangle_{L,\sigma}}\cdot\frac{L((\psi^\tau\chi)^\sigma,1)}{\Omega_\sigma}\right)_\sigma \tag{6.6}$$

up to multiplication by an element in $\mathcal{L}^\times$. Combining the first assertion of Theorem 5.3.2, (6.5) and (6.6), we conclude that

$$\left(\frac{L'(\lambda^\sigma,1)}{\Omega_\sigma\cdot\langle y_\lambda,y'_\lambda\rangle_{L,\sigma}}\right)_\sigma \in L^\times \subset L^\times_{\mathbb{C}}.$$

Since $\mathrm{rank}_{\mathcal{O}_L}A(K) = 1$ by part (i) and $y_\lambda\notin A(K)_{\mathrm{tors}}$ (given that $y_{g,\chi}\notin B_{g,\chi}(K)_{\mathrm{tors}}$ by Theorem 6.1.3 and the fact that $\mathrm{ord}_{s=1}L(g/K,\chi,s)=1$), we deduce that

$$\left(\frac{L'(\lambda^\sigma,1)}{\Omega_\sigma\cdot\mathrm{Reg}^\sigma(A)}\right)_\sigma \in L^\times \subset L^\times_{\mathbb{C}},$$

which together with the equalities $L(A/K) = \prod_\sigma L(\lambda^\sigma,s)$ and $\Omega(A) = \prod_\sigma \Omega_\sigma$ (see[BF24, Prop. 3.3]) yields the result. $\square$

Proof of Theorem B(iii). Since $\mathrm{rank}_{\mathcal{O}_{\mathcal{L}}}B_{g,\chi}(K) = 1$ and $y_{g,\chi}\notin B_{g,\chi}(K)_{\mathrm{tors}}$, the definition of $\mathrm{Reg}(B_{g,\chi})$ gives

$$\langle y_{g,\chi},y'_{g,\chi}\rangle_{\mathcal{L}} = \mathrm{Reg}(B_{g,\chi})\cdot[B_{g,\chi}(K):\mathcal{O}_{\mathcal{L}}\cdot y_{g,\chi}]\cdot[B^\vee_{g,\chi}(K):\mathcal{O}_{\mathcal{L}}\cdot y'_{g,\chi}].$$

Thus letting $\mathfrak{P}$ be any prime of $\mathcal{L}$ above p, by Theorem 6.1.3 we have

$$\mathrm{ord}_{\mathfrak{P}}\left(\frac{L'(g/K,\chi,1)}{\Omega^{\mathrm{can}}_g\cdot\mathrm{Reg}(B_{g,\chi})}\right) = \mathrm{ord}_{\mathfrak{P}}\big([B_{g,\chi}(K)\otimes_{\mathcal{O}_{\mathcal{L}}}\mathscr{O}_{\mathfrak{P}}:\mathscr{O}_{\mathfrak{P}}\cdot y_{g,\chi}]\cdot[B^\vee_{g,\chi}(K)\otimes_{\mathcal{O}_{\mathcal{L}}}\mathscr{O}_{\mathfrak{P}}:\mathscr{O}_{\mathfrak{P}}\cdot y'_{g,\chi}]\big). \tag{6.7}$$

Since $\#\text{Ш}(B_{g,\chi}/K)[\mathfrak{P}^\infty]<\infty$ as shown in the proof of part (i), the Kummer map gives isomorphisms

$$B_{g,\chi}(K)\otimes_{\mathcal{O}_{\mathcal{L}}}\mathscr{O}_{\mathfrak{P}}\to\mathrm{Sel}_{\mathrm{BK}}(K,T_{g,\chi,\mathfrak{P}}(1)),\quad B^\vee_{g,\chi}(K)\otimes_{\mathcal{O}_{\mathcal{L}}}\mathscr{O}_{\mathfrak{P}}\to\mathrm{Sel}_{\mathrm{BK}}(K,T^\tau_{g,\chi,\mathfrak{P}}(1)).$$

By the isomorphism $\mathrm{Sel}_{\mathrm{BK}}(K,T^\tau_{g,\chi,\mathfrak{P}}(1))\simeq\mathrm{Sel}_{\mathrm{BK}}(K,T_{g,\chi,\mathfrak{P}}(1))$ given by the action of complex conjugation, (6.7) can therefore be rewritten as

$$\mathrm{ord}_{\mathfrak{P}}\left(\frac{L'(g/K,\chi,1)}{\Omega^{\mathrm{can}}_g\cdot\mathrm{Reg}(B_{g,\chi})}\right) = 2\cdot\mathrm{length}_{\mathscr{O}_{\mathfrak{P}}}\big(\mathrm{Sel}_{\mathrm{BK}}(K,T_{g,\chi,\mathfrak{P}}(1))/\mathscr{O}_{\mathfrak{P}}\cdot z_{g,\chi}\big),$$

which by Theorem C becomes the equality

$$\mathrm{ord}_{\mathfrak{P}}\left(\frac{L'(g/K,\chi,1)}{\Omega^{\mathrm{can}}_g\cdot\mathrm{Reg}(B_{g,\chi})}\right) = \mathrm{ord}_{\mathfrak{P}}\left(\frac{\#\text{Ш}_{\mathrm{BK}}(W_{g,\chi,\mathfrak{P}}(1)/K)}{\#\mathrm{H}^0(K,W_{g,\chi,\mathfrak{P}}(1))^2}\right)+\sum_{w\in\Sigma,w\nmid p}\mathrm{ord}_{\mathfrak{P}}(c_w(W_{g,\chi,\mathfrak{P}}(1)/K)). \tag{6.8}$$

On the other hand, in the same manner as in the proof of part (ii), from the factorization $L(g/K,\chi,s) = L(\lambda,s)L(\psi^\tau\chi,s)$ and Proposition 6.1.2, and using Theorem 5.3.2 for the second equality, we obtain

$$\begin{aligned}\mathrm{ord}_{\mathfrak{P}}\left(\frac{L'(g/K,\chi,1)}{\Omega_g^{\mathrm{can}}\cdot\mathrm{Reg}(B_{g,\chi})}\right) &= \mathrm{ord}_{\mathfrak{P}}\left(\frac{L'(\lambda,1)}{\Omega\cdot\mathrm{Reg}(A)}\right) + \mathrm{ord}_{\mathfrak{P}}\left(\frac{L(\psi^\tau\chi,1)}{\Omega}\right)\\ &= \mathrm{ord}_{\mathfrak{P}}\left(\frac{L'(\lambda,1)}{\Omega\cdot\mathrm{Reg}(A)}\right)\\ &\quad + \mathrm{ord}_{\mathfrak{P}}\left(\frac{\#Ш_{\mathrm{BK}}(W_{\psi^\tau\chi,\mathfrak{P}}(1)/K)}{\#\mathrm{H}^0(K,W_{\psi^\tau\chi,\mathfrak{P}}(1))^2}\right) + \sum_{w\in\Sigma, w\nmid p}\mathrm{ord}_{\mathfrak{P}}(c_w(W_{\psi^\tau\chi,\mathfrak{P}}(1)/K)).\end{aligned} \tag{6.9}$$

Combining (6.8) and (6.9) and using the decomposition $W_{g,\chi}\simeq W_\lambda(1)\oplus W_{\psi^\tau\chi}(1)$ we thus arrive at

$$\mathrm{ord}_{\mathfrak{P}}\left(\frac{L'(\lambda,1)}{\Omega\cdot\mathrm{Reg}(A)}\right) = \mathrm{ord}_{\mathfrak{P}}\left(\frac{\#Ш_{\mathrm{BK}}(W_{\lambda,\mathfrak{P}}(1)/K)}{\#\mathrm{H}^0(K,W_{\lambda,\mathfrak{P}}(1))^2}\right) + \sum_{w\in\Sigma, w\nmid p}\mathrm{ord}_{\mathfrak{P}}(c_w(W_{\lambda,\mathfrak{P}}(1)/K)). \tag{6.10}$$

Noting that the right-hand side of (6.10) can be rewritten as

$$\mathrm{ord}_{\mathfrak{P}}\left(\frac{\#Ш(A/K)[\mathfrak{P}^\infty]}{\#A(K)_{\mathrm{tors}}\cdot\#A^\vee(K)_{\mathrm{tors}}}\right) + \sum_{w\in\Sigma, w\nmid p}\mathrm{ord}_{\mathfrak{P}}(\mathrm{Tam}(A/K)),$$

this concludes the proof of Theorem B. □

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