Congruence ideals associated to Yoshida lifts

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Seminar on Geometry and Arithmetic

December 2020

*joint work with Ming-Lun Hsieh

Herbrand–Ribet theorem

- Let p denote an odd prime.
- Let A denote the p-primary part of the class group of $\mathbb{Q}(\zeta_p)$.

$$\omega : \mathrm{Gal}\left(\mathbb{Q}(\zeta_p)/\mathbb{Q}\right) \cong (\mathbb{Z}/p\mathbb{Z})^\times \hookrightarrow \mathbb{Z}_p^\times.$$

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Theorem (Herbrand–Ribet)

Let k be an even integer between 2 and $p-1$. The following statements are equivalent.

1. *p divides the numerator of $\zeta(1-k)$.*
2. *$A_{p-k} \neq 0$.*

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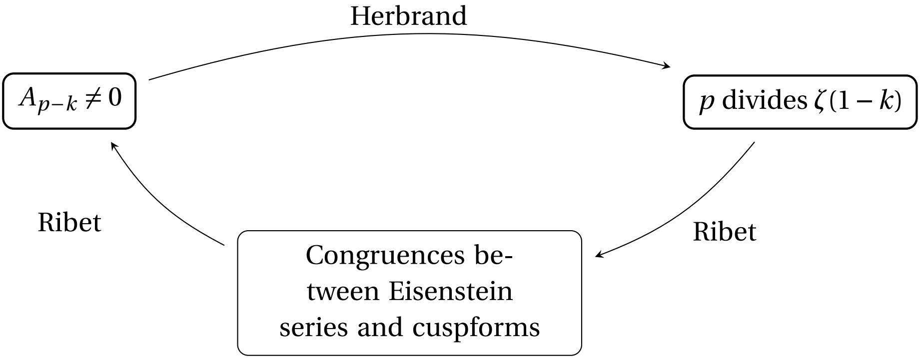

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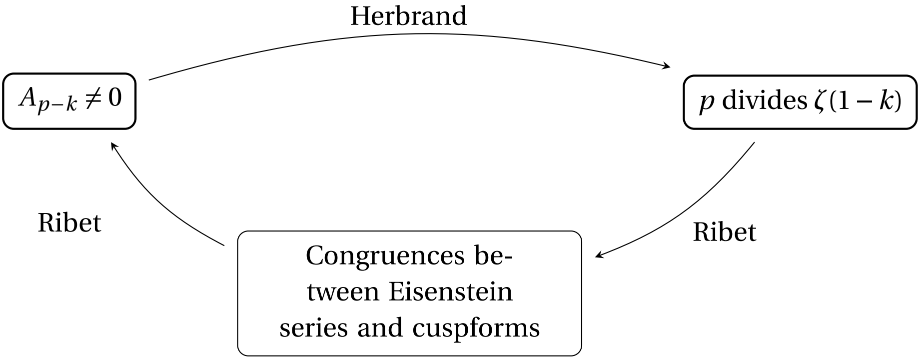

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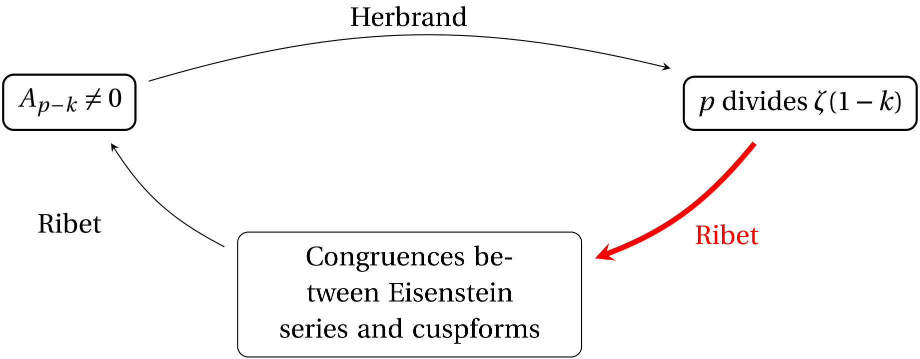

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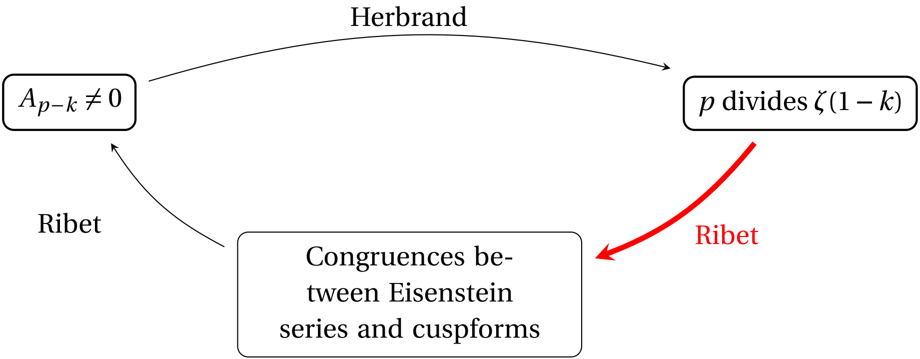
If p divides $\zeta(1-k)$, then there exists a cuspform $f_k \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ such that

$$E_k \equiv f_k \mod p.$$

Here,

$$E_k(q) = \frac{\zeta(1-k)}{2} + q + \sum_{n\geq 2} \sigma_{k-1}(n) q^n.$$



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For example, take $p = 691$ and $k = 12$.

$$\zeta(1-12) = \frac{691}{32760}.$$

Then, $A_{691-12} \cong \mathbb{Z}/691\mathbb{Z}$.

The q-expansion $E_{12}(q)$

$\frac{691}{65520} + q + 2049q^2 + 177148q^3 + 4196353q^4 + 48828126q^5 + O(q^6)$.

The q-expansion of $f_{12}(q)$:

$0 + q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + O(q^6)$.

Their difference divided by 691:

$\frac{1}{65520} + 3q^2 + 256q^3 + 6075q^4 + 70656q^5 + O(q^6)$.

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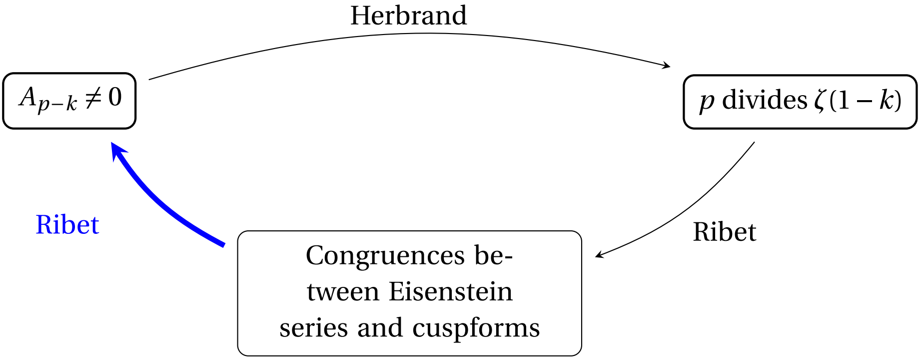
In this talk, we will focus on a generalization of …



Quick sketch of the remaining steps in Ribet's converse

1. Use class field theory.
   - Interpret non-vanishing of A_{p-k} in terms of non-vanishing of a Galois cohomology group.

$$\ker\left(H^1\left(\mathbb{Q}_S/\mathbb{Q}, \mathbb{F}_p(\omega^{p-k})\right) \to H^1\left(I_p, \mathbb{F}_p(\omega^{p-k})\right)\right).$$

2. Interpret the first Galois cohomology group in terms of $\mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$-equivariant extensions:

$$0 \longrightarrow \mathbb{F}_p \longrightarrow E \longrightarrow \mathbb{F}_p(\omega^{p-k}) \longrightarrow 0,$$

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Quick sketch: How to produce such an extension?

3. (Deligne) There exists a 2-dimensional $\mathbb{Q}_p$-vector space with a $\mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$-action with the property that

$$\mathrm{Trace}(\mathrm{Frob}_l) = a_l(f_k), \qquad \forall\ l \neq p.$$

4. One can choose Galois stable $\mathbb{Z}_p$-lattices inside this $\mathbb{Q}_p$-vector space, but there need not be a unique choice.

5. Ribet showed that one can choose such a lattice so that its mod-p reduction is a non-split extension of the form

$$0 \longrightarrow \mathbb{F}_p \longrightarrow E \longrightarrow \mathbb{F}_p(\omega^{p-k}) \longrightarrow 0.$$

6. Using a theorem of (Deligne–Rapoport) Mazur–Wiles, one can show that that there exists a unramified-at-p-quotient isomorphic to $\mathbb{F}_p$.

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Formalizing congruences (Hida, Doi–Hida, Ribet)

- T: the local component of the Hecke algebra, corresponding to E_k, acting on $M_k(\mathrm{SL}_2(\mathbb{Z}))$.

$$\begin{aligned} \phi_{E_k} : \mathrm{T} &\twoheadrightarrow \mathbb{Z}_p, \\ T_l &\to 1 + l^{k-1}. \end{aligned}$$

- $\mathrm{T}_{\mathrm{cusp}}$: the max. quotient of T acting faithfully on $S_k(\mathrm{SL}_2(\mathbb{Z}))$.

$$\mathrm{T} \twoheadrightarrow \mathrm{T}_{\mathrm{cusp}}$$

- The image of $\ker(\phi_{E_k})$ in $\mathrm{T}_{\mathrm{cusp}}$ is called the Eisenstein ideal.

$$\begin{array}{ccc} \mathrm{T} & \xrightarrow{\ker(\phi_{E_k})} & \mathbb{Z}_p \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{T}_{\mathrm{cusp}} & \longrightarrow & \star \end{array}$$

$$\frac{\mathrm{T}_{\mathrm{cusp}}}{\mathrm{Eis}} \cong \frac{\mathbb{Z}_p}{(\text{cong ideal})}.$$

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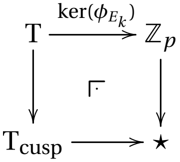

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Classical Iwasawa theory

- In classical Iwasawa theory, we study arithmetic invariants over p-adic families.

$$\mathbb{Q}(\zeta_p) \subset \mathbb{Q}(\zeta_{p^2}) \subset \cdots \subset \mathbb{Q}(\zeta_{p^{n+1}}) \subset \cdots \subset \mathbb{Q}(\zeta_{p^\infty})$$

$$A_{(n)} := p\text{-primary part of the class group of } \mathbb{Q}(\zeta_{p^{n+1}}),$$

$$A_{(n)} = \bigoplus_{i=0}^{p-2} A_{(n),i}.$$

Theorem (Iwasawa, Ferrero–Washington)

Fix $0 \le i \le p-2$. For sufficiently large n, there exist constants λ_i and ν_i such that

$$|A_{(n),i}| = p^{a_{n,i}}, \text{ where } a_{n,i} = \lambda_i n + \nu_i.$$

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- Fix i. Study

$$X_i := \varprojlim A_{(n),i}$$

as a module over $\mathbb{Z}_p[[t]]$.

- Iwasawa proved his growth formula by proving that X_i is a torsion $\mathbb{Z}_p[[t]]$-module.
- To torsion $\mathbb{Z}_p[[t]]$-modules, one can associate a characteristic ideal. If $\mathscr{P} = (\beta)$ is a height one prime ideal, we have

$$(X_i)_{\mathscr{P}} \cong \bigoplus_s \frac{\mathbb{Z}_p[[t]]_{\mathscr{P}}}{(\beta)^{a_s}}. \qquad \mathrm{Char}(X_i) := (\prod_{\mathscr{P}} \prod \beta^{a_s}).$$

Theorem (Mazur–Wiles, Ohta, Rubin)

Fix an odd integer $3 \leq i \leq p-2$.

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Here, θ_i *is the Kubota–Leopoldt* p*-adic* L*-function.*

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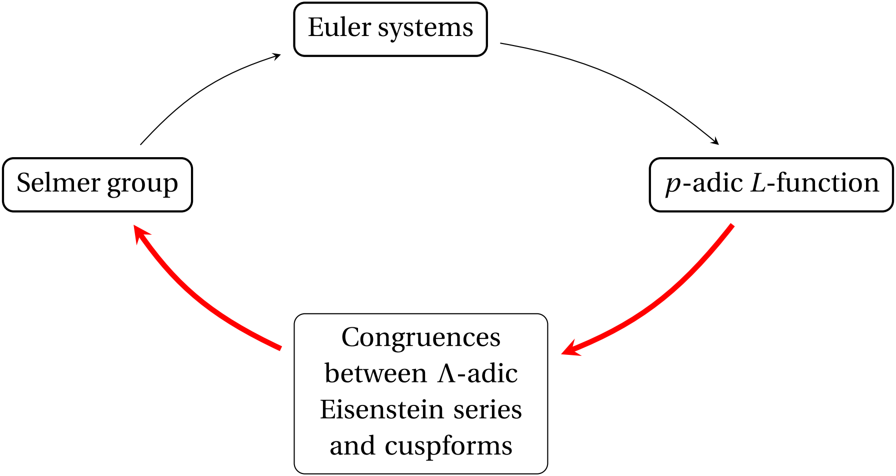

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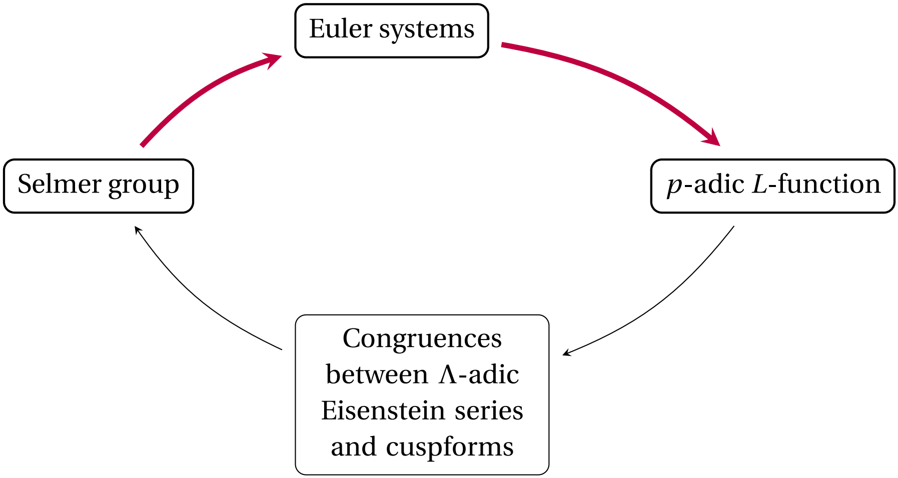

Theorem (Rubin, Kolvyagin, Thaine)

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Iwasawa's (open) question on cyclicity

For each odd integer $3 \le i \le p-2$, is

$$X_i \cong \frac{\mathbb{Z}_p[[t]]}{(\theta_i)}?$$

- Iwasawa proved that this question has an affirmative answer if one assumes the Kummer–Vandiver conjecture.
- Kummer–Vandiver conjecture states that for

$$\lambda_j = \nu_j = 0, \qquad \text{for all even } 0 \le j \le p-3.$$

- Ohta (Kurihara, Harder–Pink) has general results in this direction when the Hecke algebras are Gorenstein.
- Wake has shown pseudo-cyclicity assuming Greenberg's pseudo-nullity conjecture.

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- Greenberg's pseudo-nullity conjecture states that

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Iwasawa's (open) question on cyclicity

For each odd integer $3 \le i \le p-2$, is

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Λ-adic congruence ideals

$$\begin{array}{ccc} \mathbf{T} & \xrightarrow{\phi} & \Lambda \\ \downarrow & \ulcorner & \downarrow \\ \mathbf{T}_{\perp} & \longrightarrow & \star \end{array}$$

$$\frac{\mathbf{T}_{\perp}}{?} \cong \frac{\Lambda}{\text{(cong ideal)}}.$$

- Emerton (level 1) and Ohta (general level) studied the map ϕ associated to the Λ-adic Eisenstein series so that $\perp =$ cusp and $? =$ Eis.

- Hida–Tilouine have studied the map ϕ associated to Λ-adic CM forms to prove one divisibility towards anti-cyclotomic main conjectures.

- In our work, we study the congruence ideal associated to the map ϕ coming from a p-adic family (over GSp_4) of Yoshida lifts. In this case, $?$ is called Yoshida ideal (Agarwal–Klosin).

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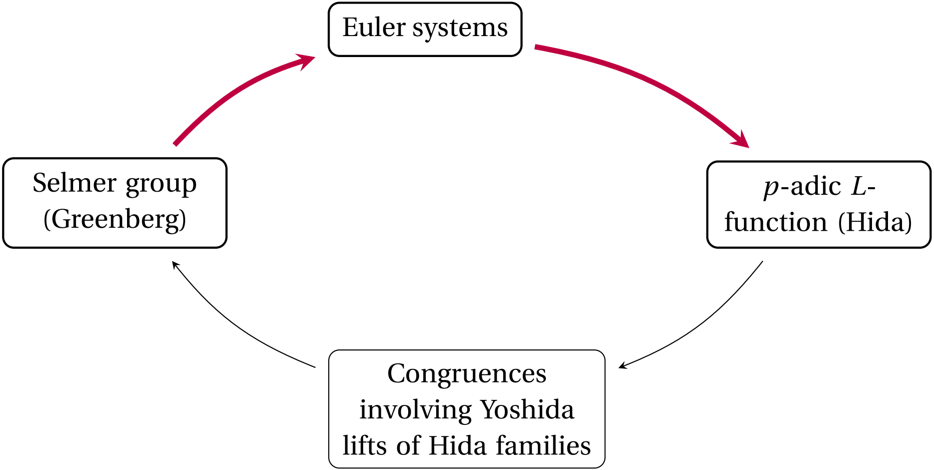

Theorem (Lei–Loeffler–Zerbes, Kings–Loeffler–Zerbes)

$$\mathrm{Char}(\mathrm{Sel}) \supset (\theta).$$

- We will assume throughout that we have the inclusion afforded by Euler systems.

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- The method of congruences introduces a third ideal and leads to the opposite inclusion:

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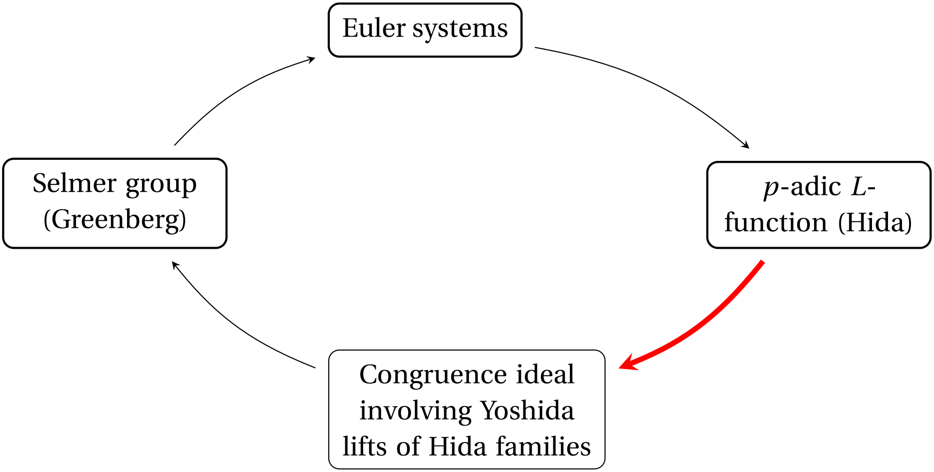
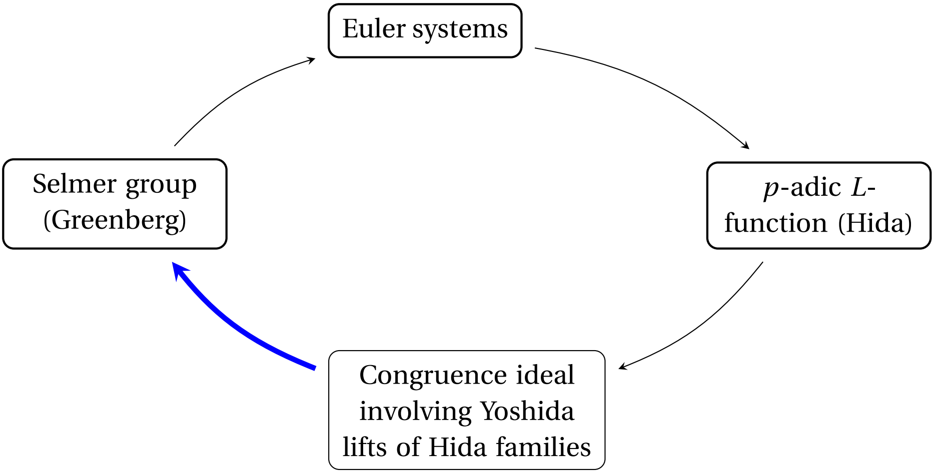


Theorem (ongoing work of Hsieh-Liu)

$$\text{cong ideal} \subset (\theta).$$



What we want to prove is

$$\mathrm{Char}(\mathrm{Sel}) \subset \text{cong ideal}.$$

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Ordinary modular forms

- f_{new} be a cuspidal eigen newform in $S_k(\Gamma_1(N))$ with weight ≥ 2.
- We assume throughout that p does not divide N.
- We will also assume that f_{new} is p-ordinary. That is, the p-adic valuation of $a_p(f_{\text{new}})$ is zero.
- Let f be the ordinary p-stabilization of f_{new}. This is a cuspform in $S_k(\Gamma_1(Np))$.
- f is an eigenform for the Hecke operators T_l and the diamond operators S_l for l not divinding N along with the U_p operator.
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- f determines a ring homomorphism:

$$\begin{aligned} \phi : \mathrm{T} &\to \overline{\mathbb{Z}}_p, \\ T_l &\to a_l(f), \\ S_l &\to l^{k-1}\psi_f(l), \\ U_p &\to a_p(f). \end{aligned}$$

- Abstract Hecke polynomial for a prime l not dividing Np:

$$x^2 - T_l x + S_l = 0.$$

Theorem (Shimura, Deligne)

There exists a Galois representation $\rho_f : \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ *such that*

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Hida families (GL_2)

Suppose F and G are two Hida families passing through two ordinary p-stabilizations f_0 and g_0.

$$\phi : \mathbf{T} \to \mathbf{I} \tag{1}$$

- $\mathbf{T}$, $\mathbf{I}$ are finitely generated over a subring over $\mathbb{Z}_p[[t]]$.
- $\mathbf{T}$: Λ-adic Hecke algebra generated by the Hecke operators T_l and diamond operators S_l for primes l not divinding N and the U_p operator acting on the space of Λ-adic cuspforms.
- $\mathbf{T}$ is reduced.
- $\mathbf{I}$ is the integral closure of $\mathbf{T}/\eta$, for some minimal prime ideal η.
- There exists a dense set of height one prime ideals $\mathfrak{p}_k$ in $\mathbf{T}$ (with $k \geq 2$) containing η and $(1+t)^m - (1+p)^{mp^{k-1}}$, for some $m \geq 1$, such that
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Rankin–Selberg main conjectures

Theorem (Hida)

There exists a continuous Galois representation

$$\rho_F : \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q}) \to \mathrm{GL}_2(\mathrm{Frac}(\mathbf{I}_F))$$

such that specialization of ρ_F at a classical prime is isomorphic to Shimura–Deligne Galois representation.

Suppose we assume the following condition.

(irr) — The residual representations $\overline{\rho}_{f_0}, \overline{\rho}_{g_0}$ are irreducible.

(Nyssen, Rouquier) ρ_F and ρ_G can be chosen to lie inside $\mathrm{GL}_2(\mathbf{I}_F)$ and $\mathrm{GL}_2(\mathbf{I}_G)$ respectively.

We are interested in studying the Iwasawa main conjecture associated to the Galois representation given by the acton of $\mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$ on

$$\mathrm{Hom}(\mathbf{L}_F, \mathbf{L}_G).$$

(mult-free) — $\overline{\rho}_{f_0} \ncong \overline{\rho}_{g_0}$.

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There exists a continuous Galois representation

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p-distinguished condition (for GSp_4)

$$\overline{\rho}_{f_0}|_{G_{\mathbb{Q}_p}} \sim \begin{bmatrix} \chi_{f_0} & * \\ 0 & \epsilon_{f_0} \end{bmatrix},$$

$$\overline{\rho}_{g_0}|_{G_{\mathbb{Q}_p}} \sim \begin{bmatrix} \chi_{g_0} & * \\ 0 & \epsilon_{g_0} \end{bmatrix}.$$

The four characters

$$\chi_{f_0}, \quad \chi_{g_0}, \quad \epsilon_{f_0}, \quad \epsilon_{g_0}$$

are all distinct.

Warm up question

Suppose you have a Hida family F passing through a CM form f with weight ≥ 2. Are all the classical specializations (with weight ≥ 2) of the Hida family F CM forms?

- Answer is Yes.
- This is due to Hida.
  - Hida proved that there exists a unique Hida family passing through classical points with weight ≥ 2.
  - Hida explicitly constructed a CM Hida family.
- Challenge/Puzzle: How does one answer the above question by simply considering the Galois representation ρ_F?
  - If the Galois representation ρ_f, associated to a classical specialization of weight ≥ 2, induced from a Hecke character of an imaginary quadratic field, then is ρ_F also induced from a Λ-adic Hecke character?

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Suppose you have a Hida family F passing through an Eisenstein series with weight ≥ 2. Are all the classical specializations (with weight ≥ 2) of the Hida family F Eisenstein?

- Answer is Yes (Hida).
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$$\mathrm{GSp}_4 = \left\{ g \in \mathrm{GL}_4,\ \text{such that } g^T \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} g = \lambda(g) \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} \right\}.$$

If $ad - bc = a'd' - b'c'$, then

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Suppose f_0 and g_0 satisfy:

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(weight) — $\mathrm{weight}(f_0) \equiv \mathrm{weight}(g_0) \mod 2(p-1)$.

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- (Laumon, Taylor, Weissauer) One can attach a Galois representation $\mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q}) \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_p)$ to Siegel cuspidal eigen forms over GSp_4 with weight (κ_1, κ_2) with $\kappa_1 \geq \kappa_2 \geq 3$.
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Hida theory for GSp_4 (Pilloni)

- Abstract Hecke polynomial at l:

$$X^4 - T_{l,2}X^3 + l(T_{l,1} + (l^2+1)S_l)X^2 - l^3 T_{l,2}S_l X + l^6 S_l^2. \quad (2)$$

- A Siegel cusp form with weight (κ_1, κ_2) is p-ordinary if the p-adic valuations of the Hecke polynomial at p:

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- The Yoshida lift of f_0 and g_0 turns out to be p-ordinary.
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Hida theory for GSp_4 (Pilloni)

- Abstract Hecke polynomial at l:

$$X^4 - T_{l,2}X^3 + l(T_{l,1} + (l^2+1)S_l)X^2 - l^3 T_{l,2}S_l X + l^6 S_l^2. \tag{2}$$

- A Siegel cusp form with weight (κ_1, κ_2) is p-ordinary if the p-adic valuations of the Hecke polynomial at p:

$$0, \qquad \kappa_2 - 2, \qquad \kappa_1 - 1, \qquad \kappa_1 + \kappa_2 - 3.$$

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Congruence ideals associated to Yoshida lifts

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- For simplicity, assume ϕ is surjective.
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- So far, we only know that it passes through one classical Yoshida lift.
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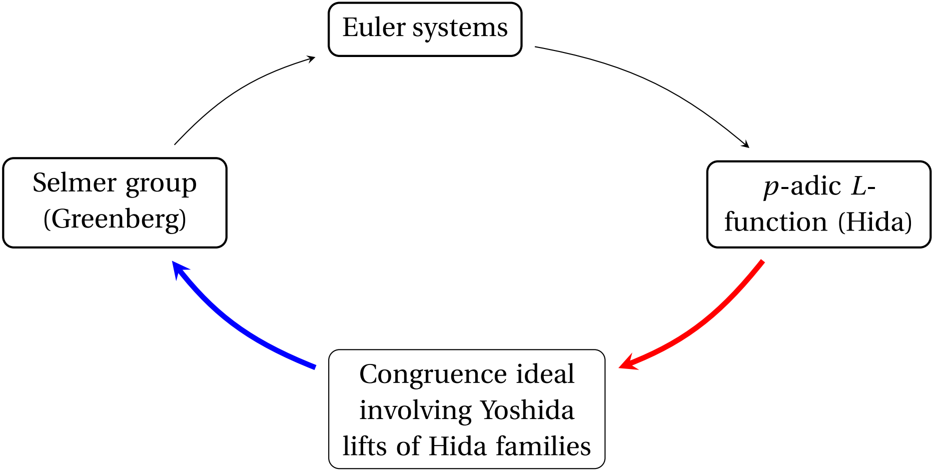
Ongoing work of Hsieh–Liu

One approach: a direct automorphic construction of the desired p-adic family of Siegel forms.



Question

Suppose you have a GSp_4 Hida family passing through the Yoshida lift of f_0 and g_0. Do almost all relevant classical specializations of the Hida family correspond to Yoshida lifts?

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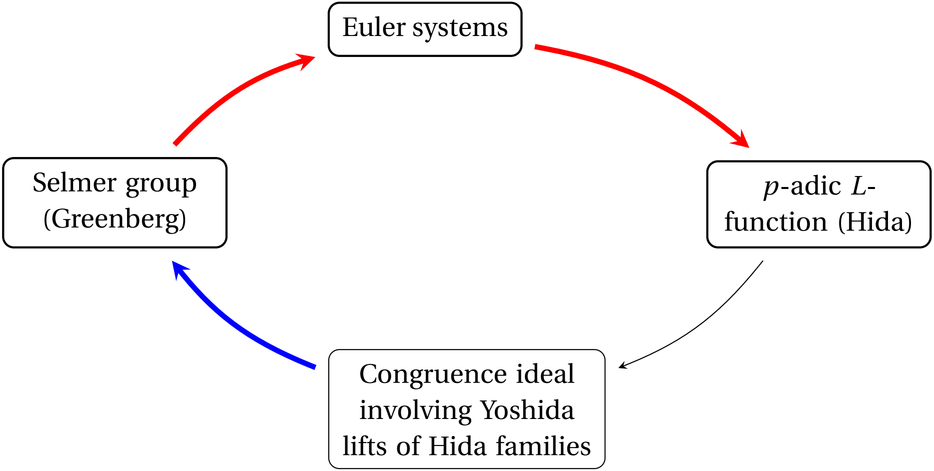

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Main results: p-adic families of Hecke eigensystems

Henceforth, we work under all the assumptions stated above.

Theorem (Hsieh–P.)

- *The p-adic family of Hecke eigensystems*

$$\phi : \mathbb{T} \to \mathbb{I}$$

corresponds to the Yoshida lifts of F and G. That is, for all good primes l,

$$\phi(\text{Hecke poly at } l) = \text{Char poly of } \mathrm{Frob}_l \text{ on } \rho_{F,G}.$$

- *Every irreducible component of $\mathbb{T}_\perp$ does not correspond to a Yoshida lift of Hida families.*

Here, $\rho_{F,G} = \rho_F \oplus \rho_G(\sqrt{\kappa_F/\kappa_G})$.

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Main results towards Rankin–Selberg main conjectures

Theorem (Hsieh–P.)

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An application of studying specialization of Selmer groups then leads to the following result.

Theorem (Hsieh–P.)

Suppose that

$$\text{cong ideal} \subset \theta_{F,G}, \tag{4}$$

$$\theta_{F,G,s} \subset \mathrm{Char}\Big(\mathrm{Sel}_{\mathbf{F},G}(\mathbb{Q}_{\mathrm{cyc}})^\vee\Big). \tag{5}$$

Then, the three-variable and two-variable main conjectures hold.

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Main result towards pseudocyclicity

Theorem (Hsieh–P.)

Suppose that the two algebraic p-adic L-functions

$$\Theta_{\mathbf{F},G,s}, \qquad \Theta_{F,\mathbf{G},s}, \tag{6}$$

have no common irreducible factor.
Then, the $\mathbb{I}[[t]]$-module $\mathrm{Sel}_{\mathbf{F},G}(\mathbb{Q}_{\mathrm{cyc}})^\vee$ is pseudo-cyclic. That is, for all height one primes $\mathscr{P}$,

$$\left(\mathrm{Sel}_{\mathbf{F},G}(\mathbb{Q}_{\mathrm{cyc}})^\vee\right)_{\mathscr{P}} \cong \frac{\mathbb{I}[[t]]_{\mathscr{P}}}{(\Theta_{\mathbf{F},G,s})}.$$

Ingredients in the proof

- Use the theory of pseudo-representations of Bellaïche–Chenevier.

$$\text{globally} \sim \begin{bmatrix} a_{11}(F) & a_{12}(F) & b_{13} & b_{14} \\ a_{21}(F) & a_{22}(F) & b_{23} & b_{24} \\ c_{31} & c_{32} & d_{33}(G) & d_{34}(G) \\ c_{41} & c_{42} & d_{43}(G) & d_{44}(G) \end{bmatrix}$$

- To construct the global extension, we need to use the fact that Yoshida ideal contains the reducibility ideal.
- To show that these desired extensions satisfy the local Selmer condition, we need to use a result of Urban.

$$\text{locally} \sim \begin{bmatrix} \psi_1\chi_p^{\kappa_1+\kappa_2-3} & * & * & * \\ 0 & \psi_2\chi_p^{\kappa_1-1} & * & * \\ 0 & 0 & \psi_3\chi_p^{\kappa_2-2} & * \\ 0 & 0 & 0 & \psi_4 \end{bmatrix}$$

Thank you.