DIAGONAL CYCLES AND ANTICYCLOTOMIC IWASAWA THEORY OF MODULAR FORMS

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Abstract. We construct a new anticyclotomic Euler system (in the sense of Jetchev–Nekovář–Skinner) for the Galois representation $V_{f,\chi}$ attached to a newform f of weight $k \geq 2$ twisted by an anticyclotomic Hecke character χ. We then show some arithmetic applications of the constructed Euler system, including new results on the Bloch–Kato conjecture in ranks zero and one, and a divisibility towards the Iwasawa–Greenberg main conjecture for $V_{f,\chi}$.

In particular, in the case where the base-change of f to our imaginary quadratic field has root number $+1$ and χ has higher weight (which implies that the complex L-function $L(V_{f,\chi}, s)$ vanishes at the center), our results show that the Bloch–Kato Selmer group of $V_{f,\chi}$ is nonzero, and if a certain distinguished class $\kappa_{f,\chi}$ is nonzero, then the Selmer group is one-dimensional. Such applications to the Bloch–Kato conjecture for $V_{f,\chi}$ were left wide open by the earlier approaches using Heegner cycles and/or Beilinson–Flach classes. Our construction is based instead on a generalisation of the Gross–Kudla–Schoen diagonal cycles.

Contents

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Introduction

Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_0(N_f))$ be an elliptic newform of even weight $k \geq 2$, and let $p \nmid 6N_f$ be a prime. Let $K/\mathbf{Q}$ be an imaginary quadratic field in which p splits. Let L be a number field containing K and the Fourier coefficients of f, and let $\mathfrak{P}$ be a prime of L above p at which f is ordinary, i.e. $v_{\mathfrak{P}}(a_p) = 0$. Let χ be an anticyclotomic Hecke character of K, and consider the conjugate self-dual $G_K = \mathrm{Gal}(\overline{\mathbf{Q}}/K)$-representation

$$V_{f,\chi} := V_f^\vee(1 - k/2) \otimes \chi^{-1},$$

where $V_f^\vee$ is the contragredient of Deligne's $\mathfrak{P}$-adic Galois representation associated to f.

We prove, among other results, the following applications to the Bloch–Kato conjecture for $V_{f,\chi}$: Under mild hypothesis on f and χ, the nonvanishing of the Rankin–Selberg L-function $L(f/K, \chi, s)$ at the center $s = k/2$ implies that the dimension of the associated Bloch–Kato Selmer group is 0; and when this central L-value vanishes, the nonvanishing of a distinguished class $\kappa_{f,\chi}$ implies that the dimension of the associated Bloch–Kato Selmer group is 1. These results are applications of the main contribution of this paper, which is the construction of a new anticyclotomic Euler system for $V_{f,\chi}$.

Our construction is based on a generalisation of the diagonal cycles introduced by Gross–Kudla [GK92] and Gross–Schoen [GS95], and studied more recently by Darmon–Rotger and Bertolini–Seveso–Venerucci (see [BDR+22]).

0.1. **Main results.** Assume that the discriminant D_K of K satisfies $(D_K, N_f) = 1$. Write $N_f = N^+N^-$ with N^+ (resp. N^-) divisible only by primes that are split (resp. inert) in K, and assume that

$$N^- \text{ is squarefree.} \tag{0.1}$$

Denote by $\nu(N^-)$ the number of prime factors of N^-, and assume also that

$$\chi \text{ has conductor } c\mathcal{O}_K \text{ with } (c, pN_f) = 1. \tag{0.2}$$

Under hypotheses (0.1) and (0.2), it is known that the sign $\epsilon(f, \chi)$ in the functional equation for $L(f/K, \chi, s)$ (relating its values at s and $k - s$) depends only on the global root number of the base-change of f to K, given by

$$\epsilon(f/K) = -(-1)^{\nu(N^-)},$$

and the infinity type $(-j, j)$ of χ. According to these, the values of $\epsilon(f, \chi)$ are as in the following table[1]:

[1] Because $L(f/K, \chi, s) = L(f/K, \chi^{\mathbf{c}}, s)$, where $\chi^{\mathbf{c}}$ is the composition of χ with the action of complex conjugation, without loss of generality we may assume $j \geq 0$.

	$\epsilon(f/K)=-1$	$\epsilon(f/K)=+1$
$0\leq j<k/2$	-1	$+1$
$j\geq k/2$	$+1$	-1

0.1.1. *The Euler system.* Throughout the remainder of this Introduction, assume that f and K satisfy the following hypotheses:

(h1) f is ordinary and non-Eisenstein at $\mathfrak{P}$;
(h2) $p=\mathfrak{p}\overline{\mathfrak{p}}$ splits in K;
(h3) $p\nmid h_K$, where h_K is the class number of K.

For every positive integer n, let $K[n]$ be the maximal p-subextension of the ring class field of K of conductor n. Denote by $\mathcal{N}$ the set of squarefree products of primes $\mathfrak{l}\subset\mathcal{O}_K$ split in K with $\ell=N_{K/\mathbf{Q}}(\mathfrak{l})$ prime to p.

Theorem A (Theorem 2.3.2)**.** *There exists a family of cohomology classes*

$$z_{f,\chi,\mathfrak{m},r}\in H^1(K[mp^r],T_{f,\chi})$$

indexed by the ideals $\mathfrak{m}\in\mathcal{N}$ *with* $m=N_{K/\mathbf{Q}}(\mathfrak{m})$ *coprime to* p *and* $r\geq 0$*, where* $T_{f,\chi}$ *is a certain* G_K*-stable* $\mathcal{O}$*-lattice inside* $V_{f,\chi}$*, such that*

$$\mathrm{cor}_{K[mp^{r+1}]/K[mp^r]}(z_{f,\chi,\mathfrak{m},r+1})=z_{f,\chi,\mathfrak{m},r}$$

for all $r\geq 0$*, and for every split prime* $\mathfrak{l}$ *of* $\mathcal{O}_K$ *of norm* ℓ *with* $(\ell,mp)=1$ *we have the tame norm relation*

$$\mathrm{cor}_{K[m\ell p^r]/K[mp^r]}(z_{f,\chi,\mathfrak{m}\mathfrak{l},r})=P_{\mathfrak{l}}(\mathrm{Frob}_{\mathfrak{l}})\,z_{f,\chi,\mathfrak{m},r},$$

where $P_{\mathfrak{l}}(X)=\det(1-\mathrm{Frob}_{\mathfrak{l}}X\,|\,V_{f,\chi}^{\vee}(1))$*, and* $\mathrm{Frob}_{\mathfrak{l}}$ *is the geometric Frobenius.*

The system of classes of Theorem A defines an *anticyclotomic Euler system* in the sense of Jetchev–Nekovář–Skinner [JNS] for the conjugate self-dual representation $V_{f,\chi}$. Significantly extending Kolyvagin's methods, the general theory developed in *op. cit.* provides a machinery that bounds Selmer groups for conjugate self-dual representations V from the input of a non-trivial anticyclotomic Euler system for V. The Selmer group being bounded depends on the local condition at p satisfied by the Euler system classes, and by varying certain elements in the construction of $z_{f,\chi,\mathfrak{m},r}$, we produce in fact two different anticyclotomic Euler systems for $V_{f,\chi}$, differing by their local conditions at the primes above p.

To describe this, recall that by $\mathfrak{P}$-ordinarity of f, the Galois representation $V_f^{\vee}$ restricted to a decomposition group $G_{\mathbf{Q}_p}\subset G_{\mathbf{Q}}$ fits into a short exact sequence

$$0\to V_f^{\vee,+}\to V_f^{\vee}\to V_f^{\vee,-}\to 0,$$

where $V_f^{\vee,\pm}\simeq L_{\mathfrak{P}}$, with the $G_{\mathbf{Q}_p}$-action on $V_f^{\vee,-}$ given by the unramified character sending the arithmetic Frobenius Frob_p^{-1} to α_p, the unit root of $x^2-a_px+p^{k-1}$. Put

$$V_{f,\chi}^{\pm}:=V_f^{\vee,\pm}(1-k/2)\otimes\chi^{-1}.$$

Then more generally, we construct:

- An anticyclotomic Euler system $\{z_{f,\chi,\mathfrak{m},r}^{\mathrm{ord},\mathrm{ord}}\}_{\mathfrak{m},r}$ with the local condition at the primes $w|p$ defined by
$$H^1_{\mathrm{ord}}(K[mp^r]_w,V_{f,\chi}):=\ker\bigl(H^1(K[mp^r]_w,V_{f,\chi})\to H^1(K[mp^r]_w,V_{f,\chi}^{-})\bigr).$$
- An anticyclotomic Euler system $\{z_{f,\chi,\mathfrak{m},r}^{\mathrm{rel},\mathrm{str}}\}_{\mathfrak{m},r}$ for the local condition at the primes $w|p$ defined by
$$\begin{cases}H^1(K[mp^r]_w,V_{f,\chi}) & \text{if } w|\mathfrak{p},\\ 0 & \text{if } w|\overline{\mathfrak{p}}.\end{cases}$$

Using the Panchiskin condition, it can be shown that at least one of these classes land in the Bloch–Kato Selmer group $\mathrm{Sel}_{\mathrm{BK}}(K[mp^r],V_{f,\chi})$, namely the class

$$\kappa_{f,\chi,\mathfrak{m},r}:=\begin{cases}z_{f,\chi,\mathfrak{m},r}^{\mathrm{rel},\mathrm{str}} & \text{if } j\geq k/2,\\ z_{f,\chi,\mathfrak{m},r}^{\mathrm{ord},\mathrm{ord}} & \text{if } 0\leq j<k/2.\end{cases}$$

0.1.2. *Applications to the Bloch–Kato conjecture in rank* 1. Put

$$\kappa_{f,\chi} := \mathrm{cor}_{K[1]/K}(\kappa_{f,\chi,(1),0}) \in \mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi})$$

From the general Euler system machinery of [JNS] applied to the construction of Theorem A we deduce in particular the following result. Let $\mathcal{O}$ be the ring of integers of $L_{\mathfrak{P}}$. We say that f has *big image* if for a certain Galois stable $\mathcal{O}$-lattice $T_f^\vee \subset V_f^\vee$, the image of $G_{\mathbf{Q}}$ in $\mathrm{Aut}_{\mathcal{O}}(T_f^\vee)$ contains a conjugate of $\mathrm{SL}_2(\mathbf{Z}_p)$.

Theorem B (Theorem 5.7.1)**.** *Let $f \in S_k(\Gamma_0(N_f))$ be a newform and χ an anticyclotomic Hecke character of K as above of infinity type $(-j, j)$. Assume that*

$$\epsilon(f/K) = +1 \quad \textit{and} \quad j \geq k/2,$$

which implies $L(f/K, \chi, k/2) = 0$. Assume also that $p > k-2$, $\bar{\rho}_f$ is p-distinguished, and f has big image. Then

$$\dim_{L_{\mathfrak{P}}} \mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi}) \geq 1.$$

Moreover, if the class $\kappa_{f,\chi}$ is nonzero, then

$$\mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi}) = L_{\mathfrak{P}} \cdot \kappa_{f,\chi}.$$

By the Gross–Zagier formula for the modified diagonal cycles introduced in [GK92, GS95] (a special case of the arithmetic Gan–Gross–Prasad conjecture for $\mathrm{SO}(3)\times\mathrm{SO}(4)$) proved by Yuan–Zhang–Zhang [YZZ] in certain cases, the non-triviality of $\kappa_{f,\chi}$ is expected to be governed by the nonvanishing of $L'(f/K, \chi, k/2)$, and hence Theorem B provides strong evidence towards the Bloch–Kato conjecture for $V_{f,\chi}$ in analytic rank 1.

Our methods also yield an analogue of Theorem B in the "indefinite case" $\epsilon(f/K) = -1$ and $0 \leq j < k/2$ (indeed, for $N^- = 1$ this follows immediately from Theorem 6.6.1), but in this case such result can also be obtained from the Euler system of (generalised) Heegner cycles [Nek92, CH18a].

0.1.3. *Applications to the Bloch–Kato conjecture in rank* 0. We now turn our attention to the cases where $\epsilon(f, \chi) = +1$, so the central value $L(f/K, \chi, k/2)$ is expected to be generically nonzero. Put

$$\kappa'_{f,\chi,\mathfrak{m},r} := \begin{cases} z_{f,\chi,\mathfrak{m},r}^{\mathrm{ord},\mathrm{ord}} & \text{if } j \geq k/2, \\ z_{f,\chi,\mathfrak{m},r}^{\mathrm{rel},\mathrm{str}} & \text{if } 0 \leq j < k/2, \end{cases}$$

and $\kappa'_{f,\chi} := \mathrm{cor}_{K[1]/K}(\kappa'_{f,\chi,(1),0})$. Building on the reciprocity law for diagonal cycles by Bertolini–Seveso–Venerucci [BSV22], we show that the class $\kappa'_{f,\chi}$ is non-crystalline at p precisely when $L(f/K, \chi, k/2) \neq 0$. Together with the machinery of [JNS] applied to the anticyclotomic Euler system of Theorem A extending $\kappa'_{f,\chi}$, we thus deduce in particular the following cases of the Bloch–Kato conjecture in analytic rank 0.

Theorem C (Theorem 5.5.1)**.** *Let $f \in S_k(\Gamma_0(N_f))$ be a newform and χ an anticyclotomic Hecke character of K as above. Assume that $\epsilon(f/K) = +1$ and $p > k-2$. Then*

$$L(f/K, \chi, k/2) \neq 0 \quad \Longrightarrow \quad \mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi}) = 0,$$

and hence the Bloch–Kato conjecture holds in this case.

Note that the nonvanishing of $L(f/K, \chi, k/2)$ implies that $\epsilon(f, \chi) = +1$, and so without loss of generality, the character χ in Theorem C may be assumed to have infinity type $(-j, j)$ with $0 \leq j < k/2$. Similarly as in the rank 1 case, our methods also yield an analogue of Theorem C in the indefinite case (see Theorem 6.5.1).

Finally, we note that results also include the proof of a divisibility towards the anticyclotomic Iwasawa Main Conjecture for $V_{f,\chi}$, giving in particular a new proof of the main result of [BD05] (see Theorem 5.6.1).

0.2. **Relation to previous works.** Starting with the landmark results by Gross–Zagier and Kolyvagin [GZ86, Kol88] (see also [BD90]), and followed by their vast generalisations by Zhang [Zha97], Tian [Tia03], Nekovář [Nek07], Yuan–Zhang–Zhang [YZZ13] and others, the Euler system of Heegner points and Heegner cycles has been a key ingredient in the study of the arithmetic of $V_{f,\chi}$ under the Heegner hypothesis

$$\epsilon(f/K) = -1.$$

Classical Heegner cycles account for the cases where the anticyclotomic character χ has finite order (i.e., $j = 0$), but using their new variant by Bertolini–Darmon–Prasanna [BDP13], one obtains classes controlling the arithmetic of $\mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi})$ in the following cases:

$$\text{(1}^{\text{st}}\text{ quadrant)} \qquad \epsilon(f/K) = -1, \quad 0 \leq j < k/2.$$

In another major advance, Bertolini–Darmon [BD05] exploited congruences between modular forms on different quaternion algebras and the Cerednik–Drinfeld theory of interchange of invariants to realise the Galois representation (on finite quotients of) $T_{f,\chi}$ in the torsion of the Jacobian of certain Shimura curves. This allowed them to still use the Heegner point construction in situations where $\epsilon(f/K) = +1$. Together with the extension to higher weights by Chida–Hsieh [CH15], these methods yielded a proof of many cases of the Bloch–Kato conjecture in analytic rank 0 when

$$\text{(2}^{\text{nd}}\text{ quadrant)} \qquad \epsilon(f/K) = +1, \quad 0 \leq j < k/2.$$

under a certain "level-raising" hypothesis. More recently, the Euler system of Beilinson–Flach classes constructed by Lei–Loeffler–Zerbes [LLZ14, LLZ15] and Kings–Loeffler–Zerbes [KLZ17, KLZ20] (inspired in part by earlier results of Bertolini–Darmon–Rotger [BDR15a, BDR15b]) provided an alternative approach to similar rank 0 results under some hypotheses (among other applications).

On the other hand, exploiting the variation of (generalised) Heegner cycles in p-adic families, the first author and Hsieh [CH18a, Cas20], and more recently Kobayashi [Kob], obtained results on the Bloch–Kato conjecture for $V_{f,\chi}$ in rank 0 in the cases

$$\text{(3}^{\text{rd}}\text{ quadrant)} \qquad \epsilon(f/K) = -1, \quad j \geq k/2.$$

Contrastingly, in the cases where

$$\text{(4}^{\text{th}}\text{ quadrant)} \qquad \epsilon(f/K) = +1, \quad j \geq k/2,$$

the conjectures of Beilinson–Bloch and Bloch–Kato predict the existence of non-trivial classes in $\mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi})$ coming from geometry (since $\epsilon(f,\chi) = -1$ and therefore $L(f/K,\chi,k/2) = 0$), but the construction of such classes seems to fall outside of all the aforementioned methods.

The anticyclotomic Euler system constructed in this paper allows us to fill this gap, while also providing a new approach to the aforementioned results in other cases:

	$\epsilon(f/K) = -1$	$\epsilon(f/K) = +1$
$0 \leq j < k/2$	1^{st} quadrant [Kol88], [Tia03], [Nek07], etc. Theorem 6.6.1	2^{nd} quadrant [BD05], [CH15], [KLZ17], etc. Theorem 5.5.1
$j \geq k/2$	3^{rd} quadrant [CH18a], [Cas20], [Kob], etc. Theorem 6.5.1	4^{th} quadrant – Theorem 5.7.1

In future work, we intend to generalise our construction to totally real fields, a setting in which one finds even more cases where the arithmetic of Rankin–Selberg convolutions falls outside the scope of Heegner cycles and/or Beilinson–Flach classes.

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Part 1. The Euler system

1. Preliminaries

1.1. **Modular curves and Hecke operators.** We give a precise description of the modular curves and Hecke operators that will appear in our construction. The main references for this section are [Kat04, §2], [BSV22, §2], and [ACR21, §2], where more details can be found.

1.1.1. *Modular curves.* Let M, N, u, v be positive integers such that $M + N \geq 5$. Define $Y(M, N)$ to be the affine modular curve over $\mathbf{Z}[1/MN]$ representing the functor

$$S \mapsto \left\{ \begin{array}{l} \text{isomorphism classes of triples } (E, P, Q) \text{ where } E \text{ is an elliptic curve over } S, \\ P,\ Q \text{ are sections of } E \text{ over } S \text{ such that } M \cdot P = N \cdot Q = 0; \text{ and the map} \\ \mathbf{Z}/M\mathbf{Z} \times \mathbf{Z}/N\mathbf{Z} \to E, \text{ sending } (a, b) \mapsto a \cdot P + b \cdot Q \text{ is injective} \end{array} \right\}$$

for $\mathbf{Z}[1/MN]$-schemes S. More generally, define the affine modular curve $Y(M(u), N(v))$ over $\mathbf{Z}[1/MNuv]$ representing the functor

$$S \mapsto \left\{ \begin{array}{l} \text{isomorphism classes of quintuples } (E, P, Q, C, D) \text{ where } (E, P, Q) \text{ is as above,} \\ P \in C \text{ is a cyclic subgroup of } E \text{ of order } Mu, \\ Q \in D \text{ is a cyclic subgroup of } E \text{ of order } Nv \text{ such that} \\ C \text{ is complementary to } Q \text{ and } D \text{ is complementary to } P \end{array} \right\}$$

for $\mathbf{Z}[1/MNuv]$-schemes S. When either $u = 1$ or $v = 1$, we drop them from the notation.

Let $\mathbf{H}$ be the Poincaré upper half-plane and define the modular group:

$$\Gamma(M(u), N(v)) = \left\{ \gamma \in \mathrm{SL}_2(\mathbf{Z}) \text{ such that } \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod \begin{pmatrix} M & Mu \\ Nv & N \end{pmatrix} \right\}.$$

The Riemann surface $Y(M, N)(\mathbf{C})$ admits a complex uniformisation:

$$\begin{array}{ccc} (\mathbf{Z}/M\mathbf{Z})^\times \times \Gamma(M, N)\backslash\mathbf{H} & \xrightarrow{\sim} & Y(M, N)(\mathbf{C}) \\ (m, z) & \mapsto & (\mathbf{C}/\mathbf{Z} + \mathbf{Z}z, mz/M, 1/N), \end{array}$$

and similarly for $Y(M(u), N(v))(\mathbf{C})$.

Let ℓ be a prime. There is an isomorphism of $\mathbf{Z}[1/\ell MN]$-schemes:

$$\begin{array}{lll} \varphi_\ell : Y(M, N(\ell)) & \to & Y(M(\ell), N) \\ \quad (E, P, Q, C) & \mapsto & (E/NC, P + NC, \ell^{-1}(Q) \cap C + NC, (\ell^{-1}(\mathbf{Z} \cdot P + NC)/NC)), \end{array}$$

which under the complex uniformisation is induced by the map $(m, z) \mapsto (m, \ell \cdot z)$.

1.1.2. *Degeneracy maps.* We have the natural degeneracy maps

$$\begin{array}{ccccc} Y(M, N\ell) & \xrightarrow{\mu_\ell} & Y(M, N(\ell)) & \xrightarrow{\nu_\ell} & Y(M, N) \\ & & \Big\downarrow{\scriptstyle \varphi_\ell} & & \\ Y(M\ell, N) & \xrightarrow{\check{\mu}_\ell} & Y(M(\ell), N) & \xrightarrow{\check{\nu}_\ell} & Y(M, N), \end{array}$$

where $\mu_\ell(E, P, Q) = (E, P, \ell \cdot Q, \mathbf{Z} \cdot Q)$, $\nu_\ell(E, P, Q, C) = (E, P, Q)$, and $\check{\mu}_\ell$, $\check{\nu}_\ell$ are defined similarly. Put

$$\begin{aligned} \mathrm{pr}_1 := \nu_\ell \circ \mu_\ell : Y(M, N\ell) &\to Y(M, N), \\ (E, P, Q) &\mapsto (E, P, \ell \cdot Q) \end{aligned}$$

and

$$\begin{aligned}\mathrm{pr}_\ell := \check{\nu}_\ell \circ \varphi_\ell \circ \mu_\ell : Y(M, N\ell) &\to Y(M, N)\\ (E, P, Q) &\mapsto (E/N\mathbf{Z}\cdot Q, P + N\mathbf{Z}\cdot Q, Q + N\mathbf{Z}\cdot Q).\end{aligned}$$

On the complex upper half plane $\mathbf{H}$, the map pr_1 (resp. pr_ℓ) is induced by the identity (resp. multiplication by ℓ). Moreover, $\mu_\ell, \check{\mu}_\ell, \nu_\ell, \check{\nu}_\ell, \mathrm{pr}_1, \mathrm{pr}_\ell$ are all finite étale morphisms of $\mathbf{Z}[1/MN\ell]$-schemes.

1.1.3. *Relative Tate modules and Hecke operators.* Let S be a $\mathbf{Z}[1/MN\ell p]$-scheme where p is a fixed prime. For each $\mathbf{Z}[1/MN\ell p]$-scheme X, denote the base change $X_S = X \times_{\mathbf{Z}[1/MN\ell p]} S$. Notate $A = A_X$ to be either the locally constant sheaf $\mathbf{Z}/p^m\mathbf{Z}(j)$ or the locally constant p-adic sheaf $\mathbf{Z}_p(j)$ on $X_{\text{ét}}$ for some fixed $j \in \mathbf{Z}$ and $m \geq 1$.

For the ease of notation, we may write $\cdot$ for $M(u), N(v)$ (i.e. $Y(\cdot) = Y(M(u), N(v))$). Denote by $E(\cdot)$ the universal elliptic curve over $Y(\cdot)$. Then one obtains a natural degree ℓ isogeny of universal elliptic curves under the base change by $\varphi_\ell^* E(M(\ell), N) \to Y(M, N(\ell))$:

$$\lambda_\ell : E(M, N(\ell)) \to \varphi_\ell^*(E(M(\ell), N).$$

Denote by $v_\cdot : E(\cdot)_S \to Y(\cdot)_S$ the structure map. We also use $\nu_\ell, \check{\nu}_\ell$ and λ_ℓ for the base change to S of the corresponding degeneracy maps. Set:

$$\mathscr{T}_\cdot(A) = R^1 v_{\cdot *}\mathbf{Z}_p(1) \otimes_{\mathbf{Z}_p} A \text{ and } \mathscr{T}_\cdot^*(A) = \mathrm{Hom}_A(\mathscr{T}_\cdot(A), A)$$

where $R^q v_{\cdot *}$ is the q-th right derivative of $v_{\cdot *} : E(\cdot)_{\text{ét}} \to Y(\cdot)_{\text{ét}}$. When $A = \mathbf{Z}_p$, this gives the relative Tate module of the universal elliptic curve, in which case we will drop the $\mathbf{Z}_p$ from the notation.

Fix an integer $r \geq 0$. The (perfect) cup product pairing combined with the relative trace

$$\mathscr{T}_\cdot \otimes_{\mathbf{Z}_p} \mathscr{T}_\cdot \to R^2 v_{\cdot *}\mathbf{Z}_p(2) \cong \mathbf{Z}_p(1)$$

allows us to identify $\mathscr{T}_\cdot(-1)$ with $\mathscr{T}_\cdot^*$. Put

$$\mathscr{L}_{\cdot,r}(A) = \mathrm{Tsym}^r_A \mathscr{T}_\cdot(A), \quad \mathscr{S}_{\cdot,r}(A) = \mathrm{Symm}^r_A \mathscr{T}_\cdot^*(A),$$

where $\mathrm{Tsym}^r_R M$ is the R-submodule of the symmetric tensors in $M^{\otimes r}$, $\mathrm{Symm}^r_R M$ is the maximal symmetric quotient of $M^{\otimes r}$, and M is any finite free module over a profinite $\mathbf{Z}_p$-algebra R. When the level is clear, we shall simplify the notations, e.g. writing:

$$\mathscr{L}_r(A) = \mathscr{L}_{M(u),N(v),r}(A), \quad \mathscr{L}_r = \mathscr{L}_r(\mathbf{Z}_p), \quad \mathscr{S}_r(A) = \mathscr{S}_{M(u),N(v),r}(A), \quad \mathscr{S}_r = \mathscr{S}_r(\mathbf{Z}_p). \tag{1.1}$$

Let $\mathscr{F}^r_\cdot$ be either $\mathscr{L}_{\cdot,r}(A)$ or $\mathscr{S}_{\cdot,r}(A)$. Then there are natural isomorphisms of sheaves

$$\nu_\ell^*(\mathscr{F}^r_{M,N}) \cong \mathscr{F}^r_{M,N(\ell)}, \quad \check{\nu}_\ell^*(\mathscr{F}^r_{M,N}), \cong \mathscr{F}^r_{M(\ell),N},$$

and these induce pullback maps

$$H^i_{\text{ét}}(Y(M, N(\ell))_S, \mathscr{F}^r_{M,N(\ell)}) \xleftarrow{\nu_\ell^*} H^i_{\text{ét}}(Y(M, N)_S, \mathscr{F}^r_{M,N}) \xrightarrow{\check{\nu}_\ell^*} H^i_{\text{ét}}(Y(M(\ell), N)_S, \mathscr{F}^r_{M(\ell),N})$$

and traces

$$H^i_{\text{ét}}(Y(M, N(\ell))_S, \mathscr{F}^r_{M,N(\ell)}) \xrightarrow{\nu_{\ell *}} H^i_{\text{ét}}(Y(M, N)_S, \mathscr{F}^r_{M,N}) \xleftarrow{\check{\nu}_{\ell *}} H^i_{\text{ét}}(Y(M(\ell), N)_S, \mathscr{F}^r_{M(\ell),N}).$$

The finite étale isogeny λ_ℓ induces morphisms

$$\lambda_{\ell *} : \mathscr{F}^r_{M,N(\ell)} \to \varphi_\ell^*(\mathscr{F}^r_{M(\ell),N}), \quad \lambda_\ell^* : \varphi_\ell^*(\mathscr{F}^r_{M(\ell),N}) \to \mathscr{F}^r_{M,N(\ell)}$$

and this allows us to define a pushforward

$$\Phi_{\ell *} := \varphi_{\ell *} \circ \lambda_{\ell *} : H^i_{\text{ét}}(Y(M, N(\ell))_S, \mathscr{F}^r_{M,N(\ell)}) \to H^i_{\text{ét}}(Y(M(\ell), N)_S, \mathscr{F}^r_{M(\ell),N})$$

and a pullback

$$\Phi_\ell^* := \lambda_\ell^* \circ \varphi_\ell^* : H^i_{\text{ét}}(Y(M(\ell), N)_S, \mathscr{F}^r_{M(\ell),N}) \to H^i_{\text{ét}}(Y(M, N(\ell))_S, \mathscr{F}^r_{M,N(\ell)}).$$

The Hecke operator T_ℓ and the dual Hecke operator T'_ℓ acting on $H^i_{\text{ét}}(Y(M, N)_S, \mathscr{F}^r_{M,N})$ are defined by

$$T_\ell := \check{\nu}_{\ell *} \circ \Phi_{\ell *} \circ \nu_\ell^*, \quad T'_\ell := \nu_{\ell *} \circ \Phi_\ell^* \circ \check{\nu}_\ell^*.$$

Remark 1.1.1. Note the relations
$$\deg(\mu_\ell)T_\ell = \mathrm{pr}_{\ell *}\circ \mathrm{pr}_1^*, \quad \deg(\mu_\ell)T'_\ell = \mathrm{pr}_{1*}\circ \mathrm{pr}_\ell^*,$$
as follow immediately from the definitions.

For $d \in (\mathbf{Z}/MN\mathbf{Z})^*$, the diamond operator $\langle d\rangle$ on $Y(\cdot)$ is defined in terms of moduli by
$$(E,P,Q,C,D)\mapsto (E, d^{-1}\cdot P, d\cdot Q, C, D).$$
This is also a unique diamond operator $\langle d\rangle$ on the universal elliptic curve making the following diagram cartesian:
$$\begin{tikzcd} E(\cdot)_S \arrow[r, "\langle d\rangle"] \arrow[d, "v."'] & E(\cdot)_S \arrow[d, "v."] \\ Y(\cdot)_S \arrow[r, "\langle d\rangle"] & Y(\cdot)_S \end{tikzcd} ,$$
and this induces automorphisms $\langle d\rangle = \langle d\rangle^*$ and $\langle d\rangle' = \langle d\rangle_*$ on $H^i_{\text{ét}}(Y(\cdot)_S, \mathscr{F}_.)$.

For any profinite $\mathbf{Z}_p$-algebra R and finite free R-module M, the evaluation map induces a perfect pairing
$$\mathrm{Tsym}^r_R M \otimes_R \mathrm{Symm}^r_R M^* \to R,$$
where $M^* = \mathrm{Hom}_R(M,\mathbf{Z}_p)$. This gives a perfect pairing $\mathscr{L}_r \otimes_{\mathbf{Z}_p} \mathscr{S}_r \to \mathbf{Z}_p$, and therefore a cup product
$$\langle\cdot,\cdot\rangle : H^1_{\text{ét}}(Y(\cdot)_{\overline{\mathbf{Q}}}, \mathscr{L}_r(1)) \otimes_{\mathbf{Z}_p} H^1_{\text{ét},c}(Y(\cdot)_{\overline{\mathbf{Q}}}, \mathscr{S}_r) \to H^2_{\text{ét}}(Y(\cdot)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \cong \mathbf{Z}_p,$$
which is perfect after inverting p. Moreover, the Hecke operators T_ℓ, T'_ℓ, $\langle d\rangle$, $\langle d\rangle'$ induce endomorphisms on the compactly supported cohomology groups $H^1_{\text{ét},c}(Y(\cdot)_{\overline{\mathbf{Q}}}, \mathscr{S}_r)$, and by construction, (T_ℓ, T'_ℓ) and $(\langle d\rangle, \langle d\rangle')$ are adjoint pairs under $\langle\cdot,\cdot\rangle$. The Eichler–Shimura isomorphism [Shi94]
$$H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}}, \mathscr{L}_r) \otimes_{\mathbf{Z}_p} \mathbf{C} \cong M_{r+2}(N,\mathbf{C}) \oplus \overline{S_{r+2}(N,\mathbf{C})}$$
commutes with the action of the Hecke operators on both sides.

1.2. **Galois representations associated to newforms.** Let $f = \sum_{n=1}^\infty a_n q^n$ be a normalised newform of weight $k \geq 2$, level $\Gamma_1(N_f)$, and nebentype χ_f. Let $p \nmid N_f$ be a prime. Fix embeddings $i_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $i_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Let $L/\mathbf{Q}$ be a finite extension containing all values $i_\infty^{-1}(a_n)$ and $i_\infty^{-1}\circ\chi_f$. Let $\mathfrak{P}$ be the prime of L above p with respect to i_p. Then Eichler–Shimura and Deligne construct a p-adic Galois representation associated to f:
$$\rho_{f,\mathfrak{P}} : G_{\mathbf{Q}} \to \mathrm{GL}_2(L_{\mathfrak{P}})$$
which is unramified outside pN_f, and characterised by the property for all finite primes $\ell \nmid pN_f$,
$$\mathrm{trace}(\rho_{f,\mathfrak{P}}(\mathrm{Frob}_\ell)) = i_p(a_\ell), \quad \det(\rho_{f,\mathfrak{P}}(\mathrm{Frob}_\ell)) = i_p(\chi_f(\ell)l^{k-1}),$$
where Frob_ℓ is the geometric Frobenius. Moreover, $\rho_{f,\mathfrak{P}}$ is known to be irreducible [Rib77], hence absolutely irreducible since the image of the complex conjugation has eigenvalues 1 and -1.

1.2.1. *Geometric realisations.* The representation $\rho_{f,\mathfrak{P}}$ can be realised geometrically as the largest subspace V_f of
$$H^1_{\text{ét}}(Y_1(N_f)_{\overline{\mathbf{Q}}}, \mathscr{S}_{k-2}) \otimes L_{\mathfrak{P}}$$
on which T_ℓ acts as multiplication by a_ℓ for all $\ell \nmid N_f p$ and $\langle d\rangle' = \langle d\rangle_*$ acts as multiplication by $\chi_f(d)$ for all $d \in (\mathbf{Z}/N_f\mathbf{Z})^\times$. If N is any multiple of N_f, then the above subspace with N_f replaced by N gives rise to a representation $V_f(N)$ isomorphic (non-canonically) to a finite number of copies of V_f.

The dual $V_f^\vee = \mathrm{Hom}(V_f, L_{\mathfrak{P}})$ can be interpreted as the maximal quotient of
$$H^1_{\text{ét}}(Y_1(N_f)_{\overline{\mathbf{Q}}}, \mathscr{L}_{k-2}(1)) \otimes L_{\mathfrak{P}}$$
on which the dual Hecke operator T'_ℓ acts as multiplication by a_ℓ for all $\ell \nmid N_f p$ and $\langle d\rangle = \langle d\rangle^*$ acts as multiplication by $\chi_f(d)$ for all $d \in (\mathbf{Z}/N_f\mathbf{Z})^\times$.

Let $\mathcal{O}$ be the ring of integers of $L_{\mathfrak{P}}$. In this paper we shall be mostly working with $V_f^\vee$ and the $G_{\mathbf{Q}}$-stable $\mathcal{O}$-lattice $T_f^\vee \subset V_f^\vee$ defined as the image of $H^1_{\text{ét}}(Y_1(N_f)_{\overline{\mathbf{Q}}}, \mathscr{L}_{k-2}(1)) \otimes \mathcal{O}$ in $V_f^\vee$.

1.2.2. *The p-ordinary case.* If f is ordinary at p, i.e. $i_p(a_p) \in \mathcal{O}^\times$, then the restriction of V_f to $G_{\mathbf{Q}_p} \subset G_{\mathbf{Q}}$ is reducible, fitting into and exact sequence of $L_{\mathfrak{P}}[G_{\mathbf{Q}_p}]$-modules

$$0 \to V_f^+ \to V_f \to V_f^- \to 0$$

with $\dim_{L_{\mathfrak{P}}} V_f^\pm = 1$, and which the $G_{\mathbf{Q}_p}$-action on the subspace V_f^+ given by the unramified character sending Frob_p to α_p, the unit root of $x^2 - a_p x + \chi_f(p)p^{k-1}$. By duality, we also obtain an exact sequence for $V_f^\vee$ restricted to $G_{\mathbf{Q}_p}$

$$0 \to V_f^{\vee,+} \to V_f^\vee \to V_f^{\vee,-} \to 0 \tag{1.2}$$

with $V_f^{\vee,+} \simeq (V_f^-)^\vee(1-k)(\chi_f^{-1})$, and with the $G_{\mathbf{Q}_p}$-action on the quotient $V_f^{\vee,-}$ given by the unramified character sending arithmetic Frobenius Frob_p^{-1} to α_p.

1.3. **Patched CM Hecke modules.** In this section, after explaining our conventions on Hecke characters, we recall the construction of certain patched CM Hecke modules from [LLZ15].

1.3.1. *Hecke characters and theta series.* Let K be an imaginary quadratic field in which

$$p = \mathfrak{p}\bar{\mathfrak{p}} \text{ splits},$$

with $\mathfrak{p}$ the prime of K above p induced by $i_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. We say that a Hecke character $\psi : \mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times$ has infinity type $(a, b) \in \mathbf{Z}^2$ if $\psi_\infty(x_\infty) = x_\infty^a \bar{x}_\infty^b$. Then $\psi(x)x_\infty^{-a}\bar{x}_\infty^{-b}$ is a ray class character, hence it takes value in a finite extension L/K. For $\mathfrak{P}|\mathfrak{p}$ the prime of L above p induced by i_p, we define the p-adic avatar $\psi_{\mathfrak{P}}$ of ψ as follows. Denote by $\mathrm{rec}_K : \mathbb{A}_K^\times \to G_K^{\mathrm{ab}}$ the geometrically normalised Artin reciprocity map. For $g \in G_K$, we take $x \in \mathbb{A}_K^\times$ such that $\mathrm{rec}_K(x) = g|_{K^{\mathrm{ab}}}$ and define

$$\psi_{\mathfrak{P}}(g) = i_p \circ i_\infty^{-1}(\psi(x)x_\infty^{-a}\bar{x}_\infty^{-b})x_{\mathfrak{p}}^a x_{\bar{\mathfrak{p}}}^b.$$

Since there should be no confusion, in the following we shall also use ψ to denote its p-adic avatar.

Let ψ be a Hecke character of K of infinity type $(-1, 0)$, conductor $\mathfrak{f}$, taking values in a finite extension L/K. The theta series attached to ψ is

$$\theta_\psi = \sum_{(\mathfrak{a},\mathfrak{f})=1} \psi(\mathfrak{a}) q^{N_{K/\mathbf{Q}}(\mathfrak{a})} \in S_2(\Gamma_1(N_\psi), \chi_\psi \epsilon_K)$$

where $N_\psi = N_{K/\mathbf{Q}}(\mathfrak{f})\mathrm{disc}(K/\mathbf{Q})$, χ_ψ is the unique Dirichlet character modulo $N_{K/\mathbf{Q}}(\mathfrak{f})$ such that $\psi((n)) = n\chi_\psi(n)$ for all $n \in \mathbf{Z}$ with $(n, N_{K/\mathbf{Q}}(\mathfrak{f})) = 1$, and ϵ_K is the quadratic Dirichlet character attached to K. The cuspform θ_ψ is new of level N_ψ if $\mathfrak{f}$ is the conductor of ψ, and its $\mathfrak{P}$-adic representation satisfies

$$V_{\theta_\psi} \cong \mathrm{Ind}_K^{\mathbf{Q}} L_{\mathfrak{P}}(\psi), \quad V_{\theta_\psi}^\vee \cong \mathrm{Ind}_K^{\mathbf{Q}} L_{\mathfrak{P}}(\psi^{-1}).$$

1.3.2. *Hecke algebras and norm maps.* Let $\mathfrak{n} \subset \mathcal{O}_K$ be an ideal divisible by $\mathfrak{f}$. Put $N = N_{K/\mathbf{Q}}(\mathfrak{n})\mathrm{disc}(K/\mathbf{Q})$. Let $K[\mathfrak{n}]$ be the ray class field of K with conductor $\mathfrak{n}$, and let $H_{\mathfrak{n}}$ be the ray class group of K modulo $\mathfrak{n}$. Let $K(\mathfrak{n})$ be the largest p-subextension of K contained in $K[\mathfrak{n}]$, so $\mathrm{Gal}(K(\mathfrak{n})/K) \cong H_{\mathfrak{n}}^{(p)}$ is the largest p-power quotient of $H_{\mathfrak{n}}$. For an ideal $\mathfrak{k}$ of K coprime to $\mathfrak{n}$, let $[\mathfrak{k}]$ be the class of $\mathfrak{k}$ in $H_{\mathfrak{n}}$.

Proposition 1.3.1 ([LLZ15, Prop. 3.2.1])**.** *Let $\mathbb{T}'(N)$ be the subalgebra of $\mathrm{End}_{\mathbf{Z}}(H^1(Y_1(N)(\mathbf{C}), \mathbf{Z}))$ generated by $\langle d \rangle'$ and T'_ℓ for all primes ℓ. There exists a homomorphism $\phi_{\mathfrak{n}} : \mathbb{T}'(N) \to \mathcal{O}[H_{\mathfrak{n}}]$ defined by*

$$\phi_{\mathfrak{n}}(T'_\ell) = \sum_{\mathfrak{l}} [\mathfrak{l}]\psi(\mathfrak{l}),$$

$$\phi_{\mathfrak{n}}(\langle d \rangle') = \chi_\psi(d)\epsilon_K(d)[(d)],$$

where the sum is over the ideals $\mathfrak{l} \subset \mathcal{O}_K$ with $\mathfrak{l} \nmid \mathfrak{n}$ and $N_{K/\mathbf{Q}}(\mathfrak{l}) = \ell$.

For $\mathfrak{n}' = \mathfrak{n}\mathfrak{l}$, with $\mathfrak{l}$ a prime ideal and $(\mathfrak{n}', p) = 1$, put $N' = N_{K/\mathbf{Q}}(\mathfrak{n}')\mathrm{disc}(K/\mathbf{Q})$. Following [LLZ15, §3.3], we consider the norm maps

$$\mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}'} : \mathcal{O}[H_{\mathfrak{n}'}^{(p)}] \otimes_{\mathbb{T}'(N')\otimes\mathbf{Z}_p,\phi_{\mathfrak{n}'}} H^1_{\text{ét}}(Y_1(N')_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \to \mathcal{O}[H_{\mathfrak{n}}^{(p)}] \otimes_{\mathbb{T}'(N)\otimes\mathbf{Z}_p,\phi_{\mathfrak{n}}} H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1))$$

defined by the following formulae:

- If $\mathfrak{l} \mid \mathfrak{n}$ then
$$\mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}'} = 1 \otimes \mathrm{pr}_{1*};$$
- If $\mathfrak{l} \nmid \mathfrak{n}$ is split or ramified in K, then
$$\mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}'} = 1 \otimes \mathrm{pr}_{1*} - \frac{\psi(\mathfrak{l})[\mathfrak{l}]}{\ell} \otimes \mathrm{pr}_{\ell*};$$
- If $\mathfrak{l} \nmid \mathfrak{n}$ is inert in K, say $\mathfrak{l} = (\ell)$, then
$$\mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}'} = 1 \otimes \mathrm{pr}_{1*} - \frac{\psi(\mathfrak{l})[\mathfrak{l}]}{\ell^2} \otimes \mathrm{pr}_{\ell\ell*};$$

and we extend the definition of $\mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}'}$ to any pair of ideals $\mathfrak{n} \mid \mathfrak{n}'$ by composition.

Notation 1.3.2. Let ψ be a character character of K of infinity type $(-1, 0)$ and conductor $\mathfrak{f}$. We say that ψ satisfies *Condition* ♠ if either $(p, \mathfrak{f}) = 1$, or $\mathfrak{p} \mid \mathfrak{f}$, $\bar{\mathfrak{p}} \nmid \mathfrak{f}$, and

$$\psi|_{\mathcal{O}_K^\times} \not\equiv \omega \pmod{\mathfrak{P}},$$

where ω is the Teichmüller character.

Since we assume that p splits in K, by Proposition 5.1.2 and Remark 5.1.3 in [LLZ15], if Condition ♠ holds, then for any ideal $\mathfrak{n} \subset \mathcal{O}_K$ divisible by $\mathfrak{f}$ and with $(\mathfrak{n}, \bar{\mathfrak{p}}) = 1$, the maximal ideal of $\mathbb{T}'(N)$ defined by the kernel of the composite map

$$\mathbb{T}'(N) \xrightarrow{\phi_{\mathfrak{n}}} \mathcal{O}[H_{\mathfrak{n}}] \xrightarrow{\text{aug}} \mathcal{O} \to \mathcal{O}/\mathfrak{P},$$

where $\phi_{\mathfrak{n}}$ is the map from Proposition 1.3.1, is non-Eisenstein, p-ordinary, and p-distinguished.

Theorem 1.3.3. *Let $\mathcal{A}$ be the set of ideals $\mathfrak{m} \subset \mathcal{O}_K$ with $(\mathfrak{m}, \bar{\mathfrak{p}}) = 1$, and put $\mathcal{A}_{\mathfrak{f}} = \{\mathfrak{f}\mathfrak{m} \colon \mathfrak{m} \in \mathcal{A}\}$. Suppose ψ satisfies Condition ♠. Then there is a family of $G_{\mathbf{Q}}$-equivariant isomorphisms of $\mathcal{O}[H_{\mathfrak{n}}^{(p)}]$-modules*

$$\nu_{\mathfrak{n}} : \mathcal{O}[H_{\mathfrak{n}}^{(p)}] \otimes_{\mathbb{T}'(N)\otimes\mathbf{Z}_p,\phi_{\mathfrak{n}}} H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \xrightarrow{\cong} \mathrm{Ind}_{K(\mathfrak{n})}^{\mathbf{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1})$$

indexed by $\mathfrak{n} \in \mathcal{A}_{\mathfrak{f}}$, such that for any $\mathfrak{n}, \mathfrak{n}' \in \mathcal{A}_{\mathfrak{f}}$ with $\mathfrak{n} \mid \mathfrak{n}'$ the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}[H_{\mathfrak{n}'}^{(p)}] \otimes_{\mathbb{T}'(N')\otimes\mathbf{Z}_p,\phi_{\mathfrak{n}'}} H^1_{\text{ét}}(Y_1(N')_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \arrow[r, "\nu_{\mathfrak{n}'}", "\cong"'] \arrow[d, "\mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}'}"'] & \mathrm{Ind}_{K(\mathfrak{n}')}^{\mathbf{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}) \arrow[d, "\mathrm{Norm}_{\mathfrak{n}}^{\mathfrak{n}'}"'] \\
\mathcal{O}[H_{\mathfrak{n}}^{(p)}] \otimes_{\mathbb{T}'(N)\otimes\mathbf{Z}_p,\phi_{\mathfrak{n}}} H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \arrow[r, "\nu_{\mathfrak{n}}", "\cong"'] & \mathrm{Ind}_{K(\mathfrak{n})}^{\mathbf{Q}} \mathcal{O}(\psi_{\mathfrak{P}}^{-1}),
\end{tikzcd}$$

where $\mathrm{Norm}_{\mathfrak{n}}^{\mathfrak{n}'}$ is the natural norm map.

Proof. This is a reformulation of Corollary 5.2.6 in [LLZ15] in the case where p splits in K. □

1.4. **Diagonal classes.** We sketch the construction of the diagonal classes in the triple product of modular curves $Y_1(N)$ using classical invariant theory, following Section 3 in [BSV22].

We recall some notation used in Section 1.1.3. Here, $Y_1(N) = Y_1(N)_{\mathbf{Q}}$, $E_1(N) = E_1(N)_{\mathbf{Q}}$ the universal elliptic curve over $Y_1(N)$ together with the structural map $v : E_1(N) \to Y_1(N)$. The relative Tate module of the universal elliptic curve is $\mathscr{T} = R^1 v_* \mathbf{Z}_p(1)$, and its dual is $\mathscr{T}^* = \mathrm{Hom}_{\mathbf{Z}_p}(\mathscr{T}, \mathbf{Z}_p)$. The cup product pairing combined with the relative trace

$$\mathscr{T} \otimes_{\mathbf{Z}_p} \mathscr{T} \to R^2 v_* \mathbf{Z}_p(2) \cong \mathbf{Z}_p(1)$$

gives a perfect relative Weil pairing

$$\langle , \rangle_{E_1(N)_{p^\infty}} : \mathscr{T} \otimes_{\mathbf{Z}_p} \mathscr{T} \to \mathbf{Z}_p(1),$$

which allows $\mathscr{T}(-1)$ to be identified with $\mathscr{T}^*$.

For A either the locally constant sheaf $\mathbf{Z}/p^m\mathbf{Z}(j)$ or the locally constant p-adic sheaf $\mathbf{Z}_p(j)$ on $X_{\text{ét}}$ for some fixed $m \geq 1$ and $m, j \in \mathbf{Z}$, recall that

$$\mathscr{L}_r(A) = \mathrm{Tsym}^r_A \mathscr{T}(A), \quad \mathscr{S}_r(A) = \mathrm{Symm}^r_A \mathscr{T}^*(A),$$

where given any finite free module M over a profinite $\mathbf{Z}_p$-algebra R, $\mathrm{Tsym}^r_R M$ is the R- submodule of the symmetric tensors in $M^{\otimes r}$, and $\mathrm{Symm}^r_R M$ is the maximal symmetric quotient of $M^{\otimes r}$.

For a fixed geometric point $\eta : \mathrm{Spec}(\overline{\mathbf{Q}}) \to Y_1(N)$, denote by $\mathcal{G}_\eta = \pi_1^{\text{ét}}(Y_1(N), \eta)$ the fundamental group of $Y_1(N)$ with base point η. The stalk of $\mathscr{T}$ at η, denoted $\mathscr{T}_\eta$, is a free $\mathbf{Z}_p$-module of rank 2, equipped with a continuous action of $\mathcal{G}_\eta$. Fix a choice of $\mathbf{Z}_p$-module isomorphism $\zeta : \mathscr{T}_\eta \cong \mathbf{Z}_p \oplus \mathbf{Z}_p$ such that $\langle x, y \rangle_{E_1(N)_{p^\infty}} = \zeta(x) \wedge \zeta(y)$ (where we identify $\bigwedge^2 \mathbf{Z}_p^2$ with $\mathbf{Z}_p$ via $(1,0) \wedge (0,1) = 1$). One then obtains a continuous group homomorphism:

$$\rho_\eta : \mathcal{G}_\eta \to \mathrm{Aut}_{\mathbf{Z}_p}(\mathscr{T}_\eta) \cong \mathrm{GL}_2(\mathbf{Z}_p).$$

By [FK88, Prop A I.8], the category of locally constant p-adic sheaves on $Y_1(N)_{\text{ét}}$ is equivalent to the category of p-adic representations of $\mathcal{G}_\eta$ via the map $\mathscr{F} \mapsto \mathscr{F}_\eta$. Using ρ_η, one can associate with every continuous representation of $\mathrm{GL}_2(\mathbf{Z}_p)$ over a free finite $\mathbf{Z}_p$-module M a smooth sheaf $M^{\text{ét}}$ on $Y_1(N)$ such that $M^{\text{ét}}_\eta = M$.

Let $S_i(A)$ be the set of 2-variable homogeneous polynomials of degree i in $A[x_1, x_2]$ equipped with the action of $\mathrm{GL}_2(\mathbf{Z}_p)$ by $gP(x_1, x_2) = P((x_1, x_2) \cdot g)$ for all $g \in \mathrm{GL}_2(\mathbf{Z}_p)$ and $P \in S_i(A)$. Its A-linear dual $L_i(A)$ is also equipped with a $\mathrm{GL}_2(\mathbf{Z}_p)$-action by $g\tau(P(x_1, x_2)) = \tau(g^{-1}P(x_1, x_2))$ for all $g \in \mathrm{GL}_2(\mathbf{Z}_p)$, $P \in S_i(A)$, and $\tau \in L_i(A)$. As sheaves on $Y_1(N)_{\mathbf{Q}}$, one has

$$L_i(A)^{\text{ét}} = \mathscr{L}_i(A) \quad \text{and} \quad S_i(A)^{\text{ét}} = \mathscr{S}_i(A).$$

Hence $\mathscr{T}_\eta \cong L_1(\mathbf{Z}_p)$ and $\mathbf{Z}_p(1)_\eta \cong \bigwedge^2 \mathscr{T}_\eta \cong \det^{-1}$. This implies that for any $j \in \mathbf{Z}$, and any p-adic representation M of $\mathrm{GL}_2(\mathbf{Z}_p)$:

$$H^0(\mathrm{GL}_2(\mathbf{Z}_p), M \otimes \det{}^{-j}) \hookrightarrow H^0(\mathcal{G}_\eta, M \otimes \det{}^{-j}) \cong H^0_{\text{ét}}(Y_1(N), M^{\text{ét}}(j)). \tag{1.3}$$

Assumption 1.1. Let $\mathbf{r} = (r_1, r_2, r_3)$ such that $r_i \in \mathbf{Z}_{\geq 0}$, $(r_1 + r_2 + r_3)/2 = r \in \mathbf{Z}_{\geq 0}$, and $r_i + r_j \geq r_k$ for all permutation (i, j, k) of $(1, 2, 3)$. We call this the *balanced condition*.

Under the Assumption 1.1, let

$$S_{\mathbf{r}} = S_{r_1}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} S_{r_2}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} S_{r_3}(\mathbf{Z}_p)$$

a $\mathrm{GL}_2(\mathbf{Z}_p)$-representation, and let

$$\mathscr{S}_{\mathbf{r}} = S_{\mathbf{r}}^{\text{ét}} = \mathscr{S}_{r_1}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathscr{S}_{r_2}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} \mathscr{S}_{r_3}(\mathbf{Z}_p).$$

We identify $S_{\mathbf{r}}$ with the module of 6-variable polynomials $\mathbf{Z}_p[x_1, x_2, y_1, y_2, z_1, z_2]$ which is homogeneous of degree r_1, r_2, and r_3 in the variables (x_1, x_2), (y_1, y_2), and (z_1, z_2) respectively. By the Clebsch–Gordan decomposition of classical invariant theory, the following is a $\mathrm{GL}_2(\mathbf{Z}_p)$-invariant of $S_{\mathbf{r}} \otimes \det^{-r}$:

$$\mathrm{Det}^{\mathbf{r}}_N := \det \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}^{r-r_3} \det \begin{pmatrix} x_1 & x_2 \\ z_1 & z_2 \end{pmatrix}^{r-r_2} \det \begin{pmatrix} y_1 & y_2 \\ z_1 & z_2 \end{pmatrix}^{r-r_1},$$

i.e. $\mathrm{Det}_N^{\mathbf{r}} \in H^0(\mathrm{GL}_2(\mathbf{Z}_p), S_{\mathbf{r}} \otimes \det^{-r})$ and denote its image under (1.3) as

$$\mathtt{Det}_N^{\mathbf{r}} \in H^0_{\text{ét}}(Y_1(N), \mathscr{S}_{\mathbf{r}}(r)). \tag{1.4}$$

Let $p_j : Y_1(N)^3 \to Y_1(N)$ for $j \in \{1,2,3\}$ be the natural projections and denote

$$\mathscr{S}_{[\mathbf{r}]} := p_1^* \mathscr{S}_{r_1}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} p_2^* \mathscr{S}_{r_2}(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} p_3^* \mathscr{S}_{r_3}(\mathbf{Z}_p),$$
$$\mathtt{W}_{N,\mathbf{r}} := H^3_{\text{ét}}(Y_1(N)^3_{\overline{\mathbf{Q}}}, \mathscr{S}_{[\mathbf{r}]}(r+2)).$$

Because $Y_1(N)_{\overline{\mathbf{Q}}}$ is a smooth affine curve over $\overline{\mathbf{Q}}$, we have $H^4_{\text{ét}}(Y_1(N)^3_{\overline{\mathbf{Q}}}, \mathscr{S}_{[\mathbf{r}]}(r+2)) = 0$. Hence by the Hochschild–Serre spectral sequence,

$$H^p(\mathbf{Q}, H^q_{\text{ét}}(Y_1(N)^3_{\overline{\mathbf{Q}}}, \mathscr{S}_{[\mathbf{r}]}(r+2))) \Longrightarrow H^{p+q}_{\text{ét}}(Y_1(N)^3_{\overline{\mathbf{Q}}}, \mathscr{S}_{[\mathbf{r}]}(r+2))$$

one obtains

$$\mathtt{HS} : H^4_{\text{ét}}(Y_1(N)^3, \mathscr{S}_{[\mathbf{r}]}(r+2)) \to H^1(\mathbf{Q}, \mathtt{W}_{N,\mathbf{r}}).$$

If we let $d : Y_1(N) \to Y_1(N)^3$ be the diagonal embedding, then there is a natural isomorphism $d^* \mathscr{S}_{[\mathbf{r}]} \cong \mathscr{S}_{\mathbf{r}}$ of smooth sheaves on $Y_1(N)_{\text{ét}}$. As d is an embedding of codimension 2, there is a pushforward map

$$d_* : H^0_{\text{ét}}(Y_1(N), \mathscr{S}_{\mathbf{r}}(r)) \to H^4_{\text{ét}}(Y_1(N)^3, \mathscr{S}_{\mathbf{r}}(r+2)),$$

and we define the class

$$(\mathtt{HS} \circ d_*)(\mathtt{Det}_N^{\mathbf{r}}) \in H^1(\mathbf{Q}, \mathtt{W}_{N,\mathbf{r}}).$$

Dually, by the bilinear form $\det^* : L_i(\mathbf{Z}_p) \otimes_{\mathbf{Z}_p} L_i(\mathbf{Z}_p) \to \mathbf{Z}_p \otimes \det^{-i}$ defined by $\det^*(\tau \otimes \sigma) = \tau \otimes \sigma((x_1y_2 - x_2y_1)^i)$ that becomes perfect after inverting p, we can define an isomorphism of $\mathrm{GL}_2(\mathbf{Z}_p)$-modules

$$\mathbf{s}_i : S_i(\mathbf{Q}_p) \cong L_i(\mathbf{Q}_p) \otimes \det{}^i,$$

and so $\mathbf{s}_i : \mathscr{S}_i(\mathbf{Q}_p) \cong \mathscr{L}_i(\mathbf{Q}_p) \otimes \det^i$ by the above equivalence of categories. We then similarly define the sheaves $\mathscr{L}_{\mathbf{r}}$ on $Y_1(N)$ and $\mathscr{L}_{[\mathbf{r}]}$ on $Y_1(N)^3$. Set

$$\mathtt{V}_{N,\mathbf{r}} := H^3_{\text{ét}}(Y_1(N)^3_{\overline{\mathbf{Q}}}, \mathscr{L}_{[\mathbf{r}]}(2-r)), \quad V_{N,\mathbf{r}} = \mathtt{V}_{N,\mathbf{r}} \otimes \mathbf{Q}_p.$$

Let $\mathbf{s}_{\mathbf{r}} = \mathbf{s}_{\mathbf{r}_1} \otimes \mathbf{s}_{\mathbf{r}_2} \otimes \mathbf{s}_{\mathbf{r}_3}$, which gives an isomorphism $W_{N,\mathbf{r}} \to V_{N,\mathbf{r}}$, and finally as in [BSV22] put

$$\kappa_{N,\mathbf{r}} := (\mathbf{s}_{\mathbf{r}*} \circ \mathtt{HS} \circ d_*)(\mathtt{Det}_N^{\mathbf{r}}) \in H^1(\mathbf{Q}, V_{N,\mathbf{r}}). \tag{1.5}$$

As explained in detail in [*loc. cit.*, §3.2], the class $\kappa_{N,\mathbf{r}}$ is closely related to the p-adic étale Abel–Jacobi image of the generalised Gross–Kudla–Schoen diagonal cycles on Kuga–Sato varieties studied in [DR14].

Proposition 1.4.1. *For a prime number ℓ and a positive integer m, if $(m\ell, pN) = 1$ then*

$$(\mathrm{pr}_{i*}, \mathrm{pr}_{j*}, \mathrm{pr}_{k*})\kappa_{m\ell,r} = (\bigstar)\kappa_{m,r}$$

where

(i,j,k)	$\bigstar$
$(\ell,1,1)$	$(\ell-1)(T_\ell,1,1)$
$(1,\ell,1)$	$(\ell-1)(1,T_\ell,1)$
$(1,1,\ell)$	$(\ell-1)(1,1,T_\ell)$
$(1,\ell,\ell)$	$\ell^{r-r_1}(\ell-1)(T'_\ell,1,1)$
$(\ell,1,\ell)$	$\ell^{r-r_2}(\ell-1)(1,T'_\ell,1)$
$(\ell,\ell,1)$	$\ell^{r-r_3}(\ell-1)(1,1,T'_\ell)$

If $(\ell, m) = 1$ then we also have

(i,j,k)	$\bigstar$
$(1,1,1)$	(ℓ^2-1)
(ℓ,ℓ,ℓ)	$(\ell^2-1)\ell^r$

Proof. See equations (174) and (176) in [BSV22]. □

2. Main theorems

In this section, for a newform f and two Hecke characters ψ_1, ψ_2 of an imaginary quadratic field K, using the results from [BSV22] and [LLZ15] recalled in the preceding section, we construct a family of cohomology classes for $f \otimes \psi_1\psi_2$ defined over ring class field extensions of K, and prove that they satisfy the norm relations of an anticyclotomic Euler system.

The construction is in two steps: we first give the construction and the proof of the tame norm relations in the case where $(f, \theta_{\psi_1}, \theta_{\psi_2})$ have weights $(2,2,2)$; then using the variation of diagonal cycle classes in Hida families we extend the construction to more general weights and deduce the proof of the wild norm relations.

Throughout this section we consider the following set-up. We let $f \in S_k(\Gamma_0(N_f))$ be a newform of weight $k \geq 2$, $K/\mathbf{Q}$ an imaginary quadratic field of discriminant prime to N_f, and ψ_1, ψ_2 (not necessarily primitive) Hecke characters of K of infinity type $(1-k_1, 0), (1-k_2, 0)$, with $k_1, k_2 \geq 1$, and modulus $\mathfrak{f}_1, \mathfrak{f}_2$, respectively. Denote by

$$\theta_{\psi_i} \in S_{k_i}(N_{\psi_i}, \chi_{\psi_i})$$

the associated theta series, where $N_{\psi_i} = N_{K/\mathbf{Q}}(\mathfrak{f}_i) \cdot \mathrm{disc}(K/\mathbf{Q})$ and χ_{ϕ_i} is the Dirichlet character modulo $N_{K/\mathbf{Q}}(\mathfrak{f}_i)$ defined by $\psi_i((n)) = n^{k_i - 1}\chi_{\psi_i}(n)$ for all integers n prime to $N_{K/\mathbf{Q}}(\mathfrak{f}_i)$ $(i = 1, 2)$. We assume the self-duality condition

$$\chi_{\psi_1}\chi_{\psi_2} = 1. \tag{2.1}$$

In particular, since k is even by hypothesis, condition (2.1) implies that $k_1 \equiv k_2 \pmod 2$.

Let L/K be a finite extension containing the Fourier coefficients of f, θ_{ψ_1}, and θ_{ψ_2}. Let $p \geq 5$ be a prime that splits in K and such that $(p, N_f N_{\psi_1} N_{\psi_2}) = 1$, and let $\mathfrak{P}|\mathfrak{p}$ be the primes of L/K above p determined by a fixed embedding $i_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Finally, let $L_{\mathfrak{P}}$ be the completion of L at $\mathfrak{P}$, and denote by $\mathcal{O}$ the ring of integers of $L_{\mathfrak{P}}$.

2.1. **Construction in weight** $(2,2,2)$**.** Suppose in this subsection that $k = k_1 = k_2 = 2$. Let $N = \mathrm{lcm}(N_f, N_{\psi_1}, N_{\psi_2})$ and for every positive integer m put

$$Y(m) := Y(1, Nm) = Y_1(Nm).$$

When $m = 1$, we drop it from the notation, so $Y := Y_1(N)$. We begin with the cohomology class

$$\tilde{\kappa}_m^{(1)} := \mathbf{s}_{\mathbf{r}*} \circ \mathtt{HS} \circ d_*(\mathtt{Det}^{\mathbf{r}}_{Nm}) \in H^1\big(\mathbf{Q}, H^3_{\text{ét}}(Y(m)^3_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(2)\big) \tag{2.2}$$

in the notations of Section 1.4, where $\mathbf{r} = (0,0,0)$, and put

$$\tilde{\kappa}_m^{(2)} = (\mathrm{pr}_{m*}, 1, 1)\tilde{\kappa}_m^{(1)} \in H^1\big(\mathbf{Q}, H^3_{\text{ét}}(Y_{\overline{\mathbf{Q}}} \times Y(m)^2_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(2)\big),$$

where, writing $m = \prod_i \ell_i$ as a product of (not necessarily distinct) primes, pr_{m*} is the composition of the pushforward by the degeneracy maps pr_{ℓ_i}.

Applying the Künneth decomposition theorem (see e.g. [Mil80, Thm. 22.4]) together with the natural degeneracy maps $Y(m) \to Y_1(N_{\psi_i} m)$ $(i = 1, 2)$, the class $\tilde{\kappa}_m^{(2)}$ is projected to

$$\tilde{\kappa}_m^{(3)} \in H^1\big(\mathbf{Q}, H^1_{\text{ét}}(Y_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes H^1_{\text{ét}}(Y_1(N_{\psi_1} m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes H^1_{\text{ét}}(Y_1(N_{\psi_2} m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1))(-1)\big). \tag{2.3}$$

Now we fix a test vector $\breve{f} \in S_2(N)[f]$. The maps used in the construction $\tilde{\kappa}_m^{(3)}$ are compatible with correspondences, and so after tensoring with $\mathcal{O}$ the above process gives rise to a class

$$\begin{aligned}\tilde{\kappa}^{(4)}_{f,\psi_1,\psi_2,\mathfrak{m}} \in H^1\big(\mathbf{Q}, T_f^\vee \otimes H^1_{\text{ét}}(Y_1(N_{\psi_1} m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_1} m)} \mathcal{O}[H^{(p)}_{\mathfrak{f}_1\mathfrak{m}}]\\ \otimes H^1_{\text{ét}}(Y_1(N_{\psi_2} m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_2} m)} \mathcal{O}[H^{(p)}_{\mathfrak{f}_2\overline{\mathfrak{m}}}]\big),\end{aligned}$$

where the labeled tensor products are with respect to the Hecke algebra homomorphisms

$$\phi_{\mathfrak{f}_1\mathfrak{m}} : \mathbb{T}'(N_{\psi_1} m) \to \mathcal{O}[H^{(p)}_{\mathfrak{f}_1\mathfrak{m}}], \quad \phi_{\mathfrak{f}_2\overline{\mathfrak{m}}} : \mathbb{T}'(N_{\psi_2} m) \to \mathcal{O}[H^{(p)}_{\mathfrak{f}_2\overline{\mathfrak{m}}}]$$

of Proposition 1.3.1, and we used our chosen $\breve{f}$ to take the image under the projection $H^1_{\text{ét}}(Y_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \to T_f^\vee$ in the first factor.

Since p splits in K and is coprime to $\mathfrak{f}_1\mathfrak{f}_2$, Condition ♠ is satisfied for both ψ_1 and ψ_2. Therefore by the isomorphisms from Proposition 1.3.3:

$$\nu_{\mathfrak{f}_1\mathfrak{m}} : H^1_{\text{ét}}(Y_1(N_{\psi_1}m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_1}m)} \mathcal{O}[H^{(p)}_{\mathfrak{f}_1\mathfrak{m}}] \xrightarrow{\sim} \mathrm{Ind}^{\mathbf{Q}}_{K(\mathfrak{f}_1\mathfrak{m})}\mathcal{O}(\psi_1^{-1}),$$
$$\nu_{\mathfrak{f}_2\overline{\mathfrak{m}}} : H^1_{\text{ét}}(Y_1(N_{\psi_2}m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes_{\mathbb{T}'(N_{\psi_2}m)} \mathcal{O}[H^{(p)}_{\mathfrak{f}_2\overline{\mathfrak{m}}}] \xrightarrow{\sim} \mathrm{Ind}^{\mathbf{Q}}_{K(\mathfrak{f}_2\overline{\mathfrak{m}})}\mathcal{O}(\psi_2^{-1}),$$

the class $\tilde{\kappa}^{(4)}_{f,\psi_1,\psi_2,\mathfrak{m}}$ defines an element in

$$H^1\big(\mathbf{Q}, T_f^\vee \otimes_{\mathcal{O}} \mathrm{Ind}^{\mathbf{Q}}_{K(\mathfrak{f}_1\mathfrak{m})}\mathcal{O}(\psi_1^{-1}) \otimes_{\mathcal{O}} \mathrm{Ind}^{\mathbf{Q}}_{K(\mathfrak{f}_2\overline{\mathfrak{m}})}\mathcal{O}(\psi_2^{-1})(-1)\big),$$

which under the maps induced by $H_{\mathfrak{f}_1\mathfrak{m}} \twoheadrightarrow H_{\mathfrak{m}}$ and $H_{\mathfrak{f}_2\overline{\mathfrak{m}}} \twoheadrightarrow H_{\overline{\mathfrak{m}}}$ is naturally projected to a class

$$\tilde{\kappa}^{(5)}_{f,\psi_1,\psi_2,\mathfrak{m}} \in H^1\big(\mathbf{Q}, T_f^\vee \otimes_{\mathcal{O}} \mathrm{Ind}^{\mathbf{Q}}_{K}\mathcal{O}_{\psi_1^{-1}}[H^{(p)}_{\mathfrak{m}}] \otimes_{\mathcal{O}} \mathrm{Ind}^{\mathbf{Q}}_{K}\mathcal{O}_{\psi_2^{-1}}[H^{(p)}_{\overline{\mathfrak{m}}}](-1)\big).$$

2.1.1. *Projection to ring class groups.* Directly from the definitions of the class groups involved, we deduce the commutative diagram with exact rows

$$\begin{tikzcd}
\mathcal{O}_K^\times \times \mathcal{O}_K^\times \arrow[r] & (\mathcal{O}_K/\mathfrak{m})^\times \times (\mathcal{O}_K/\overline{\mathfrak{m}})^\times \arrow[r] & H_{\mathfrak{m}} \times H_{\overline{\mathfrak{m}}} \arrow[r] & H_1 \times H_1 \arrow[r] & 1 \\
\mathcal{O}_K^\times \arrow[r] \arrow[u, "\Delta"] & (\mathcal{O}_K/m\mathcal{O}_K)^\times \arrow[r] \arrow[u, "\simeq"] & H_m \arrow[r] \arrow[u] & H_1 \arrow[r] \arrow[u, "\Delta"] & 1,
\end{tikzcd}$$

where the unlabelled vertical arrow is given by the restriction map

$$\sigma \mapsto (\sigma|_{K_{\mathfrak{m}}}, \sigma|_{K_{\overline{\mathfrak{m}}}}).$$

In particular, when $p \nmid 6h_K$, where $h_K := |H_1|$ is the class number of K, taking p-primary parts this map induces an isomorphism

$$H^{(p)}_m \xrightarrow{\simeq} H^{(p)}_{\mathfrak{m}} \times H^{(p)}_{\overline{\mathfrak{m}}}. \tag{2.4}$$

Given an integer $n > 0$, let $H[n]$ be the ring class group of K of conductor n, so $H[n] \simeq \mathrm{Pic}(\mathcal{O}_n)$ under the Artin reciprocity map, where $\mathcal{O}_n = \mathbf{Z} + n\mathcal{O}_K$ is the order of K of conductor n. Let $H[n]^{(p)}$ be the maximal p-power quotient of $H[n]$, and denote by $K[n]$ be the maximal p-extension inside the ring class field of K of conductor n, so $H[n]^{(p)} = \mathrm{Gal}(K[n]/K)$.

Proposition 2.1.1. *Suppose $p \nmid 6h_K$ and $\mathfrak{m}$ is an ideal of $\mathcal{O}_K$ of norm m divisible only by primes that are split in K. Then, identifying $H^{(p)}_m$ with $H^{(p)}_{\mathfrak{m}} \times H^{(p)}_{\overline{\mathfrak{m}}}$ as in (2.4), we have an exact sequence*

$$1 \longrightarrow (\mathbf{Z}/m\mathbf{Z})^{\times,(p)} \xrightarrow{\Delta} H^{(p)}_{\mathfrak{m}} \times H^{(p)}_{\overline{\mathfrak{m}}} \xrightarrow{\pi_\Delta} H[m]^{(p)} \longrightarrow 1,$$

where the map Δ sends $a \mapsto ([a],[a])$ for every integer a coprime to m. Moreover, if $(\ell) = \mathfrak{l}\bar{\mathfrak{l}}$ is a prime that splits in K and is coprime to m, the projection π_Δ sends

$$[\mathfrak{l}] \times [\mathfrak{l}] \mapsto \mathrm{Frob}_{\mathfrak{l}}$$

where $\mathrm{Frob}_{\mathfrak{l}}$ is the Frobenius element of $\mathfrak{l}$ in $H[m]^{(p)}$.

Proof. The first part is clear from the above discussion together with the commutative diagram with exact rows

$$\begin{tikzcd}
\mathcal{O}_K^\times \arrow[r] \arrow[d] & (\mathcal{O}_K/m\mathcal{O}_K)^\times \arrow[r] \arrow[d] & H_m \arrow[r] \arrow[d] & H_1 \arrow[r] \arrow[d] & 1 \\
\mathcal{O}_K^\times/\mathbf{Z}^\times \arrow[r] & (\mathcal{O}_K/m\mathcal{O}_K)^\times/(\mathbf{Z}/m\mathbf{Z})^\times \arrow[r] & H[m] \arrow[r] & H_1 \arrow[r] & 1,
\end{tikzcd}$$

where the vertical arrows are given by the natural projections. The second part follows from the functoriality properties of Frobenii. □

Under the hypotheses in Proposition 2.1.1, we can consider the image of $\tilde{\kappa}^{(5)}_{f,\psi_1,\psi_2,\mathfrak{m}}$ under the composite map

$$\begin{tikzcd}
\mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}}[H_{\mathfrak{m}}^{(p)}] \otimes_{\mathcal{O}} \mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_2^{-1}}[H_{\overline{\mathfrak{m}}}^{(p)}] \arrow[r, "\xi"] \arrow[dr, dotted, "\xi_\Delta"'] & \mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}\psi_2^{-1}}[H_{\mathfrak{m}}^{(p)} \times H_{\overline{\mathfrak{m}}}^{(p)}] \arrow[d, "\pi_\Delta"] \\
& \mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}\psi_2^{-1}}[H[m]^{(p)}],
\end{tikzcd} \tag{2.5}$$

where the horizontal arrow is the map determined by $\phi_1 \otimes \phi_2 \mapsto \xi(\phi_1 \otimes \phi_2)$ with $\xi(\phi_1 \otimes \phi_2)(g) = \phi_1(g_1) \otimes \phi_2(g_2)$ if $g = (g_1, g_2) \in H_{\mathfrak{m}}^{(p)} \times H_{\overline{\mathfrak{m}}}^{(p)}$, resulting in the class

$$\kappa^{(6)}_{f,\psi_1,\psi_2,\mathfrak{m}} \in H^1\big(\mathbf{Q}, T_f^\vee \otimes_{\mathcal{O}} \mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}\psi_2^{-1}}[H[m]^{(p)}](-1)\big).$$

Definition 2.1.2. Suppose $p \nmid 6h_K$ splits in K and $\mathfrak{m}$ is an ideal of $\mathcal{O}_K$ of norm m divisible only by primes that split in K. Then we define

$$\widetilde{\kappa}_{f,\psi_1,\psi_2,\mathfrak{m}} \in H^1\big(K[m], T_f^\vee(\psi_1^{-1}\psi_2^{-1})(-1)\big)$$

to be the image of $\kappa^{(6)}_{f,\psi_1,\psi_2,\mathfrak{m}}$ under the isomorphism

$$H^1\big(\mathbf{Q}, T_f^\vee \otimes_{\mathcal{O}} \mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}\psi_2^{-1}}[H[m]^{(p)}](-1)\big) \simeq H^1\big(K[m], T_f^\vee(\psi_1^{-1}\psi_2^{-1})(-1)\big)$$

given by Shapiro's lemma.

We finish this section by recording the following observation for our later use.

Lemma 2.1.3. *The following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}}[H_{\mathfrak{m}\mathfrak{l}}^{(p)}] \otimes_{\mathcal{O}} \mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_2^{-1}}[H_{\overline{\mathfrak{m}}\overline{\mathfrak{l}}}^{(p)}] \arrow[r] \arrow[d, "\mathrm{Norm}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{l}} \otimes \mathrm{Norm}_{\overline{\mathfrak{m}}}^{\overline{\mathfrak{m}}\overline{\mathfrak{l}}}"] & \mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}\psi_2^{-1}}[H[ml]^{(p)}] \arrow[d, "\mathrm{Norm}_m^{ml}"] \\
\mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}}[H_{\mathfrak{m}}^{(p)}] \otimes_{\mathcal{O}} \mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_2^{-1}}[H_{\overline{\mathfrak{m}}}^{(p)}] \arrow[r] & \mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{\psi_1^{-1}\psi_2^{-1}}[H[m]^{(p)}],
\end{tikzcd}$$

where the horizonal arrows are given by the composition ξ_Δ in (2.5).

Proof. This is clear from the explicit description of the maps involved. □

2.2. **Proof of the tame norm relations.** Let $\mathfrak{m}$ be an ideal of norm m for which we have the class $\tilde{\kappa}_{f,\psi_1,\psi_2,\mathfrak{m}}$ as in Definition 2.1.2.

Proposition 2.2.1. *Let $\mathfrak{l}$ be a prime split in K of norm ℓ coprime to mp. Then*

$$\begin{aligned}\mathrm{Norm}_{K[m]}^{K[m\ell]}(\tilde{\kappa}_{f,\psi_1,\psi_2,\mathfrak{m}\mathfrak{l}}) = (\ell-1)\bigg(a_\ell(f) - \frac{\psi_1(\mathfrak{l})\psi_2(\mathfrak{l})}{\ell}([\mathfrak{l}]\times[\mathfrak{l}]) - \frac{\psi_1(\bar{\mathfrak{l}})\psi_2(\bar{\mathfrak{l}})}{\ell}([\bar{\mathfrak{l}}]\times[\bar{\mathfrak{l}}]) \\ + (1-\ell)\frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{\ell^2}([\mathfrak{l}]\times[\bar{\mathfrak{l}}])\bigg)(\tilde{\kappa}_{f,\psi_1,\psi_2,\mathfrak{m}}).\end{aligned}$$

Proof. In the notations of Theorem 1.3.3, for any $\mathfrak{n} = \mathfrak{f}\mathfrak{m} \in \mathcal{A}_{\mathfrak{f}}$ put

$$H^1(\psi, \mathfrak{f}\mathfrak{m}) := H^1_{\text{ét}}(Y_1(N_\psi m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)) \otimes_{\mathbb{T}'(N_\psi m)} \mathcal{O}[H_{\mathfrak{f}\mathfrak{m}}^{(p)}].$$

Then from Theorem 1.3.3 and Lemma 2.1.3 we have the following commutative diagram:

$$\begin{tikzcd}
H^1\big(\mathbf{Q}, T_f^\vee \otimes H^1(\psi_1, \mathfrak{f}_1\mathfrak{m}\mathfrak{l}) \otimes H^1(\psi_2, \mathfrak{f}_2\overline{\mathfrak{m}}\bar{\mathfrak{l}})\big) \arrow[r] \arrow[d, "1 \otimes \mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{l}} \otimes \mathcal{N}_{\overline{\mathfrak{m}}}^{\overline{\mathfrak{m}}\bar{\mathfrak{l}}}"] & H^1\big(K[ml], T_f^\vee(\psi_1^{-1}\psi_2^{-1})(-1)\big) \arrow[d, "\mathrm{Norm}_m^{ml}"] \\
H^1\big(\mathbf{Q}, T_f^\vee \otimes H^1(\psi_1, \mathfrak{f}_1\mathfrak{m}) \otimes H^1(\psi_2, \mathfrak{f}_2\overline{\mathfrak{m}})\big) \arrow[r] & H^1\big(K[m], T_f^\vee(\psi_1^{-1}\psi_2^{-1})(-1)\big),
\end{tikzcd} \tag{2.6}$$

where the horizontal arrows are given by the maps induced by the composition

$$H^1(\psi_1, \mathfrak{f}_1\mathfrak{m}') \xrightarrow{\nu_{\mathfrak{f}_1\mathfrak{m}'}} \mathrm{Ind}_{K(\mathfrak{f}_1\mathfrak{m}')}^{\mathbf{Q}} \mathcal{O}(\psi_1^{-1}) \simeq \mathrm{Ind}_K^{\mathbf{Q}} \mathcal{O}_{\psi_1^{-1}}[H_{\mathfrak{f}_1\mathfrak{m}'}^{(p)}] \longrightarrow \mathrm{Ind}_K^{\mathbf{Q}} \mathcal{O}_{\psi_1^{-1}}[H_{\mathfrak{m}'}^{(p)}]$$

and likewise for $H^1(\psi_2, \mathfrak{f}_2\overline{\mathfrak{m}}')$, together with the composition ξ_Δ in (2.5).

Now, tracing through the definitions we compute:

$$\begin{aligned}
&(1 \otimes \mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{l}} \otimes \mathcal{N}_{\overline{\mathfrak{m}}}^{\overline{\mathfrak{m}}\overline{\mathfrak{l}}})(\tilde{\kappa}_{m\ell}^{(2)}) \\
&= (1 \otimes \mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{l}} \otimes \mathcal{N}_{\overline{\mathfrak{m}}}^{\overline{\mathfrak{m}}\overline{\mathfrak{l}}})(\mathrm{pr}_{m\ell*}, 1, 1)(\tilde{\kappa}_{m\ell}^{(1)}) \\
&= (\mathrm{pr}_{m*}, 1, 1)(\mathrm{pr}_{\ell*} \otimes \mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{l}} \otimes \mathcal{N}_{\overline{\mathfrak{m}}}^{\overline{\mathfrak{m}}\overline{\mathfrak{l}}})(\tilde{\kappa}_{m\ell}^{(1)}) \\
&= (\mathrm{pr}_{m*}, 1, 1)\Big(\mathrm{pr}_{\ell*} \times (1 \otimes \mathrm{pr}_{1*} - \frac{\psi_1(\mathfrak{l})[\mathfrak{l}]}{\ell} \otimes \mathrm{pr}_{\ell*}) \times (1 \otimes \mathrm{pr}_{1*} - \frac{\psi_2(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}]}{\ell} \otimes \mathrm{pr}_{\ell*})\Big)(\tilde{\kappa}_{m\ell}^{(1)}) \\
&= (\mathrm{pr}_{m*}, 1, 1)\Big((\mathrm{pr}_{\ell*}, \mathrm{pr}_{1*}, \mathrm{pr}_{1*}) - \frac{\psi_1(\mathfrak{l})[\mathfrak{l}]}{\ell}(\mathrm{pr}_{\ell*}, \mathrm{pr}_{\ell*}, \mathrm{pr}_{1*}) - \frac{\psi_2(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}]}{\ell}(\mathrm{pr}_{\ell*}, \mathrm{pr}_{1*}, \mathrm{pr}_{\ell*}) \\
&\quad + \frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{\ell^2}([\mathfrak{l}] \times [\bar{\mathfrak{l}}])(\mathrm{pr}_{\ell*}, \mathrm{pr}_{\ell*}, \mathrm{pr}_{\ell*})\Big)(\tilde{\kappa}_{m\ell}^{(1)}).
\end{aligned}$$

Together with Proposition 1.4.1, we thus obtain

$$\begin{aligned}
(1 \otimes \mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{l}} \otimes \mathcal{N}_{\overline{\mathfrak{m}}}^{\overline{\mathfrak{m}}\overline{\mathfrak{l}}})(\tilde{\kappa}_{m\ell}^{(2)}) &= (\ell - 1)(\mathrm{pr}_{m*}, 1, 1)\Big((T_\ell, 1, 1) - \frac{\psi_1(\mathfrak{l})[\mathfrak{l}]}{\ell}(1, 1, T'_\ell) - (1, T'_\ell, 1)\frac{\psi_2(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}]}{\ell} \\
&\quad + \frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{\ell^2}([\mathfrak{l}] \times [\bar{\mathfrak{l}}])(\ell + 1)\Big)(\tilde{\kappa}_m^{(1)}) \\
&= (\ell - 1)\Big((T_\ell, 1, 1) - \frac{\psi_1(\mathfrak{l})[\mathfrak{l}]}{\ell}(1, 1, T'_\ell) - (1, T'_\ell, 1)\frac{\psi_2(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}]}{\ell} \\
&\quad + \frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{\ell^2}([\mathfrak{l}] \times [\bar{\mathfrak{l}}])(\ell + 1)\Big)(\tilde{\kappa}_m^{(2)}),
\end{aligned}$$

and from this it follows that

$$\begin{aligned}
&(1 \otimes \mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\mathfrak{l}} \otimes \mathcal{N}_{\overline{\mathfrak{m}}}^{\overline{\mathfrak{m}}\overline{\mathfrak{l}}})(\tilde{\kappa}_{f,\psi_1,\psi_2,\mathfrak{m}\mathfrak{l}}^{(4)}) \\
&= (\ell - 1)\Big(a_\ell(f) - \frac{\psi_1(\mathfrak{l})[\mathfrak{l}]}{\ell}(\psi_2(\mathfrak{l})[\mathfrak{l}] + \psi_2(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}]) - (\psi_1(\mathfrak{l})[\mathfrak{l}] + \psi_1(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}])\frac{\psi_2(\bar{\mathfrak{l}})[\bar{\mathfrak{l}}]}{\ell} \\
&\quad + \frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{\ell^2}([\mathfrak{l}] \times [\bar{\mathfrak{l}}])(\ell + 1)\Big)(\tilde{\kappa}_{f,\psi_1,\psi_2,\mathfrak{m}}^{(4)}) \\
&= (\ell - 1)\Big(a_\ell(f) - \frac{\psi_1(\mathfrak{l})\psi_2(\mathfrak{l})}{\ell}([\mathfrak{l}] \times [\mathfrak{l}]) - \frac{\psi_1(\bar{\mathfrak{l}})\psi_2(\bar{\mathfrak{l}})}{\ell}([\bar{\mathfrak{l}}] \times [\bar{\mathfrak{l}}]) \\
&\quad + (1 - \ell)\frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{\ell^2}([\mathfrak{l}] \times [\bar{\mathfrak{l}}])\Big)(\tilde{\kappa}_{f,\psi_1,\psi_2,\mathfrak{m}}^{(4)}).
\end{aligned}$$

In light of the commutative diagram (2.6), this yields the result. □

Remark 2.2.2. The appearance of the factor $(\ell - 1)$ in Proposition 2.2.1 can be traced back to the relations $\deg(\mu_\ell)T_\ell = \mathrm{pr}_{\ell*} \circ \mathrm{pr}_1^*$ and $\deg(\mu_\ell)T'_\ell = \mathrm{pr}_{1*} \circ \mathrm{pr}_\ell^*$, i.e., it is caused by the degeneracy map μ_ℓ. In the next subsection we shall get rid of this extra factor.

Remark 2.2.3. We want to emphasize that Proposition 2.2.1 is the key result for the construction of our anticyclotomic Euler system for $T_f^\vee(\psi_1^{-1}\psi_2^{-1})(-1)$. Indeed, with the factor $(\ell - 1)$ stripped out, the term in the right-hand side of Proposition 2.2.1 can be massaged to agree with the local Euler factor at $\mathfrak{l}$ of the Galois representation $[T_f^\vee(\psi_1^{-1}\psi_2^{-1})(-1)]^\vee(1) = T_f(\psi_1\psi_2)(2)$, giving the correct norm relations.

2.2.1. *Removing the extra factor* $(\ell-1)$. Adapting some ideas from [DR17, §1.4], we now introduce a modification of the classes $\widetilde{\kappa}_{f,\psi_1,\psi_2,\mathfrak{m}}$ for which we can prove an analogue of Proposition 2.2.1 without the extra factor $(\ell-1)$.

We begin by noting that for any prime $\ell\nmid N$ the degeneracy maps $\mathrm{pr}_1,\mathrm{pr}_\ell : Y_1(N\ell)\to Y_1(N)$ can be factored as

$$\begin{tikzcd} Y_1(N\ell) \arrow[d, "\mu_\ell"'] \arrow[dr, "\mathrm{pr}_1"] & \\ Y(1,N(\ell)) \arrow[r, "\pi_1"'] & Y_1(N) \end{tikzcd} \qquad \begin{tikzcd} Y_1(N\ell) \arrow[d, "\mu_\ell"'] \arrow[dr, "\mathrm{pr}_\ell"] & \\ Y(1,N(\ell)) \arrow[r, "\pi_\ell"'] & Y_1(N), \end{tikzcd}$$

where π_1 and π_ℓ are a non-Galois coverings of degree $\ell+1$, and we recall that μ_ℓ is a cyclic Galois covering of degree $\ell-1$.

Denote by

$$D_m=\{(\langle d\rangle,\langle d\rangle): d\in(\mathbf{Z}/Nm\mathbf{Z})^\times,\ d\equiv 1 \pmod N\}$$

the set of diamond operators acting diagonally on $Y_1(Nm)^2$. Set

$$W_1(Nm)=(Y_1(Nm)\times Y_1(Nm))/D_m,$$

and denote by $d_m: Y_1(Nm)^2\to W_1(Nm)$ the natural projection map, which is an étale morphism of degree $\phi(m)=|(\mathbf{Z}/m\mathbf{Z})^\times|$.

Let $\tilde{\kappa}_m^{(1)}$ be as in (2.2), and denote by

$$\kappa_m^{(1)}\in H^1\big(\mathbf{Q},H^3_{\text{ét}}(Y(1,N(m))_{\overline{\mathbf{Q}}}\times W_1(Nm)_{\overline{\mathbf{Q}}},\mathbf{Z}_p)(2)\big)$$

the image of $(\mu_{m*},1,1)(\tilde{\kappa}_m^{(1)})$ under the natural map induced by d_m; thus the class $\kappa_m^{(1)}$ is defined by the relation

$$(\mu_{m*},d_{m*})\tilde{\kappa}_m^{(1)}=\phi(m)\kappa_m^{(1)}. \tag{2.7}$$

Proposition 2.2.4. *For a prime number ℓ and a positive integer m such that $(m,\ell)=1$ and $(m\ell,pN)=1$, we have*

$$(\pi_{i*},\mathrm{pr}_{j*},\mathrm{pr}_{k*})\kappa_{m\ell}^{(1)}=(\bigstar)\kappa_m^{(1)},$$

where

(i,j,k)	$\bigstar$	(i,j,k)	$\bigstar$
$(\ell,1,1)$	$(T_\ell,1,1)$	$(\ell,1,\ell)$	$(1,T'_\ell,1)$
$(1,\ell,1)$	$(1,T_\ell,1)$	$(\ell,\ell,1)$	$(1,1,T'_\ell)$
$(1,1,\ell)$	$(1,1,T_\ell)$	$(1,1,1)$	$(\ell+1)$
$(1,\ell,\ell)$	$(T'_\ell,1,1)$	(ℓ,ℓ,ℓ)	$(\ell+1)$

Proof. Directly from the definitions we find

$$\begin{aligned}(\mu_{m*},d_{m*})(\mathrm{pr}_{\ell*},\mathrm{pr}_{1*},\mathrm{pr}_{1*})\tilde{\kappa}_{m\ell}^{(1)}&=(\pi_{\ell*},\mathrm{pr}_{1*},\mathrm{pr}_{1*})(\mu_{m\ell*},d_{m\ell*})\tilde{\kappa}_m^{(1)}\\&=\phi(m\ell)(\pi_{\ell*},\mathrm{pr}_{1*},\mathrm{pr}_{1*})\kappa_m^{(1)},\end{aligned}$$

while on the other hand, from Proposition 1.4.1 and (2.7) we have

$$\begin{aligned}(\mu_{m*},d_{m*})(\mathrm{pr}_{\ell*},\mathrm{pr}_{1*},\mathrm{pr}_{1*})\tilde{\kappa}_{m\ell}^{(1)}&=(\mu_{m*},d_{m*})(\ell-1)(T_\ell,1,1)\tilde{\kappa}_m^{(1)}\\&=\phi(m)(\ell-1)(T_\ell,1,1)\kappa_m^{(1)}.\end{aligned}$$

Since $\phi(m\ell)=(\ell-1)\phi(m)$ under our assumptions, this shows the result in the case $(i,j,k)=(\ell,1,1)$ and the other cases are shown in the same manner. $\square$

Now we want to proceed as above to obtain from the new $\kappa_m^{(1)}$ a construction of classes satisfying the correct norm relations (i.e., without the factor $\ell-1$). This requires a careful study of the étale cohomology of the quotient $Y(1,N(m))\times W_1(Nm)$.

We begin with the Hochschild–Serre spectral sequence:

$$E_2^{p,q} = H^p\big(D_m, H^q_{\text{ét},c}(Y(1,N(m))_{\overline{\mathbf{Q}}} \times Y_1(Nm)^2_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)\big) \Rightarrow H^{p+q}_{\text{ét},c}\big(Y(1,N(m))_{\overline{\mathbf{Q}}} \times W_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p\big).$$

This yields the exact sequence

$$E \longrightarrow H^3_{\text{ét},c}\big(Y(1,N(m))_{\overline{\mathbf{Q}}} \times W_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p\big) \xrightarrow{(1,d_m^*)} E_2^{0,3} \xrightarrow{d_2^{0,3}} E_2^{2,2}, \tag{2.8}$$

where E is naturally identified with a subquotient of $E_2^{1,2} \oplus E_2^{2,1}$. Thus we see that the difference between the two middle pieces are classes coming from $H^q_{\text{ét},c}(Y(1,N(m))_{\overline{\mathbf{Q}}} \times Y_1(Nm)^2_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)$ with $0 \le q \le 2$. From the Künneth decomposition, each of these classes will have a factor from either $H^0_{\text{ét},c}(Y(1,N(m))_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)$ or $H^0_{\text{ét},c}(Y_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)$, and so they will be annihilated after localisation at a non-Eisenstein ideal $\mathcal{I}$ of $\mathbb{T}'_{Nm}$, therefore obtaining an isomorphism

$$H^3_{\text{ét},c}\big(Y(1,N(m))_{\overline{\mathbf{Q}}} \times W_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p\big)_{\mathcal{I}} \xrightarrow{(1,d_m^*)} H^3_{\text{ét},c}\big(Y(1,N(m))_{\overline{\mathbf{Q}}} \times Y_1(Nm)^2_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)\big)^{D_m}_{\mathcal{I}}$$

after localizing (2.8) at $\mathcal{I}$. By Poincaré duality, from (2.8) we obtain a map

$$H^3_{\text{ét}}\big(Y(1,N(m))_{\overline{\mathbf{Q}}} \times W_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p\big) \xleftarrow{(1,d_{m*})} H^3_{\text{ét}}\big(Y(1,N(m))_{\overline{\mathbf{Q}}} \times Y_1(Nm)^2_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)\big)_{D_m} \tag{2.9}$$

whose kernel and cokernel will also be annihilated by localization at a non-Eisenstein ideal $\mathcal{I}$. Additionally, we recall the following lemma:

Lemma 2.2.5. *For $\mathcal{I}$ a non-Eisenstein maximal ideal of $\mathbb{T}'_N$, we have natural isomorphisms*

$$H^1_c(Y_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)_{\mathcal{I}} \xrightarrow{\sim} H^1(X_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)_{\mathcal{I}} \xrightarrow{\sim} H^1(Y_1(N)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)_{\mathcal{I}}.$$

Proof. Immediate from the Manin–Drinfeld theorem (see e.g. [LLZ14, Prop. 4.1.3]). □

Hence from (2.9), Lemma 2.2.5, and the Künneth decomposition, it follows that after localization at a non-Eisenstein maximal ideal $\mathcal{I}$ we get a natural map

$$\begin{aligned} H^3_{\text{ét}}(Y(1,N(m))_{\overline{\mathbf{Q}}} \times W_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)_{\mathcal{I}} \xrightarrow{(1,d_{m*}^{-1})} & H^1_{\text{ét}}(Y(1,N(m))_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)_{\mathcal{I}} \otimes \\ & H^1_{\text{ét}}(Y_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)_{\mathcal{I}} \otimes_{D_m} H^1_{\text{ét}}(Y_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p))_{\mathcal{I}}. \end{aligned} \tag{2.10}$$

Next, for ψ a Hecke character of K of conductor $\mathfrak{f}$ and $\mathfrak{m}$ an ideal of K of norm m coprime to p divisible only by primes that split in K, we let $\mathcal{I}_{\mathfrak{fm}}$ be the kernel of the composite map

$$\mathbb{T}'_{Nm} \xrightarrow{\phi_{\mathfrak{fm}}} \mathcal{O}[H_{\mathfrak{fm}}] \xrightarrow{\text{aug}} \mathcal{O} \longrightarrow \mathcal{O}/\mathfrak{P}.$$

By [LLZ15, Prop 5.1.2], the maximal ideal $\mathcal{I}_{\mathfrak{fm}}$ is non-Eisenstein, p-ordinary and p-distinguished. At a later step, we shall look at the module

$$\mathcal{O}[H^{(p)}_{\mathfrak{fm}}] \otimes_{\mathbb{T}'(Nm)\otimes\mathbf{Z}_p,\phi_{\mathfrak{fm}}} H^1_{\text{ét}}(Y_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1)),$$

and it is clear that the map from $H^1_{\text{ét}}(Y_1(Nm)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p(1))$ to this module factors through completion at $\mathcal{I}_{\mathfrak{fm}}$. Moreover, assuming that f is non-Eisenstein modulo $\mathfrak{P}$ (i.e. $T_f^\vee$ is residually irreducible), we can choose an auxiliary prime $q \nmid Nmp$ with $1+q-a_q(f) \in \mathbf{Z}_p^\times$ and $\frac{1+q-T_q}{1+q-a_q(f)} \notin \mathcal{I}_{\mathfrak{fm}}$, hence defining an invertible element after localization at $\mathcal{I}_{\mathfrak{fm}}$ that fixes the f-isotypic component we are interested in.

Now we put $\kappa_m^{(2)} := (\pi_{m*}, 1, 1)\kappa_m^{(1)}$, and define

$$\kappa_m^{(3)} \in H^1\big(\mathbf{Q}, H^1_{\text{ét}}(Y(1,N(m))_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)_{\mathcal{I}} \otimes H^1_{\text{ét}}(Y_1(N_{\psi_1}m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)_{\mathcal{I}} \otimes_{D_m} H^1_{\text{ét}}(Y_1(N_{\psi_2}m)_{\overline{\mathbf{Q}}}, \mathbf{Z}_p)_{\mathcal{I}}(-1)\big),$$

with $\mathcal{I} = \mathcal{I}_{\mathfrak{fm}}$, to be the image of $\kappa_m^{(2)}$ under the map (2.10) composed with the natural degeneracy maps $Y_1(Nm) \to Y_1(N_{\psi_i}m)$ $(i = 1,2)$.

Note that taking $D_m^{(p)}$-coinvariants (where $D_m^{(p)}$ denotes the p-part of D_m) is compatible with the diagonal map ξ_Δ in (2.5), since by Theorem 1.3.1 for $(\langle d\rangle, \langle d\rangle) \in D_m^{(p)}$ we have $\phi_{\mathfrak{m}}(\langle d\rangle') \times \phi_{\overline{\mathfrak{m}}}(\langle d\rangle') =$

$[d] \times [d] \in H_{\mathfrak{m}}^{(p)} \times H_{\bar{\mathfrak{m}}}^{(p)}$, and this is in the kernel of π_Δ. Thus applying to $\kappa_m^{(3)}$ the same process we used above to go from $\tilde{\kappa}_m^{(3)}$ to the class $\tilde{\kappa}_{f,\psi_1,\psi_2,\mathfrak{m}}$ of Definition 2.1.2 we obtain

$$\kappa_{f,\psi_1,\psi_2,\mathfrak{m}} \in H^1\big(K[m], T_f^\vee(\psi_1^{-1}\psi_2^{-1})(-1)\big).$$

Proposition 2.2.6. *Suppose f is non-Eisenstein modulo $\mathfrak{P}$. Let $\mathfrak{m}$ be an ideal of $\mathcal{O}_K$ of norm m divisible only by primes split in K, and let $\mathfrak{l}$ be a prime split in K of norm ℓ coprime to mp. Then*

$$\begin{aligned}\mathrm{Norm}_{K[m]}^{K[m\ell]}(\kappa_{f,\psi_1,\psi_2,\mathfrak{m}\mathfrak{l}}) = \bigg(a_\ell(f) - \frac{\psi_1(\mathfrak{l})\psi_2(\mathfrak{l})}{\ell}([\mathfrak{l}]\times[\mathfrak{l}]) - \frac{\psi_1(\bar{\mathfrak{l}})\psi_2(\bar{\mathfrak{l}})}{\ell}([\bar{\mathfrak{l}}]\times[\bar{\mathfrak{l}}]) \\ + (1-\ell)\frac{\psi_1(\mathfrak{l})\psi_2(\bar{\mathfrak{l}})}{\ell^2}([\mathfrak{l}]\times[\bar{\mathfrak{l}}])\bigg)(\kappa_{f,\psi_1,\psi_2,\mathfrak{m}}).\end{aligned}$$

Proof. After the above discussion, the same calculation as in the proof of Proposition 2.2.1 applies, replacing the use of Proposition 1.4.1 by Proposition 2.2.4. □

Thus we arrive at the following theorem:

Theorem 2.2.7. *Suppose $p \nmid 6h_K$ and f is non-Eisenstein modulo $\mathfrak{P}$. Let $\mathfrak{m}$ run over the ideals of $\mathcal{O}_K$ divisible only by primes that are split in K with $m = N_{K/\mathbf{Q}}(\mathfrak{m})$ coprime to p. Then there exists a collection of cohomology classes*

$$z_{f,\psi_1,\psi_2,\mathfrak{m}} \in H^1\big(K[m], T_f^\vee(\psi_1^{-1}\psi_2^{-1})(-1)\big)$$

such that for every split prime $\mathfrak{l}$ of $\mathcal{O}_K$ of norm ℓ with $(\ell, mp) = 1$ we have the norm relation

$$\mathrm{Norm}_{K[m]}^{K[m\ell]}(z_{f,\psi_1,\psi_2,\mathfrak{m}\mathfrak{l}}) = P_{\mathfrak{l}}(\mathrm{Frob}_{\mathfrak{l}})(z_{f,\psi_1,\psi_2,\mathfrak{m}}),$$

where $P_{\mathfrak{l}}(X) = \det(1 - X \cdot \mathrm{Frob}_{\mathfrak{l}} \,|\, T_f(\psi_1\psi_2)(2))$.

Proof. Denote by $Q_{\mathfrak{l}}$ the factor appearing in the right-hand side of Proposition 2.2.6. Recalling that $[\mathfrak{l}]\times[\mathfrak{l}]$ corresponds to $\mathrm{Frob}_{\mathfrak{l}} \in H[m]^{(p)}$ under the map π_Δ of Proposition 2.1.1, we find the following congruences as endomorphisms of $H^1(K[m], T_f^\vee(\psi_1^{-1}\psi_2^{-1})(-1))$:

$$\begin{aligned}&-\psi_1\psi_2(\mathfrak{l})([\mathfrak{l}]\times[\mathfrak{l}])\cdot Q_{\mathfrak{l}} \\ &\equiv -a_\ell(f)\psi_1\psi_2(\mathfrak{l})([\mathfrak{l}]\times[\mathfrak{l}]) + \frac{\psi_1\psi_2(\mathfrak{l})^2}{\ell}([\mathfrak{l}]\times[\mathfrak{l}])^2 + \frac{\psi_1\psi_2((\ell))}{\ell}([\ell]\times[\ell]) \\ &\equiv P_{\mathfrak{l}}(\mathrm{Frob}_{\mathfrak{l}}) \pmod{\ell - 1},\end{aligned}$$

using the relation $\psi_1\psi_2((\ell)) = \chi_{\psi_1}\chi_{\psi_2}(\ell)\ell^2 = \ell^2$ and the fact that $[\ell]\times[\ell]$ is in the kernel of π_Δ for the second congruence. Therefore, by Lemma 9.6.1 and 9.6.3 in [Rub00], the existence of classes $z_{f,\psi_1,\psi_2,\mathfrak{m}}$ satisfying the stated norm relations follows from Proposition 2.2.6. □

Remark 2.2.8. Similar to what we did for $\mathfrak{l}$ a split prime of K, when $\mathfrak{l} = (\ell)$ is inert in K, we also obtain such a norm relation like in Theorem 2.2.7. Remember that in this case, we push forward from level $N\ell^2$ to level N. First, note that the norm map from Proposition 1.3.1 is then given by

$$\mathcal{N}_{\mathfrak{n}}^{\mathfrak{n}(\ell)} = 1 \otimes \mathrm{pr}_{1*} - \frac{\psi(\ell)[(\ell)]}{\ell^2} \otimes \mathrm{pr}_{\ell\ell*}$$

Second, to calculate $(1 \otimes \mathcal{N}_{\mathfrak{m}}^{\mathfrak{m}\ell} \otimes \mathcal{N}_{\bar{\mathfrak{m}}}^{\bar{\mathfrak{m}}\ell})(\kappa_{m\ell}^{(2)})$, just like in Proposition 2.2.1, we use the table in Proposition 1.4.1 together with

$$\begin{aligned}(\mathrm{pr}_{\ell*}, \mathrm{pr}_{1*}, \mathrm{pr}_{1*})(T_\ell, 1, 1)\kappa_{m\ell}^{(2)} &= \{(T_\ell^2, 1, 1) - (\ell+1)(\langle\ell\rangle, 1, 1)\}\kappa_m^{(2)}, \\ (\mathrm{pr}_{\ell*}, \mathrm{pr}_{\ell*}, \mathrm{pr}_{1*})(1, 1, T'_\ell)\kappa_{m\ell}^{(2)} &= \{(1, 1, T'^2_\ell) - (\ell+1)(1, 1, \langle\ell\rangle')\}\kappa_m^{(2)},\end{aligned}$$

and arrive at

$$\mathrm{Norm}_{K[m]}^{K[m\ell]}(\kappa^{(6)}_{f,\psi_1,\psi_2,\mathfrak{m}\mathfrak{l}}) = (\ell-1)\left(a_\ell(f)^2 - (\ell+1) - \frac{2(\ell+1)}{\ell}[\ell]\times[\ell] + (\ell+1)[\ell]\times[\ell]\right)(\kappa^{(6)}_{f,\psi_1,\psi_2,\mathfrak{m}}).$$

Instead of Proposition 2.1.1, we use the following exact sequence

$$1 \longrightarrow H_m \xrightarrow{\Delta} H_m \times H_m \longrightarrow H_m \longrightarrow 1,$$

combining with the quotient $H_m \to H[m]$, which makes $[\ell] \times [\ell]$ acting trivially on the cohomology class. After removing the extra factor $(\ell - 1)$ and multiplying with -1 on the RHS factor, we obtain the correct Euler factor modulo $\ell^2 - 1$:

$$P_{\mathfrak{l}}(\mathrm{Frob}_{\mathfrak{l}}) = 2 + 2\ell - a_\ell(f)^2.$$

Note that $(\ell+1)/\ell \equiv (\ell+1)\ell = \ell^2 + \ell \equiv 1 + \ell \pmod{\ell^2 - 1}$ and the twist $\psi_1(\ell)\psi_2(\ell)/\ell^2 = 1$.

2.3. **Construction for general weights and wild norm relations.** Keeping the setting introduced at the beginning of §2, we now extend the construction of the preceding subsection to arbitrary weights $k \geq 1$ and $k_1, k_2 \geq 1$, and prove that the resulting classes $\kappa_{f,\psi_1,\psi_2,\mathfrak{m}}$ also satisfy the wild norm relations (i.e., they are universal norms for the anticyclotomic $\mathbf{Z}_p$-extension of K). In this subsection, we assume in addition that

$$p = \mathfrak{p}\bar{\mathfrak{p}} \text{ splits in } K, \tag{2.11}$$

with $\mathfrak{p}$ the prime of K above p induced by our fixed embedding $i_p : \overline{\mathbf{Q}}_p \hookrightarrow \overline{\mathbf{Q}}_p$, and that

$$p \nmid h_K, \tag{2.12}$$

where h_K is the class number of K.

Let $\Gamma_{\mathfrak{p}}$ the Galois group of the unique $\mathbf{Z}_p$-extension of K unramified outside $\mathfrak{p}$, and let ψ_0 be the unique Hecke character of K of infinity type $(-1, 0)$, conductor $\mathfrak{p}$, and whose $\mathfrak{P}$-adic avatar factors through $\Gamma_{\mathfrak{p}}$. (As noted in [BL18, §3.2.1], the uniqueness of ψ_0 is a consequence of our assumption (2.12).) The characters ψ_1, ψ_2 fixed at the beginning of this section can be uniquely written as

$$\psi_i = \xi_i \psi_0^{k_i - 1},$$

where ξ_i is a ray class character of K of conductor dividing $\mathfrak{f}_i\mathfrak{p}$. Viewing ψ_0 and ξ_i as characters on $H_{\mathfrak{f}_i\mathfrak{p}^\infty}$ (noting that $\Gamma_{\mathfrak{p}}$ is a quotient the latter group), we consider the formal q-expansion

$$\boldsymbol{\theta}_{\xi_i}(q) = \sum_{(\mathfrak{a}, \mathfrak{f}_i\mathfrak{p})=1} \xi_i\psi_0(\mathfrak{a})[\mathfrak{a}]q^{N_{K/\mathbf{Q}}(\mathfrak{a})} \in \Lambda_{\mathfrak{p}}[\![q]\!],$$

where $\Lambda_{\mathfrak{p}} = \mathcal{O}[\![\Gamma_{\mathfrak{p}}]\!]$. Identifying $\Gamma_{\mathfrak{p}}$ with $\Gamma = 1 + p\mathbf{Z}_p$ via the (geometrically normalised) local Artin map, inducing an identification $\Lambda_{\mathfrak{p}} \simeq \mathcal{O}[\![\Gamma]\!]$, then $\boldsymbol{\theta}_{\xi_i}$ is the Hida family passing through θ_{ψ_i}, in the sense that the specialisation of $\boldsymbol{\theta}_{\xi_i}$ at weight k_i and trivial character recovers the ordinary p-stabilization of θ_{ψ_i} (see §4.1.1 below for our conventions regarding Hida families).

In the following, we let $\boldsymbol{f}$ be the Hida family associated to f, and

$$(\boldsymbol{g}, \boldsymbol{h}) = (\boldsymbol{\theta}_{\xi_1}, \boldsymbol{\theta}_{\xi_2})$$

be the CM Hida families associated to ψ_1 and ψ_2, respectively. We also use κ_f, κ_g, and κ_h to denote the Dirichlet characters modulo p giving the p-part of the tame characters of $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{h}$, respectively.

Let $\Lambda = \mathbf{Z}_p[\![\Gamma]\!]$. For each $i \in \mathbf{Z}/(p-1)\mathbf{Z}$ denote by $\kappa_i : \mathbf{Z}_p^\times \to \Lambda^\times$ the character $z \mapsto \omega^i(z)[\langle z \rangle]$, where $\omega : \mathbf{Z}_p^\times \to \mathbf{Z}_p^\times$ is the reduction map composed with the Teichmüller lift, and $\langle z \rangle = z\omega^{-1}(z) \in 1 + p\mathbf{Z}_p$.

Set $\mathrm{T} = \mathbf{Z}_p^\times \times \mathbf{Z}_p$, $\mathrm{T}' = p\mathbf{Z}_p \times \mathbf{Z}_p^\times$. Let $\mathfrak{m}_\Lambda$ be the maximal ideal of Λ, denote by $\mathrm{Cont}(\mathbf{Z}_p, \Lambda)$ the Λ-module of continuous functions on $\mathbf{Z}_p$ with values in Λ, and put $\kappa = \kappa_i$ for some $i \in \mathbf{Z}/(p-1)\mathbf{Z}$. We consider the Λ-modules

$$\mathcal{A}_\kappa = \left\{ f : \mathrm{T} \to \Lambda \mid f(1, z) \in \mathrm{Cont}(\mathbf{Z}_p, \Lambda) \text{ and } f(a \cdot t) = \kappa(a) \cdot f(t) \text{ for all } a \in \mathbf{Z}_p^\times,\, t \in \mathrm{T} \right\},$$
$$\mathcal{A}'_\kappa = \left\{ f : \mathrm{T}' \to \Lambda \mid f(pz, 1) \in \mathrm{Cont}(\mathbf{Z}_p, \Lambda) \text{ and } f(a \cdot t) = \kappa(a) \cdot f(t) \text{ for all } a \in \mathbf{Z}_p^\times,\, t \in \mathrm{T}' \right\}$$

equipped with the $\mathfrak{m}_\Lambda$-adic topology, and set

$$\mathcal{D}_\kappa = \mathrm{Hom}_{\mathrm{cont},\Lambda}(\mathcal{A}_\kappa, \Lambda), \quad \mathcal{D}'_\kappa = \mathrm{Hom}_{\mathrm{cont},\Lambda}(\mathcal{A}'_\kappa, \Lambda)$$

equipped with the weak-$*$ topology.

As in [BSV22, Eq. (81)], the evaluation $\mathcal{A}^{\cdot}_{\kappa} \otimes_{\Lambda} \mathcal{D}^{\cdot}_{\kappa} \to \Lambda$ gives rise to a Λ-module homomorphism

$$\xi_i : H^1(\Gamma, \mathcal{A}_\kappa) \to \mathrm{Hom}_\Lambda(H^1_c(\Gamma, \mathcal{D}_\kappa), \Lambda).$$

Similarly, the determinant map $\det : \mathrm{T} \times \mathrm{T}' \to \mathbf{Z}_p^\times$ defined by $\det((x_1, x_2), (y_1, y_2)) = x_1y_2 - x_2y_1$, composed with $\kappa_i : \mathbf{Z}_p^\times \to \Lambda^\times$ gives rise to

$$\zeta_i : \mathrm{Hom}_\Lambda(H^1_c(\Gamma, \mathcal{D}_\kappa), \Lambda) \to H^1(\Gamma, \mathcal{D}'_\kappa)(-\kappa_i).$$

Then for any weight $k = r + 2 \geq 2$ with $k \equiv i \pmod{p-1}$ we have specialization maps

$$\rho_r : H^1(\Gamma, \mathcal{A}_\kappa) \to H^1_{\text{ét}}(Y_{\overline{\mathbf{Q}}}, \mathscr{S}_r), \quad \rho_r : H^1(\Gamma, \mathcal{D}'_\kappa) \to H^1_{\text{ét}}(Y_{\overline{\mathbf{Q}}}, \mathscr{L}_r)$$

fitting into the commutative diagram

$$\begin{tikzcd} H^1(\Gamma, \mathcal{A}_\kappa) \arrow[r, "\zeta_i \circ \xi_i"] \arrow[d, "\rho_r"] & H^1(\Gamma, \mathcal{D}'_\kappa)(-\kappa_i) \arrow[d, "\rho_r"] \\ H^1_{\text{ét}}(Y_{\overline{\mathbf{Q}}}, \mathscr{S}_r) \arrow[r, "\mathtt{s}_r"] & H^1_{\text{ét}}(Y_{\overline{\mathbf{Q}}}, \mathscr{L}_r)(-r). \end{tikzcd}$$

To ease notation, set $Y(m,p) = Y(1, Nm(p))$ and denote by $\Gamma(m,p)$ the associated congruence subgroup. Adopting the notations from [BSV22, §8.1] (but working the modules of continuous functions $\mathcal{A}^{\cdot}_{\kappa}$ and their duals $\mathcal{D}^{\cdot}_{\kappa}$ as above, rather than the analogous spaces of locally analytic functions considered in [BSV22]), we denote by

$$\widetilde{\boldsymbol{\kappa}}^{(1)}_m \in H^1\big(\mathbf{Q}, H^1(\Gamma(m,p), \mathcal{D}'_{\kappa_f}) \hat{\otimes}_{\mathcal{O}} H^1(\Gamma(m,p), \mathcal{D}'_{\kappa_g}) \hat{\otimes}_{\mathcal{O}} H^1(\Gamma(m,p), \mathcal{D}'_{\kappa_h})(2 - \kappa^*_{\boldsymbol{fgh}})\big) \tag{2.13}$$

the image of the element $\mathbf{Det}^{\boldsymbol{fgh}}_{Nm,p} \in H^0_{\text{ét}}(Y(m,p), \boldsymbol{\mathcal{A}}'_{\kappa_f} \otimes \boldsymbol{\mathcal{A}}_{\kappa_g} \otimes \boldsymbol{\mathcal{A}}_{\kappa_h}(-\kappa^*_{\boldsymbol{fgh}}))$ defined in [BSV22, §8.1] under the composition

$$\begin{aligned} &H^0_{\text{ét}}\big(Y(m,p), \boldsymbol{\mathcal{A}}'_{\kappa_f} \otimes \boldsymbol{\mathcal{A}}_{\kappa_g} \otimes \boldsymbol{\mathcal{A}}_{\kappa_h}(-\kappa^*_{\boldsymbol{fgh}})\big) \\ &\xrightarrow{d_*} H^4_{\text{ét}}\big(Y(m,p)^3, \boldsymbol{\mathcal{A}}'_{\kappa_f} \boxtimes \boldsymbol{\mathcal{A}}_{\kappa_g} \boxtimes \boldsymbol{\mathcal{A}}_{\kappa_h}(-\kappa^*_{\boldsymbol{fgh}}) \otimes \mathbf{Z}_p(2)\big) \\ &\xrightarrow{\mathtt{HS}} H^1(\mathbf{Q}, H^3_{\text{ét}}(Y(m,p)^3_{\overline{\mathbf{Q}}}, \boldsymbol{\mathcal{A}}'_{\kappa_f} \boxtimes \boldsymbol{\mathcal{A}}_{\kappa_g} \boxtimes \boldsymbol{\mathcal{A}}_{\kappa_h})(2 + \kappa^*_{\boldsymbol{fgh}})) \\ &\xrightarrow{\mathtt{K}} H^1\big(\mathbf{Q}, H^1(\Gamma(m,p), \mathcal{A}'_{\kappa_f}) \hat{\otimes}_{\mathcal{O}} H^1(\Gamma(m,p), \mathcal{A}_{\kappa_g}) \hat{\otimes}_{\mathcal{O}} H^1(\Gamma(m,p), \mathcal{A}_{\kappa_h})(2 + \kappa^*_{\boldsymbol{fgh}})\big) \\ &\xrightarrow{(w_p \otimes 1 \otimes 1)_*} H^1\big(\mathbf{Q}, H^1(\Gamma(m,p), \mathcal{A}_{\kappa_f}) \hat{\otimes}_{\mathcal{O}} H^1(\Gamma(m,p), \mathcal{A}_{\kappa_g}) \hat{\otimes}_{\mathcal{O}} H^1(\Gamma(m,p), \mathcal{A}_{\kappa_h})(2 + \kappa^*_{\boldsymbol{fgh}})\big) \\ &\xrightarrow{\mathtt{s}_{\boldsymbol{fgh}}} H^1\big(\mathbf{Q}, H^1(\Gamma(m,p), \mathcal{D}'_{\kappa_f}) \hat{\otimes}_{\mathcal{O}} H^1(\Gamma(m,p), \mathcal{D}'_{\kappa_g}) \hat{\otimes}_{\mathcal{O}} H^1(\Gamma(m,p), \mathcal{D}'_{\kappa_h})(2 - \kappa^*_{\boldsymbol{fgh}})\big), \end{aligned}$$

where $\mathtt{s}_{\boldsymbol{fgh}}$ is the tensor of the compositions $\zeta_i \circ \xi_i$ for $i = \kappa_f, \kappa_g, \kappa_h$.

As we did in §2.2.1, replacing the second and third copies of $Y(m,p)$ in the above construction by the quotient $(Y(m,p) \times Y(m,p))/D_m$, where D_m is the group of diamond operators, we obtain the class

$$\boldsymbol{\kappa}^{(1)}_m \in H^1\big(\mathbf{Q}, H^1(\tilde{\Gamma}(m,p), \mathcal{D}'_{\kappa_f}) \hat{\otimes}_{\mathcal{O}} H^1(\Gamma(m,p), \mathcal{D}'_{\kappa_g}) \hat{\otimes}_{\mathcal{O}[D_m]} H^1(\Gamma(m,p), \mathcal{D}'_{\kappa_h})(2 - \kappa^*_{\boldsymbol{fgh}})\big) \tag{2.14}$$

determined by the relation $\phi(m)\boldsymbol{\kappa}^{(1)}_m = (\mu_{m*}, d_{m*})\tilde{\boldsymbol{\kappa}}^{(1)}_m$, where $\tilde{\Gamma}(m,p) = \Gamma(1, N(mp))$, and we put

$$\boldsymbol{\kappa}^{(2)}_m := (\pi_{m*}, 1, 1)\boldsymbol{\kappa}^{(1)}_m. \tag{2.15}$$

Proposition 2.3.1. *For a prime number ℓ and a positive integer m with $(m\ell, pN) = 1$ we have*

$$(\pi_{i*}, \mathrm{pr}_{j*}, \mathrm{pr}_{k*})\boldsymbol{\kappa}^{(1)}_{m\ell} = (\bigstar)\boldsymbol{\kappa}^{(1)}_m,$$

where

(i,j,k)	$\bigstar$
$(\ell,1,1)$	$(T_\ell,1,1)$
$(1,\ell,1)$	$(1,T_\ell,1)$
$(1,1,\ell)$	$(1,1,T_\ell)$
$(1,\ell,\ell)$	$\kappa^*_{\boldsymbol{fgh}}(\ell)\kappa^*_{\boldsymbol{f}}(\ell)^{-1}(T'_\ell,1,1)$
$(\ell,1,\ell)$	$\kappa^*_{\boldsymbol{fgh}}(\ell)\kappa^*_{\boldsymbol{g}}(\ell)^{-1}(1,T'_\ell,1)$
$(\ell,\ell,1)$	$\kappa^*_{\boldsymbol{fgh}}(\ell)\kappa^*_{\boldsymbol{h}}(\ell)^{-1}(1,1,T'_\ell)$

If we also have that $(\ell,m)=1$ *then*

(i,j,k)	$\bigstar$
$(1,1,1)$	$(\ell+1)$
(ℓ,ℓ,ℓ)	$(\ell+1)\kappa^*_{\boldsymbol{fgh}}(\ell)$

Proof. With π_i replaced by pr_i and the classes $\boldsymbol{\kappa}_m^{(1)}$ replaced by $\tilde{\boldsymbol{\kappa}}_m^{(1)}$, the stated relations with an extra factor of $\ell-1$ follow immediately from equations (174) and (176) in [BSV22] (adding the prime ℓ to the level, rather than p). The stated relations for $\boldsymbol{\kappa}_m^{(1)}$ then follow in the same way as in Proposition 2.2.4. $\square$

Assume that

$$\xi_i\psi_0 \not\equiv \omega \pmod{\mathfrak{P}}$$

for $i=1,2$. Then for every $r\geq 0$ Condition $\spadesuit$ holds, and so by Theorem 1.3.3, for every ideal $\mathfrak{m}$ of norm m coprime to p the Hecke algebra homomorphism

$$\phi_{\mathfrak{f}_i\mathfrak{m}\mathfrak{p}^r}:\mathbb{T}(1,N_{\psi_i}mp^r)'\to\mathcal{O}[H_{\mathfrak{f}_i\mathfrak{m}\mathfrak{p}^r}]$$

associated to $\xi_i\psi_0$ induces an isomorphism

$$\nu_{\mathfrak{f}_i\mathfrak{m}\mathfrak{p}^r}:H^1_{\text{ét}}(Y_1(N_{\psi_i}mp^r)_{\overline{\mathbf{Q}}},\mathbf{Z}_p(1))\otimes\mathcal{O}[H^{(p)}_{\mathfrak{f}_i\mathfrak{m}\mathfrak{p}^r}]\xrightarrow{\simeq}\mathrm{Ind}^{\mathbf{Q}}_{K(\mathfrak{f}_i\mathfrak{m}\mathfrak{p}^r)}\mathcal{O}((\xi_i\psi_0)^{-1})$$

safisfying the natural compatibility as r varies. On the other hand, as noted in [BSV22, p. 38], from a slight variant of Lemma 6.8 in [GS93] we obtain a $G_{\mathbf{Q}}$-module isomorphisms

$$H^1(\Gamma(1,N_{\psi_i}m(p)),\mathcal{D}'_{\kappa_i})(1)\simeq e_i\varprojlim_r H^1_{\text{ét}}(Y_1(N_{\psi_i}mp^r)_{\overline{\mathbf{Q}}},\mathbf{Z}_p(1)). \tag{2.16}$$

Therefore combining (2.16) with the inverse limit of the isomorphisms $\nu_{\mathfrak{f}_i\mathfrak{m}\mathfrak{p}^r}$ we obtain the $G_{\mathbf{Q}}$-equivariant isomorphisms

$$\nu_{\mathfrak{f}_i\mathfrak{m}\mathfrak{p}^\infty}:H^1(\Gamma(1,N_{\psi_i}m(p)),\mathcal{D}'_{\kappa_i})\otimes\mathcal{O}[\![H^{(p)}_{\mathfrak{f}_i\mathfrak{m}\mathfrak{p}^\infty}]\!]\xrightarrow{\simeq}\mathrm{Ind}^{\mathbf{Q}}_{K(\mathfrak{f}_i\mathfrak{m})}\Lambda_{\mathfrak{p}}((\xi_i\psi_0)^{-1}).$$

Composing these with the natural degeneracy maps from level $\Gamma(m,p)$ to level $\Gamma(1,N_{\psi_i}m(p))$ and the projection $H^{(p)}_{\mathfrak{f}_i\mathfrak{m}}\to H^{(p)}_{\mathfrak{m}}$, we then obtain $G_{\mathbf{Q}}$-equivariant maps

$$\begin{aligned}H^1(\Gamma(m,p),\mathcal{D}'_{\kappa_1})\otimes\mathcal{O}[\![H^{(p)}_{\mathfrak{f}_1\mathfrak{m}\mathfrak{p}^\infty}]\!]&\to\mathrm{Ind}^{\mathbf{Q}}_K\mathcal{O}_{(\xi_1\psi_0)^{-1}}[H^{(p)}_{\mathfrak{m}}][\![\Gamma_{\mathfrak{p}}]\!],\\ H^1(\Gamma(m,p),\mathcal{D}'_{\kappa_2})\otimes\mathcal{O}[\![H^{(p)}_{\mathfrak{f}_2\overline{\mathfrak{m}}\mathfrak{p}^\infty}]\!]&\to\mathrm{Ind}^{\mathbf{Q}}_K\mathcal{O}_{(\xi_2\psi_0)^{-1}}[H^{(p)}_{\overline{\mathfrak{m}}}][\![\Gamma_{\mathfrak{p}}]\!].\end{aligned} \tag{2.17}$$

Similarly as in §2.1, the maps used in the construction of the class $\boldsymbol{\kappa}_m^{(2)}$ in (2.15) are compatible under correspondences. Hence after tensoring with $\mathcal{O}[H^{(p)}_{\mathfrak{f}_1\mathfrak{m}\mathfrak{p}^r}]$ and $\mathcal{O}[H^{(p)}_{\mathfrak{f}_2\overline{\mathfrak{m}}\mathfrak{p}^r}]$ using the maps $\phi_{\mathfrak{f}_1\mathfrak{m}\mathfrak{p}^r}$ and $\phi_{\mathfrak{f}_2\mathfrak{m}\mathfrak{p}^r}$, respectively, and letting $r\to\infty$, the same construction gives rise to a class

$$\begin{aligned}\boldsymbol{\kappa}^{(3)}_{\psi_1,\psi_2,\mathfrak{m}}\in H^1\big(\mathbf{Q},H^1(\Gamma(1,N(p)),\mathcal{D}'_{\kappa_f})\hat{\otimes}_{\mathcal{O}}(H^1(\Gamma(m,p),\mathcal{D}'_{\kappa_g})\otimes\mathcal{O}[\![H^{(p)}_{\mathfrak{f}_1\mathfrak{m}\mathfrak{p}^\infty}]\!])\\ \hat{\otimes}_{\mathcal{O}[D_m]}(H^1(\Gamma(m,p),\mathcal{D}'_{\kappa_h})\otimes\mathcal{O}[\![H^{(p)}_{\mathfrak{f}_2\overline{\mathfrak{m}}\mathfrak{p}^\infty}]\!])(2-\kappa^*_{\boldsymbol{fgh}})\big).\end{aligned}$$

Now let $\breve{f}$ be a level-N test vector for f, and consider the associated specialization map

$$\pi_f:H^1(\Gamma(1,N(p),\mathcal{D}'_{\kappa_f})(1)\to T_f^\vee. \tag{2.18}$$

Then taking the image of $\boldsymbol{\kappa}^{(3)}_{\psi_1,\psi_2,\mathfrak{m}}$ under the natural maps induced by (2.17) and (2.18) we obtain

$$\boldsymbol{\kappa}^{(4)}_{f,\psi_1,\psi_2,\mathfrak{m}} \in H^1\big(\mathbf{Q}, T_f^\vee \otimes_{\mathcal{O}} (\mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{(\xi_1\psi_0)^{-1}}[H_{\mathfrak{m}}^{(p)}][\![\Gamma_{\mathfrak{p}}]\!])\hat{\otimes}_{\mathcal{O}[D_m]}(\mathrm{Ind}_K^{\mathbf{Q}}\mathcal{O}_{(\xi_2\psi_0)^{-1}}[H_{\overline{\mathfrak{m}}}^{(p)}][\![\Gamma_{\mathfrak{p}}]\!])(-1-\kappa^*_{\boldsymbol{fgh}})\big),$$

which after specializing the third factor to weight k_2, i.e. taking the projection

$$\mathcal{O}_{(\xi_2^{-1}\psi_0)^{-1}}[H_{\overline{\mathfrak{m}}}^{(p)}][\![\Gamma_{\mathfrak{p}}]\!] \to \mathcal{O}_{\psi_2^{-1}}[H_{\overline{\mathfrak{m}}}^{(p)}],$$

and applying the diagonal map ξ_Δ in (2.5) finally gives rise to the class

$$\boldsymbol{\kappa}^{(5)}_{f,\psi_1,\psi_2,\mathfrak{m}} \in H^1\big(\mathbf{Q}, T_f^\vee(1-k/2) \otimes_{\mathcal{O}} \mathrm{Ind}_{K[m]}^{\mathbf{Q}}\Lambda_{\mathcal{O}}(\psi_1^{-1}\psi_2^{-1}\kappa_{\mathrm{ac}}^{(k_1+k_2-4)/2}\boldsymbol{\kappa}_{\mathrm{ac}}^{-1})(1-(k_1+k_2)/2)\big). \tag{2.19}$$

Here, we identify $\Gamma^- = \mathrm{Gal}(K_\infty^-/K)$ with the anti-diagonal in $(1+p\mathbf{Z}_p)\times(1+p\mathbf{Z}_p) \simeq \mathcal{O}_{K,\mathfrak{p}}^{(1)} \times \mathcal{O}_{K,\overline{\mathfrak{p}}}^{(1)}$ via the geometric normalised Artin map, and define

$$\kappa_{ac} : \Gamma^- \to \mathbf{Z}_p^\times, \qquad ((1+p)^{-1/2},(1+p)^{1/2}) \mapsto (1+p),$$
$$\boldsymbol{\kappa}_{ac} : \Gamma^- \to \Lambda^\times, \qquad ((1+p)^{-1/2},(1+p)^{1/2}) \mapsto [(1+p)].$$

Then, for T an $\mathcal{O}$-lattice inside a G_K-representation V, by Shapiro's lemma we have

$$H^1\big(K, T\hat{\otimes}_{\mathcal{O}}\Lambda_{\mathcal{O}}(\boldsymbol{\kappa}_{\mathrm{ac}}^{-1})\big) \simeq H^1_{\mathrm{Iw}}(K[p^\infty],T),$$

where $H^1_{\mathrm{Iw}}(K[p^\infty],T) := \varprojlim_r H^1(K[p^r],T)$ with limit under the corestriction maps. Thus in the following we shall view the class $\boldsymbol{\kappa}^{(5)}_{f,\psi_1,\psi_2,\mathfrak{m}}$ in (2.19) as an Iwasawa cohomology class

$$\boldsymbol{\kappa}_{f,\psi_1,\psi_2,\mathfrak{m}} \in H^1_{\mathrm{Iw}}\big(K[mp^\infty], T_f^\vee(1-k/2)\otimes\psi_1^{-1}\psi_2^{-1}\kappa_{\mathrm{ac}}^{(k_1+k_2-4)/2}(1-(k_1+k_2)/2)\big) \tag{2.20}$$

for the self-dual representation $T_f^\vee(1-k/2)$ twisted by the Hecke character $\psi_1^{-1}\psi_2^{-1}\mathbf{N}^{1-(k_1+k_2)/2}$, which is anticyclotomic and of infinity type $((k_1+k_2)/2-1, -(k_1+k_2)/2+1)$. For the ease of notation, put

$$T_{f,\psi_1,\psi_2} = T_f^\vee(1-k/2)\otimes\psi_1^{-1}\psi_2^{-1}(1-(k_1+k_2)/2). \tag{2.21}$$

Thus we arrive at the following key result.

Theorem 2.3.2. *Suppose $p \nmid 6h_K$ and f is non-Eisenstein modulo $\mathfrak{P}$. Let $\mathfrak{m}$ run over the ideals of $\mathcal{O}_K$ divisible only by primes that are split in K with $m = N_{K/\mathbf{Q}}(\mathfrak{m})$ coprime to p. Then there exists a collection of Iwasawa cohomology classes*

$$\mathbf{z}_{f,\psi_1,\psi_2,\mathfrak{m}} \in H^1_{\mathrm{Iw}}\big(K[mp^\infty], T_{f,\psi_1,\psi_2}\big)$$

such that for every split prime $\mathfrak{l}$ of $\mathcal{O}_K$ of norm ℓ with $(\ell, mp) = 1$ we have the norm relation

$$\mathrm{Norm}_{K[m]}^{K[m\ell]}(\mathbf{z}_{f,\psi_1,\psi_2,\mathfrak{m}\mathfrak{l}}) = P_{\mathfrak{l}}(\mathrm{Frob}_{\mathfrak{l}})(\mathbf{z}_{f,\psi_1,\psi_2,\mathfrak{m}}),$$

where $P_{\mathfrak{l}}(X) = \det(1 - X\cdot\mathrm{Frob}_{\mathfrak{l}} \,|\, (T_{f,\psi_1,\psi_2})^\vee(1))$.

Proof. This follows from a direct adaptation of the proof of Theorem 2.2.7. The only difference is that this time we also invoke [Rub00, Thm 6.3.5] to go from a collection of Iwasawa cohomology classes for the twist

$$T_f^\vee(1-k/2)\otimes\psi_1^{-1}\psi_2^{-1}\kappa_{\mathrm{ac}}^{(k_1+k_2-4)/2}(1-(k_1+k_2)/2) = T_{f,\psi_1,\psi_2}\otimes\kappa_{\mathrm{ac}}^{(k_1+k_2-4)/2}$$

with the stated norm relations, to a similar collection of cohomology classes for T_{f,ψ_1,ψ_2}. □

3. Anticyclotomic Euler systems

In this section we show that the system of classes constructed in Theorem 2.3.2 (and a variant thereof) land in certain Selmer groups defined in the style of Greenberg [Gre94]. As a result, our classes form an anticyclotomic Euler system in the sense of Jetchev–Nekovář–Skinner [JNS]. We then record the bounds on different Selmer groups that follow by applying their machinery to our construction.

Throughout we let $f \in S_k(\Gamma_0(N_f))$ be a p-ordinary newform of weight $k \geq 2$ with $p \nmid N_f$, and $K/\mathbf{Q}$ be an imaginary quadratic field of discriminant prime to N_f in which $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits.

3.1. **Selmer groups.** Let χ be an anticyclotomic Hecke character of K of infinity type $(-j, j)$ for some $j \geq 0$, and consider the conjugate self-dual G_K-representation

$$V_{f,\chi} := V_f^\vee(1 - k/2) \otimes \chi^{-1}.$$

Given a prime v of K above p and a G_{K_v}-stable subspace $\mathscr{F}_v^+(V_{f,\chi}) \subset V_{f,\chi}$, we put $\mathscr{F}_v^-(V_{f,\chi}) = V_{f,\chi}/\mathscr{F}_v^+(V_{f,\chi})$.

Definition 3.1.1. Let L be a finite extension of K, and fix $\mathscr{F} = \{\mathscr{F}_v^+(V_{f,\chi})\}_{v|p}$. The associated *Greenberg Selmer group* $\mathrm{Sel}_{\mathscr{F}}(L, V_{f,\chi})$ is defined by

$$\mathrm{Sel}_{\mathscr{F}}(L, V_{f,\chi}) := \ker\biggl\{H^1(L, V_{f,\chi}) \to \prod_w \frac{H^1(L_w, V_{f,\chi})}{H^1_{\mathscr{F}}(L_w, V_{f,\chi})}\biggr\},$$

where w runs over the finite primes of L, and the local conditions are given by

$$H^1_{\mathscr{F}}(L_w, V_{f,\chi}) = \begin{cases} \ker\bigl\{H^1(L_w, V_{f,\chi}) \to H^1(L_w^{\mathrm{ur}}, V_{f,\chi})\bigr\} & \text{if } w \nmid p, \\ \ker\bigl\{H^1(L_w, V_{f,\chi}) \to H^1(L_w, \mathscr{F}_v^-(V_{f,\chi}))\bigr\} & \text{if } w \mid v \mid p. \end{cases}$$

Given any lattice $T_{f,\chi} \subset V_{f,\chi}$, we let $H^1_{\mathscr{F}}(L_w, T_{f,\chi})$ be the inverse image of $H^1_{\mathscr{F}}(L_w, V_{f,\chi})$ under the natural map $H^1(L_w, T_{f,\chi}) \to H^1(L_w, V_{f,\chi})$, and define $\mathrm{Sel}_{\mathscr{F}}(L, T_{f,\chi})$ in the same manner; and given any $\mathbf{Z}_p$-extension $L_\infty = \bigcup_n L_n$ of L, we put

$$\mathrm{Sel}_{\mathscr{F}}(L_\infty, T_{f,\chi}) := \varprojlim_n \mathrm{Sel}_{\mathscr{F}}(L_n, T_{f,\chi}),$$

with limit with respect to corestriction, and also put $\mathrm{Sel}_{\mathscr{F}}(L_\infty, V_{f,\chi}) := \mathrm{Sel}_{\mathscr{F}}(L_\infty, T_{f,\chi}) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ (which is independent of the chosen $T_{f,\chi}$).

We shall be particularly interested in the following two instances of these definitions:

- The *relaxed-strict Selmer group* $\mathrm{Sel}_{\mathrm{rel,str}}(L, V_{f,\chi})$ obtained by taking

$$\mathscr{F}_v^+(V_{f,\chi}) = \begin{cases} V_{f,\chi} & \text{if } v = \mathfrak{p}, \\ 0 & \text{if } v = \bar{\mathfrak{p}}. \end{cases}$$

- The *ordinary Selmer group* $\mathrm{Sel}_{\mathrm{ord,ord}}(L, V_{f,\chi})$. Since f is p-ordinary, upon restriction to $G_{\mathbf{Q}_p} \subset G_{\mathbf{Q}}$ the Galois representation $V_f^\vee$ fits into a short exact sequence

$$0 \to V_f^{\vee,+} \to V_f^\vee \to V_f^{\vee,-} \to 0$$

with $V_f^{\vee,\pm}$ one-dimensional, and with the $G_{\mathbf{Q}_p}$-action on $V_f^{\vee,-}$ being unramified (see §1.2.2). Then $\mathrm{Sel}_{\mathrm{ord,ord}}(L, V_{f,\chi})$ is the Greenberg Selmer group defined by

$$\mathscr{F}_v^+(V_{f,\chi}) = V_{f,\chi}^+ := V_f^{\vee,+}(1 - k/2) \otimes \chi^{-1} \tag{3.1}$$

for all $v \mid p$.

Following [BK90], we also define the Selmer group $\mathrm{Sel}_{\mathrm{BK}}(L, V_{f,\chi})$ by

$$\mathrm{Sel}_{\mathrm{BK}}(L, V_{f,\chi}) := \ker\biggl\{H^1(L, V_{f,\chi}) \to \prod_w \frac{H^1(L_w, V_{f,\chi})}{H^1_f(L_w, V_{f,\chi})}\biggr\},$$

where as before w runs over the finite primes of L, and the local conditions are given by

$$H^1_f(L_w, V_{f,\chi}) = \begin{cases} \ker\bigl\{H^1(L_w, V_{f,\chi}) \to H^1(L_w^{\mathrm{ur}}, V_{f,\chi})\bigr\} & \text{if } w \nmid p, \\ \ker\bigl\{H^1(L_w, V_{f,\chi}) \to H^1(L_w, V_{f,\chi} \otimes \mathbf{B}_{\mathrm{cris}})\bigr\} & \text{if } w \mid p, \end{cases}$$

with $\mathbf{B}_{\mathrm{cris}}$ being Fontaine's crystalline period ring. The local conditions $H^1_f(L_w, T_{f,\chi}) \subset H^1(L_w, T_{f,\chi})$ are then defined by propagation.

For our later convenience, we now recall the well-known relation between these different Selmer groups. Here we shall adopt the convention that the p-adic cyclotomic character has Hodge–Tate weight -1. Thus, since χ has infinity type $(-j, j)$ (see §1.3.1 for our convention regarding infinity types), the p-adic avatar of χ has Hodge–Tate weight j at $\mathfrak{p}$ and $-j$ at $\bar{\mathfrak{p}}$.

Lemma 3.1.2. *For any finite extension L of K we have*

$$\mathrm{Sel}_{\mathrm{BK}}(L, V_{f,\chi}) = \begin{cases} \mathrm{Sel}_{\mathrm{rel,str}}(L, V_{f,\chi}) & \text{if } j \geq k/2, \\ \mathrm{Sel}_{\mathrm{ord,ord}}(L, V_{f,\chi}) & \text{if } 0 \leq j < k/2. \end{cases}$$

Proof. Combining the results of [Nek00, (3.1)-(3.2)] and [Fla90, Lem. 2, p. 125], for every prime $w|v|p$ of $L/K/\mathbf{Q}$ we have

$$H^1_f(L_w, V_{f,\chi}) = \mathrm{im}\{H^1(L_w, \mathrm{Fil}^1_v(V_{f,\chi})) \to H^1(L_w, V_{f,\chi})\},$$

where $\mathrm{Fil}^1_v(V_{f,\chi}) \subset V_{f,\chi}$ is a G_{K_v}-stable subspace (assuming it exists) such that the Hodge–Tate weights of $\mathrm{Fil}^1_v(V_{f,\chi})$ (resp. $V_{f,\chi}/\mathrm{Fil}^1_v(V_{f,\chi})$) are all < 0 (resp. ≥ 0).

Now, the Hodge–Tate weights of $V^+_{f,\chi}$ and $V^-_{f,\chi} := V_{f,\chi}/V^+_{f,\chi}$ at the primes of K above p are given by:

	$V^+_{f,\chi}$	$V^-_{f,\chi}$
HT weight at $\mathfrak{p}$	$-j-k/2$	$-j-1-k/2$
HT weight at $\bar{\mathfrak{p}}$	$j-k/2$	$j-1+k/2$

and so we find that $\mathrm{Fil}^1_{\mathfrak{p}}(V_{f,\chi}) = V_{f,\chi}$ and $\mathrm{Fil}^1_{\bar{\mathfrak{p}}}(V_{f,\chi}) = 0$ when $j \geq k/2$, and $\mathrm{Fil}^1_{\mathfrak{p}}(V_{f,\chi}) = \mathrm{Fil}^1_{\bar{\mathfrak{p}}}(V_{f,\chi}) = V^+_{f,\chi}$ when $0 \leq j < k/2$, yielding the equalities in the lemma. $\square$

For $A_{f,\chi} := \mathrm{Hom}_{\mathbf{Z}_p}(T_{f,\chi}, \mu_{p^\infty})$, and a choice of Galois stable subspaces $\mathscr{F} = \{\mathscr{F}^+_v(V_{f,\chi})\}_{v|p}$, we define the associated *dual Selmer group* $\mathrm{Sel}_{\mathscr{F}^*}(L, A_{f,\chi})$ by

$$\mathrm{Sel}_{\mathscr{F}^*}(L, A_{f,\chi}) := \ker\left\{H^1(L, A_{f,\chi}) \to \prod_w \frac{H^1(L_w, A_{f,\chi})}{H^1_{\mathscr{F}^*}(L_w, A_{f,\chi})}\right\},$$

where $H^1_{\mathscr{F}^*}(L_w, A_{f,\chi})$ is the orthogonal complement of $H^1_{\mathscr{F}}(L_w, T_{f,\chi})$ under local Tate duality

$$H^1(L_w, T_{f,\chi}) \times H^1(L_w, A_{f,\chi}) \to \mathbf{Q}_p/\mathbf{Z}_p.$$

In particular, we find that:

- The dual Selmer group of $\mathrm{Sel}_{\mathrm{rel,str}}(L, T_{f,\chi})$ consists of classes that are unramified outside p and have the strict (resp. relaxed) condition at the primes $w|\mathfrak{p}$ (resp. $w|\bar{\mathfrak{p}}$); we shall denote this by $\mathrm{Sel}_{\mathrm{str,rel}}(L, A_{f,\chi})$.
- The dual Selmer group of $\mathrm{Sel}_{\mathrm{ord,ord}}(L, T_{f,\chi})$ consists of classes that are unramified outside p, and land in the image of the natural map

$$H^1(L_w, \mathscr{F}^+_v(A_{f,\chi})) \to H^1(L_w, A_{f,\chi}), \quad \mathscr{F}^+_v(A_{f,\chi}) := \mathrm{Hom}_{\mathbf{Z}_p}(\mathscr{F}^-_v(T_{f,\chi}), \mu_{p^\infty}),$$

for $w|v|p$; we shall denote this by $\mathrm{Sel}_{\mathrm{ord,ord}}(L, A_{f,\chi})$.

3.2. **Local conditions at** p. Let ψ_1, ψ_2 be Hecke characters of K of infinity type $(1-k_1, 0), (1-k_2, 0)$ with $k_1, k_2 \geq 1$, and whose central characters satisfy $\chi_{\psi_1}\chi_{\psi_2} = 1$.

By Theorem 2.3.2 we have classes

$$\mathbf{z}_{f,\psi_1,\psi_2} \in H^1_{\mathrm{Iw}}(K[mp^\infty], T_{f,\psi_1,\psi_2}),$$

where $T_{f,\psi_1,\psi_2} = T_f^\vee(1-k/2) \otimes \psi_1^{-1}\psi_2^{-1}(1-(k_1+k_2)/2)$. Replacing the map ξ in (2.5) by the map $\phi_1 \otimes \phi_2 \mapsto \xi^{\mathbf{c}}(\phi_1 \otimes \phi_2)$ with $\xi^{\mathbf{c}}(\phi_1 \otimes \phi_2)(g) = \phi_1(g_1) \otimes \phi_2^{\mathbf{c}}(g_2)$ for $g = (g_1, g_2) \in H^{(p)}_{\mathfrak{m}} \times H^{(p)}_{\bar{\mathfrak{m}}}$, the same construction gives rise to classes

$${}^{\mathbf{c}}\mathbf{z}_{f,\psi_1,\psi_2} \in H^1_{\mathrm{Iw}}(K[mp^\infty], T_{f,\psi_1,\psi_2^{\mathbf{c}}}),$$

where $T_{f,\psi_1,\psi_2^{\mathbf{c}}} := T_f^\vee(1-k/2) \otimes \psi_1^{-1}\psi_2^{-\mathbf{c}}(1-(k_1+k_2)/2))$, satisfying the same norm-compatibility relations as in Theorem 2.3.2.

Proposition 3.2.1. *For all ideals* $\mathfrak{m}$ *of* $\mathcal{O}_K$ *divisible only by primes split in* K *with* $m = N_{K/\mathbf{Q}}(\mathfrak{m})$ *coprime to* p*, the classes* $\mathbf{z}_{f,\psi_1,\psi_2,\mathfrak{m}}$ *of Theorem 2.3.2 satisfy*

$$\mathbf{z}_{f,\psi_1,\psi_2,\mathfrak{m}} \in \mathrm{Sel}_{\mathrm{rel,str}}(K[mp^\infty], T_{f,\psi_1,\psi_2}),$$

and similarly ${}^{\mathbf{c}}\mathbf{z}_{f,\psi_1,\psi_2,\mathfrak{m}} \in \mathrm{Sel}_{\mathrm{ord,ord}}(K[mp^\infty], T_{f,\psi_1,\psi_2^{\mathbf{c}}})$.

Proof. We shall adopt some of the notations introduced later in Section 4. Let $\boldsymbol{f}, \boldsymbol{g} = \boldsymbol{\theta}_{\xi_1}(S_1), \boldsymbol{h} = \boldsymbol{\theta}_{\xi_2}(S_2)$ be the Hida families associated to $f, \theta_{\psi_1}, \theta_{\psi_2}$, respectively. By [BSV22, Cor. 8.2], after projection to $\mathbf{V}^\dagger$ the class $\kappa_m^{(1)}$ in (2.14) lands in the balanced Selmer group $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}^\dagger)$ (see Definition 4.2.2). Using that the big Galois representations associated to $\boldsymbol{g}$ and $\boldsymbol{h}$ are both induced from K, upon restriction to G_K the triple tensor product $\mathbf{V}^\dagger$ specialised to f decomposes as

$$\mathbf{V}^\dagger_{Q_0}|_{G_K} = \big(T_f^\vee(1-k/2) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\big) \oplus \big(T_f^\vee(1-k/2) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_2}^{1-\mathbf{c}}\big), \tag{3.2}$$

and as in the proof of Proposition 5.3.1, from Shapiro's lemma we find that the local condition $\mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger_{Q_0})$ cutting out the specialised balanced Selmer group at p corresponds to

$$\begin{aligned}
\mathscr{F}_{\mathfrak{p}}^{\mathrm{bal}}(\mathbf{V}^\dagger_{Q_0}|_{G_K}) &= \big(T_f^\vee(1-k/2) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\big) \oplus \big(T_f^{\vee,+}(1-k/2) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\big),\\
\mathscr{F}_{\bar{\mathfrak{p}}}^{\mathrm{bal}}(\mathbf{V}^\dagger_{Q_0}|_{G_K}) &= \{0\} \qquad\qquad\qquad\qquad\qquad\qquad\quad \oplus \big(T_f^{\vee,+}(1-k/2) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\big).
\end{aligned}$$

Since the diagonal map ξ_Δ in (2.5) (resp. its twisted variant $\xi_\Delta^{\mathbf{c}}$) has the effect of projecting onto the first (resp. second) direct summand in (3.2), this shows that the classes $\mathbf{z}_{f,\psi_1,\psi_2,\mathfrak{m}}$ (resp. ${}^{\mathbf{c}}\mathbf{z}_{f,\psi_1,\psi_2,\mathfrak{m}}$) satisfy the relaxed-strict (resp. ordinary-ordinary) condition at the primes above p. On the other hand, at the primes $w \nmid p$, because V_{f,ψ_1,ψ_2} is conjugate self-dual and pure of weight -1, we see that

$$H^0(K[mp^r]_w, V_{f,\psi_1,\psi_2}) = H^2(K[mp^r]_w, V_{f,\psi_1,\psi_2}) = 0$$

for all r, and therefore $H^1(K[mp^r]_w, V_{f,\psi_1,\psi_2}) = 0$ by Tate's local Euler characteristic formula. This shows that $H^1(K[mp^r]_w, T_{f,\psi_1,\psi_2})$ is torsion, and as a result the inclusion

$$\mathrm{res}_w(\mathbf{z}_{f,\psi_1,\psi_2,\mathfrak{m}}) \in \varprojlim_r H^1_f(K[mp^r]_w, T_{f,\psi_1,\psi_2})$$

follows automatically. Similarly, we see that the classes ${}^{\mathbf{c}}\mathbf{z}_{f,\psi_1,\psi_2,\mathfrak{m}}$ are unramifed outside p, and since we have shown that they are ordinary at the primes above p, the result follows. □

3.3. **Applying the general machinery.** We give some direct arithmetic applications that follow by applying to our construction the general Euler system machinery of Jetchev–Nekovář–Skinner [JNS]. Later in the paper, by exploiting the relation between our Euler system classes and special values of complex and p-adic L-functions, we shall deduce from these results applications to the Bloch–Kato conjecture and the anticyclotomic Iwasawa main conjecture.

For every ideal $\mathfrak{m} \subset \mathcal{O}_K$ as in Theorem 2.3.2, denote by

$$z_{f,\psi_1,\psi_2,\mathfrak{m}} \in \mathrm{Sel}_{\mathrm{rel,str}}(K[m], T_{f,\psi_1,\psi_2})$$

the image of $\mathbf{z}_{f,\psi_1,\psi_2,\mathfrak{m}}$ under the projection $\mathrm{Sel}_{\mathrm{rel,str}}(K[mp^\infty], T_{f,\psi_1,\psi_2}) \to \mathrm{Sel}_{\mathrm{rel,str}}(K[m], T_{f,\psi_1,\psi_2})$, and put

$$z_{f,\psi_1,\psi_2} := \mathrm{cor}_{K[1]/K}(z_{f,\psi_1,\psi_2,1}) \in \mathrm{Sel}_{\mathrm{rel,str}}(K, T_{f,\psi_1,\psi_2}).$$

Similarly, projecting to $K[m]$ the class ${}^{\mathbf{c}}\mathbf{z}_{f,\psi_1,\psi_2,\mathfrak{m}}$ define

$${}^{\mathbf{c}}z_{f,\psi_1,\psi_2,\mathfrak{m}} \in \mathrm{Sel}_{\mathrm{ord,ord}}(K[m], T_{f,\psi_1,\psi_2}),$$

and put ${}^{\mathbf{c}}z_{f,\psi_1,\psi_2} := \mathrm{cor}_{K[1]/K}({}^{\mathbf{c}}z_{f,\psi_1,\psi_2,1})$.

3.3.1. *Rank one results.*

Theorem 3.3.1. *Assume that f is not of CM-type and is non-Eisenstein at $\mathfrak{P}$, and that $p \nmid h_K$.*

(I) *If $z_{f,\psi_1,\psi_2} \neq 0$, then $\mathrm{Sel}_{\mathrm{rel,str}}(K, V_{f,\psi_1,\psi_2})$ is one-dimensional.*
(II) *If ${}^{\mathbf{c}}z_{f,\psi_1,\psi_2} \neq 0$, then $\mathrm{Sel}_{\mathrm{ord,ord}}(K, V_{f,\psi_1,\psi_2^{\mathbf{c}}})$ is one-dimensional.*

Proof. Denote by $\mathcal{N}$ the set of squarefree products of primes $\mathfrak{l}$ of $\mathcal{O}_K$ split in K and with $\ell = N_{K/\mathbf{Q}}(\mathfrak{l})$ coprime to p. By Theorem 2.3.2 and Proposition 3.2.1, the system of classes

$$\left\{ z_{f,\psi_1,\psi_2,\mathfrak{m}} \in \mathrm{Sel}_{\mathrm{rel,str}}(K[m], T_{f,\psi_1,\psi_2}) \;:\; \mathfrak{m} \in \mathcal{N} \right\} \tag{3.3}$$

forms an anticyclotomic Euler system in the sense of Jetchev–Nekovář–Skinner [JNS] for the relaxed-strict Greenberg Selmer group. Hence from their general results the one-dimensionality of $\mathrm{Sel}_{\mathrm{rel,str}}(K, V_{f,\psi_1,\psi_2})$ follows from the nonvanishing of z_{f,ψ_1,ψ_2} provided the G_K-representation $V = V_{f,\psi_1,\psi_2}$ satisfies the following hypotheses:

(i) V is absolutely irreducible;
(ii) There is an element $\sigma \in G_K$ fixing $K[1]K(\mu_{p^\infty}, (\mathcal{O}_K^\times)^{1/p^\infty})$ such that $V/(\sigma-1)V$ is one-dimensional;
(iii) There is an element $\gamma \in G_K$ fixing $K[1]K(\mu_{p^\infty}, (\mathcal{O}_K^\times)^{1/p^\infty})$ such that $V^{\gamma=1} = 0$.

Since we assume that f is not of CM-type, hypotheses (i)–(iii) follow easily from Momose's big image results [Mom81] as in [LLZ15, Prop. 7.1.4], whence the first part of the theorem; the proof of the second part is the same. □

3.3.2. *Iwasawa-theoretic results.* Denote by K_∞^- the anticyclotomic $\mathbf{Z}_p$-extension $K[p^\infty]/K$, and put $\Lambda_K^- = \mathbf{Z}_p[\![\mathrm{Gal}(K_\infty^-/K)]\!]$. Let $\mathbf{z}_{f,\psi_1,\psi_2,1}$ be the Λ_K^--adic class of Theorem 2.3.2 of conductor $\mathfrak{m} = (1)$, and put

$$\mathbf{z}_{f,\psi_1,\psi_2} := \mathrm{cor}_{K[1]/K}(\mathbf{z}_{f,\psi_1,\psi_2,1}) \in \mathrm{Sel}_{\mathrm{rel,str}}(K_\infty^-, T_{f,\psi_1,\psi_2}),$$

where the inclusion follows from Proposition 3.2.1. Similarly, put

$${}^{\mathbf{c}}\mathbf{z}_{f,\psi_1,\psi_2} := \mathrm{cor}_{K[1]/K}({}^{\mathbf{c}}\mathbf{z}_{f,\psi_1,\psi_2,1}) \in \mathrm{Sel}_{\mathrm{ord,ord}}(K_\infty^-, T_{f,\psi_1,\psi_2^{\mathbf{c}}}).$$

Notation 3.3.2. As in [LLZ15, §7.1], we shall say that f *has big image at* $\mathfrak{P}$ if the image of $G_{\mathbf{Q}}$ in $\mathrm{Aut}_{\mathcal{O}}(T_f^\vee)$ contains a conjugate of $\mathrm{SL}_2(\mathbf{Z}_p)$.

We also note that, by a theorem of Ribet [Rib85], if f is not of CM-type then it has big image for all but finitely many primes of L.

Put

$$X_{\mathrm{str,rel}}(K_\infty^-, A_{f,\psi_1,\psi_2}) = \mathrm{Hom}_{\mathbf{Z}_p}\big(\varinjlim \mathrm{Sel}_{\mathrm{str,rel}}(K_n^-, A_{f,\psi_1,\psi_2}), \mathbf{Q}_p/\mathbf{Z}_p\big)$$

and likewise for $X_{\mathrm{ord,ord}}(K_\infty^-, A_{f,\psi_1,\psi_2^{\mathbf{c}}})$.

The next result can be seen as a divisibility towards an anticyclotomic Iwasawa main conjecture 'without L-functions'.

Theorem 3.3.3. *Assume that f is not of CM-type and it has big image at $\mathfrak{P}$, and that $p \nmid h_K$.*

(I) *If $\mathbf{z}_{f,\psi_1,\psi_2}$ is non-torsion, then $X_{\mathrm{str,rel}}(K_\infty^-, A_{f,\psi_1,\psi_2})$ and $\mathrm{Sel}_{\mathrm{rel,str}}(K_\infty^-, T_{f,\psi_1,\psi_2})$ both have Λ_K^--rank one, and we have the divisibility*

$$\mathrm{char}_{\Lambda_K^-}(X_{\mathrm{str,rel}}(K_\infty^-, A_{f,\psi_1,\psi_2})_{\mathrm{tors}}) \supset \mathrm{char}_{\Lambda_K^-}\left(\frac{\mathrm{Sel}_{\mathrm{rel,str}}(K_\infty^-, T_{f,\psi_1,\psi_2})}{\Lambda_K^- \cdot \mathbf{z}_{f,\psi_1,\psi_2}}\right)^2$$

in Λ_K^-.

(II) *If ${}^{\mathbf{c}}\mathbf{z}_{f,\psi_1,\psi_2}$ is non-torsion, then $X_{\mathrm{ord,ord}}(K_\infty^-, A_{f,\psi_1,\psi_2^{\mathbf{c}}})$ and $\mathrm{Sel}_{\mathrm{ord,ord}}(K_\infty^-, T_{f,\psi_1,\psi_2^{\mathbf{c}}})$ both have Λ_K^--rank one, and we have the divisibility*

$$\mathrm{char}_{\Lambda_K^-}(X_{\mathrm{ord,ord}}(K_\infty^-, A_{f,\psi_1,\psi_2^{\mathbf{c}}})_{\mathrm{tors}}) \supset \mathrm{char}_{\Lambda_K^-}\left(\frac{\mathrm{Sel}_{\mathrm{ord,ord}}(K_\infty^-, T_{f,\psi_1,\psi_2^{\mathbf{c}}})}{\Lambda_K^- \cdot {}^{\mathbf{c}}\mathbf{z}_{f,\psi_1,\psi_2}}\right)^2$$

in Λ_K^-.

Here, in both (I) *and* (II)*, the subscript* tors *denotes the* Λ_K^-*-torsion submodule.*

Proof. With notations as in the proof of Theorem 3.3.1, by Theorem 2.3.2 and Proposition 3.2.1 the system of classes

$$\left\{\mathbf{z}_{f,\psi_1,\psi_2,\mathfrak{m}} \in \mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K[mp^\infty], T_{f,\psi_1,\psi_2}) : \ \mathfrak{m} \in \mathcal{N}\right\} \tag{3.4}$$

forms a Λ_K^--adic anticyclotomic Euler system in the sense of Jetchev–Nekovář–Skinner for the relaxed-strict Selmer group, and so the non-torsionness of $\mathbf{z}_{f,\psi_1,\psi_2}$ implies the conclusions in part (I) of the theorem provided the G_K-module $T = T_{f,\psi_1,\psi_2}$ satisfies the following hypotheses:

(i) $\bar{T} := T/\mathfrak{P}T$ is absolutely irreducible;

(ii) There is an element $\sigma \in G_K$ fixing $K[1]K(\mu_{p^\infty}, (\mathcal{O}_K^\times)^{1/p^\infty})$ such that $T/(\sigma-1)T$ is free of rank 1 over $\mathcal{O}$;

(iii) There is an element $\gamma \in G_K$ fixing $K[1]K(\mu_{p^\infty}, (\mathcal{O}_K^\times)^{1/p^\infty})$ and acting as multiplication by a scalar $a_\gamma \neq 1$ on $\bar{T}$;

but these follow easily from the assumption that f has big image at $\mathfrak{P}$ (see [LLZ15, Prop. 7.1.6]). This shows part (I) of the theorem, and part (II) follows in the same manner. □

Part 2. Applications

4. Preliminaries

In this section we briefly review the unbalanced triple product p-adic L-function constructed in [Hsi21], their associated Selmer groups, and the explicit construction of certain CM Hida families. We also recall from [BSV22] the explicit reciprocity law for diagonal classes, and Greenberg's formulation of the Iwasawa main conjecture for triple products following [ACR21].

4.1. Triple product p-adic L-function.

4.1.1. *Hida families.* Let $\mathbb{I}$ be a normal domain finite flat over

$$\Lambda := \mathcal{O}[\![1 + p\mathbf{Z}_p]\!],$$

where $\mathcal{O}$ is the ring of integers of a finite extension of $\mathbf{Q}_p$. For a positive integer N prime p and a Dirichlet character $\chi : (\mathbf{Z}/Np\mathbf{Z})^\times \to \mathcal{O}^\times$, we denote by $S^o(N, \chi, \mathbb{I}) \subset \mathbb{I}[\![q]\!]$ the space of ordinary $\mathbb{I}$-adic cusp forms of tame level N and branch character χ as defined in [Hsi21, §3.1].

Denote by $\mathfrak{X}^+_{\mathbb{I}} \subset \operatorname{Spec} \mathbb{I}(\overline{\mathbf{Q}}_p)$ the set of *arithmetic points* of $\mathbb{I}$, consisting of the ring homomorphisms $Q : \mathbb{I} \to \overline{\mathbf{Q}}_p$ such that $Q|_{1+p\mathbf{Z}_p}$ is given by $z \mapsto z^{k_Q-1}\epsilon_Q(z)$ for some $k_Q \in \mathbf{Z}_{\geq 2}$ called the *weight of* Q and $\epsilon_Q(z) \in \mu_{p^\infty}$. As in [Hsi21, §3.1], we say that $\boldsymbol{f} = \sum_{n=1}^\infty a_n(\boldsymbol{f})q^n \in S^o(N, \chi, \mathbb{I})$ is a *primitive Hida family* if for every $Q \in \mathfrak{X}^+_{\mathbb{I}}$ the specialization $\boldsymbol{f}_Q$ gives the q-expansion of an ordinary p-stabilised newform of weight k_Q and tame conductor N. Attached to such $\boldsymbol{f}$ we let $\mathfrak{X}^{\rm cls}_{\mathbb{I}}$ be the set of ring homomorphisms Q as above with $k_Q \in \mathbf{Z}_{\geq 1}$ such that $\boldsymbol{f}_Q$ is the q-expansion of a classical modular form (thus $\mathfrak{X}^{\rm cls}_{\mathbb{I}}$ contains $\mathfrak{X}^+_{\mathbb{I}}$).

For $\boldsymbol{f}$ a primitive Hida family of tame level N, we let

$$\rho_{\boldsymbol{f}} : G_{\mathbf{Q}} \to \operatorname{Aut}_{\mathbb{I}}(V_{\boldsymbol{f}}) \simeq \mathrm{GL}_2(\mathbb{I})$$

denote the associated Galois representation, which here we take to be the *dual* of that in [Hsi21, §3.2]; in particular, the determinant of $\rho_{\boldsymbol{f}}$ is $\chi_{\mathbb{I}} \cdot \varepsilon_{\rm cyc}$ in the notations of *loc. cit.*, where $\varepsilon_{\rm cyc}$ is the p-adic cyclotomic character. By [Wil88, Thm. 2.2.2], restricted to $G_{\mathbf{Q}_p}$ the Galois representation $V_{\boldsymbol{f}}$ fits into a short exact sequence

$$0 \to V^+_{\boldsymbol{f}} \to V_{\boldsymbol{f}} \to V^-_{\boldsymbol{f}} \to 0,$$

where the quotient $V^-_{\boldsymbol{f}}$ is free of rank one over $\mathbb{I}$, with the $G_{\mathbf{Q}_p}$-action given by the unramified character sending an arithmetic Frobenius Frob_p^{-1} to $a_p(\boldsymbol{f})$.

Denote by $\mathbb{T}(N, \mathbb{I})$ the Hecke algebra acting on $\bigoplus_\chi S^o(N, \chi, \mathbb{I})$, where χ runs over the characters of $(\mathbf{Z}/Np\mathbf{Z})^\times$. Associated with $\boldsymbol{f}$ there is a $\mathbb{I}$-algebra homomorphism

$$\lambda_{\boldsymbol{f}} : \mathbb{T}(N, \mathbb{I}) \to \mathbb{I}$$

factoring through a local component $\mathbb{T}_{\mathfrak{m}}$. Following [Hid88a] define the *congruence ideal* $C(\boldsymbol{f})$ of $\boldsymbol{f}$ by

$$C(\boldsymbol{f}) := \lambda_{\boldsymbol{f}}(\operatorname{Ann}_{\mathbb{T}_{\mathfrak{m}}}(\ker \lambda_{\boldsymbol{f}})) \subset \mathbb{I}.$$

If the residual representation $\bar{\rho}_{\boldsymbol{f}}$ is absolutely irreducible and p-distinguished, it follows from the results of [Wil95] and [Hid88a] that $C(\boldsymbol{f})$ is generated by a nonzero element $\eta_{\boldsymbol{f}} \in \mathbb{I}$.

4.1.2. *Triple products of Hida families.* Let

$$(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in S^o(N_f, \chi_f, \mathbb{I}_f) \times S^o(N_g, \chi_g, \mathbb{I}_g) \times S^o(N_h, \chi_h, \mathbb{I}_h)$$

be a triple of primitive Hida families with

$$\chi_f\chi_g\chi_h = \omega^{2a} \text{ for some } a \in \mathbf{Z}, \tag{4.1}$$

where ω is the Teichmüller character. Put

$$\mathcal{R} = \mathbb{I}_f \hat{\otimes}_{\mathcal{O}} \mathbb{I}_g \hat{\otimes}_{\mathcal{O}} \mathbb{I}_h,$$

which is a finite extension of the three-variable Iwasawa algebra $\Lambda \hat{\otimes}_{\mathcal{O}} \Lambda \hat{\otimes}_{\mathcal{O}} \Lambda$.

Let $\mathfrak{X}^+_{\mathcal{R}} \subset \operatorname{Spec} \mathcal{R}(\overline{\mathbf{Q}}_p)$ be the weight space of $\mathcal{R}$ given by

$$\mathfrak{X}^+_{\mathcal{R}} := \left\{ \underline{Q} = (Q_0, Q_1, Q_2) \in \mathfrak{X}^+_{\mathbb{I}_f} \times \mathfrak{X}^{\mathrm{cls}}_{\mathbb{I}_g} \times \mathfrak{X}^{\mathrm{cls}}_{\mathbb{I}_h} \ : \ k_{Q_0} + k_{Q_1} + k_{Q_2} \equiv 0 \ (\mathrm{mod}\ 2) \right\}.$$

This can be written as the disjoint union $\mathfrak{X}^+_{\mathbb{I}} = \mathfrak{X}^{\mathrm{bal}}_{\mathbb{I}} \sqcup \mathfrak{X}^{\boldsymbol{f}}_{\mathbb{I}} \sqcup \mathfrak{X}^{\boldsymbol{g}}_{\mathbb{I}} \sqcup \mathfrak{X}^{\boldsymbol{h}}_{\mathbb{I}}$, where

$$\mathfrak{X}^{\mathrm{bal}}_{\mathcal{R}} := \left\{ \underline{Q} \in \mathfrak{X}^+_{\mathcal{R}} \ : \ k_{Q_0} + k_{Q_1} + k_{Q_2} > 2k_{Q_i} \text{ for all } i = 0, 1, 2 \right\}$$

is the set of *balanced* weights, where each weight k_{Q_i} is smaller than the sum of the other two, and

$$\begin{aligned} \mathfrak{X}^{\boldsymbol{f}}_{\mathcal{R}} &:= \left\{ \underline{Q} \in \mathfrak{X}^+_{\mathcal{R}} \ : \ k_{Q_0} \geq k_{Q_1} + k_{Q_2} \right\}, \\ \mathfrak{X}^{\boldsymbol{f}}_{\mathcal{R}} &:= \left\{ \underline{Q} \in \mathfrak{X}^+_{\mathcal{R}} \ : \ k_{Q_1} \geq k_{Q_0} + k_{Q_2} \right\}, \\ \mathfrak{X}^{\boldsymbol{f}}_{\mathcal{R}} &:= \left\{ \underline{Q} \in \mathfrak{X}^+_{\mathcal{R}} \ : \ k_{Q_2} \geq k_{Q_0} + k_{Q_1} \right\}, \end{aligned}$$

are the set of $\boldsymbol{f}$- (resp. $\boldsymbol{g}$-, $\boldsymbol{h}$-) *unbalanced* weights.

Let $\mathbf{V} = V_{\boldsymbol{f}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}$ be the triple tensor product Galois representation attached to $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$. Write the determinant of $\mathbf{V}$ in the form $\det \mathbf{V} = \mathcal{X}^2 \varepsilon_{\mathrm{cyc}}$ (note that this is possible by (4.1)), and put

$$\mathbf{V}^\dagger := \mathbf{V} \otimes \mathcal{X}^{-1}, \tag{4.2}$$

which is a self-dual twist of $\mathbf{V}$. Define the rank four $G_{\mathbf{Q}_p}$-invariant subspace $\mathscr{F}^{\boldsymbol{f}}_p(\mathbf{V}^\dagger) \subset \mathbf{V}^\dagger$ by

$$\mathscr{F}^{\boldsymbol{f}}_p(\mathbf{V}^\dagger) := V^+_{\boldsymbol{f}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}} \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}} \otimes \mathcal{X}^{-1}, \tag{4.3}$$

and for any $\underline{Q} = (Q_0, Q_1, Q_2) \in \mathfrak{X}^{\boldsymbol{f}}_{\mathcal{R}}$ denote by $\mathscr{F}^{\boldsymbol{f}}_p(\mathbf{V}^\dagger_{\underline{Q}}) \subset \mathbf{V}^\dagger_{\underline{Q}}$ the corresponding specialisations.

For a rational prime ℓ, let $\varepsilon_\ell(\mathbf{V}^\dagger_{\underline{Q}})$ be the epsilon factor attached to the local representation $\mathbf{V}^\dagger_{\underline{Q}}|_{G_{\mathbf{Q}_\ell}}$ (see [Tat79, p. 21]), and assume that

$$\text{for some } \underline{Q} \in \mathfrak{X}^{\boldsymbol{f}}_{\mathcal{R}}, \text{ we have } \varepsilon_\ell(\mathbf{V}^\dagger_{\underline{Q}}) = +1 \text{ for all prime factors } \ell \text{ of } N_f N_g N_h. \tag{4.4}$$

As explained in [Hsi21, §1.2], it is known that condition (4.4) is independent of $\underline{Q}$, and it implies that the sign in the functional equation for the triple product L-function (with center at $s = 0$)

$$L(\mathbf{V}^\dagger_{\underline{Q}}, s)$$

is $+1$ (resp. -1) for all $\underline{Q} \in \mathfrak{X}^{\boldsymbol{f}}_{\mathcal{R}} \cup \mathfrak{X}^{\boldsymbol{g}}_{\mathcal{R}} \cup \mathfrak{X}^{\boldsymbol{h}}_{\mathcal{R}}$ (resp. $\underline{Q} \in \mathfrak{X}^{\mathrm{bal}}_{\mathcal{R}}$).

Theorem 4.1.1. *Let $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ be a triple of primitive Hida families satisfying conditions (4.1) and (4.4). Assume in addition that:*

- *$\gcd(N_f, N_g, N_h)$ is square-free,*
- *the residual representation $\bar{\rho}_{\boldsymbol{f}}$ is absolutely irreducible and p-distinguished,*

and fix a generator $\eta_{\boldsymbol{f}}$ of the congruence ideal of $\boldsymbol{f}$. Then there exists a unique element

$$\mathscr{L}^{\boldsymbol{f}, \eta_{\boldsymbol{f}}}_p(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in \mathcal{R}$$

such that for all $\underline{Q} = (Q_0, Q_1, Q_2) \in \mathfrak{X}^{\boldsymbol{f}}_{\mathcal{R}}$ of weight (k_0, k_1, k_2) with $\epsilon_{Q_0} = 1$ we have

$$\left(\mathscr{L}^{\boldsymbol{f}, \eta_{\boldsymbol{f}}}_p(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})(\underline{Q}) \right)^2 = \Gamma_{\mathbf{V}^\dagger_{\underline{Q}}}(0) \cdot \frac{L(\mathbf{V}^\dagger_{\underline{Q}}, 0)}{(\sqrt{-1})^{2k_0} \cdot \Omega^2_{\boldsymbol{f}_{Q_0}}} \cdot \mathcal{E}_p(\mathscr{F}^{\boldsymbol{f}}_p(\mathbf{V}^\dagger_{\underline{Q}})) \cdot \prod_{q \in \Sigma_{\mathrm{exc}}} (1 + q^{-1})^2,$$

where:

- *$\Gamma_{\mathbf{V}^\dagger_{\underline{Q}}}(0) = \Gamma_{\mathbf{C}}(c_{\underline{Q}}) \Gamma_{\mathbf{C}}(c_{\underline{Q}} + 2 - k_1 - k_2) \Gamma_{\mathbf{C}}(c_{\underline{Q}} + 1 - k_1) \Gamma_{\mathbf{C}}(c_{\underline{Q}} + 1 - k_2)$, with*

$$c_{\underline{Q}} = (k_0 + k_1 + k_2 - 2)/2$$

and $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$;

- $\Omega_{\boldsymbol{f}_{Q_0}}$ *is the canonical period*

$$\Omega_{\boldsymbol{f}_{Q_0}} := (-2\sqrt{-1})^{k_0+1} \cdot \frac{\|\boldsymbol{f}^{\circ}_{Q_0}\|^2_{\Gamma_0(N_f)}}{\eta_{\boldsymbol{f}_{Q_0}}} \cdot \Big(1 - \frac{\chi'_f(p)p^{k_0-1}}{\alpha^2_{Q_0}}\Big)\Big(1 - \frac{\chi'_f(p)p^{k_0-2}}{\alpha^2_{Q_0}}\Big),$$

with $\boldsymbol{f}^{\circ}_{Q_0} \in S_{k_0}(\Gamma_0(N_f))$ *the newform of conductor* N_f *associated with* $\boldsymbol{f}_{Q_0}$*,* χ'_f *the prime-to-p part of* χ_f*, and* α_{Q_0} *the specialisation of* $a_p(\boldsymbol{f}) \in \mathbb{I}_f^{\times}$ *at* Q_0*;*

- $\mathcal{E}_p(\mathscr{F}^{\boldsymbol{f}}_p(\mathbf{V}^{\dagger}_{\underline{Q}}))$ *is the modified p-Euler factor*

$$\mathcal{E}_p(\mathscr{F}^{\boldsymbol{f}}_p(\mathbf{V}^{\dagger}_{\underline{Q}})) := \frac{L_p(\mathscr{F}^{\boldsymbol{f}}_p(\mathbf{V}^{\dagger}_{\underline{Q}}),0)}{\varepsilon_p(\mathscr{F}^{\boldsymbol{f}}_p(\mathbf{V}^{\dagger}_{\underline{Q}})) \cdot L_p(\mathbf{V}^{\dagger}_{\underline{Q}}/\mathscr{F}^{\boldsymbol{f}}_p(\mathbf{V}^{\dagger}_{\underline{Q}}),0)} \cdot \frac{1}{L_p(\mathbf{V}^{\dagger}_{\underline{Q}},0)},$$

and Σ_{exc} *is an explicitly defined subset of the prime factors of* $N_fN_gN_h$*,* [Hsi21, p. 416].

Proof. This is Theorem A in [Hsi21], which in fact proves a more general interpolation formula. □

Remark 4.1.2. The construction of the p-adic L-function $\mathscr{L}_p^{\boldsymbol{f},\eta_{\boldsymbol{f}}}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ is based on the p-adic Rankin–Selberg method of Hida [Hid88b], and the proof of the above exact interpolation formula relies on a suitable choice $(\breve{\boldsymbol{f}}^{\star},\breve{\boldsymbol{g}}^{\star},\breve{\boldsymbol{h}}^{\star})$ of level-N test vectors for $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$. In general, for any choice $(\breve{\boldsymbol{f}},\breve{\boldsymbol{g}},\breve{\boldsymbol{h}})$ of level-N test vectors, Hida's method produces an element

$$\mathscr{L}_p^{\boldsymbol{f}}(\breve{\boldsymbol{f}},\breve{\boldsymbol{g}},\breve{\boldsymbol{h}}) \in \mathrm{Frac}(\mathbb{I}_f)\hat{\otimes}_{\mathcal{O}}\mathbb{I}_g\hat{\otimes}_{\mathcal{O}}\mathbb{I}_h,$$

and by virtue of the proof of Jacquet's conjecture by Harris–Kudla [HK91], for any $\underline{Q} \in \mathcal{X}^{\boldsymbol{f}}_{\mathcal{R}}$ one can find $(\breve{\boldsymbol{f}},\breve{\boldsymbol{g}},\breve{\boldsymbol{h}})$ such that

$$\mathscr{L}_p^{\boldsymbol{f}}(\breve{\boldsymbol{f}},\breve{\boldsymbol{g}},\breve{\boldsymbol{h}})(\underline{Q}) \neq 0 \iff L(\mathbf{V}^{\dagger}_{\underline{Q}},0) \neq 0.$$

In particular, if the central L-value $L(\mathbf{V}^{\dagger}_{\underline{Q}},0)$ is nonzero, then $\mathscr{L}_p^{\boldsymbol{f}}(\breve{\boldsymbol{f}},\breve{\boldsymbol{g}},\breve{\boldsymbol{h}}) \neq 0$. (In the above notation, we have $\mathscr{L}_p^{\boldsymbol{f},\eta_{\boldsymbol{f}}}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) = \eta_{\boldsymbol{f}} \cdot \mathscr{L}_p^{\boldsymbol{f}}(\breve{\boldsymbol{f}}^{\star},\breve{\boldsymbol{g}}^{\star},\breve{\boldsymbol{h}}^{\star})$.)

4.2. **Triple product Selmer groups.** Let $\mathbf{V}^{\dagger} = \mathbf{V} \otimes \mathcal{X}^{-1}$ be the self-dual twist of the Galois representation associated to a triple of primitive Hida families $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ satisfying (4.1).

Definition 4.2.1. Put

$$\mathscr{F}^{\mathrm{bal}}_p(\mathbf{V}^{\dagger}) := \big(V_{\boldsymbol{f}} \otimes V^+_{\boldsymbol{g}} \otimes V^+_{\boldsymbol{h}} + V^+_{\boldsymbol{f}} \otimes V_{\boldsymbol{g}} \otimes V^+_{\boldsymbol{h}} + V^+_{\boldsymbol{f}} \otimes V^+_{\boldsymbol{g}} \otimes V_{\boldsymbol{h}}\big) \otimes \mathcal{X}^{-1},$$

and define the *balanced local condition* $\mathrm{H}^1_{\mathrm{bal}}(\mathbf{Q}_p,\mathbf{V}^{\dagger})$ by

$$\mathrm{H}^1_{\mathrm{bal}}(\mathbf{Q}_p,\mathbf{V}^{\dagger}) := \mathrm{im}\big(\mathrm{H}^1(\mathbf{Q}_p,\mathscr{F}^{\mathrm{bal}}_p(\mathbf{V}^{\dagger})) \to \mathrm{H}^1(\mathbf{Q}_p,\mathbf{V}^{\dagger})\big).$$

Similarly, put $\mathscr{F}^{\boldsymbol{f}}_p(\mathbf{V}^{\dagger}) := \big(V^+_{\boldsymbol{f}} \otimes V_{\boldsymbol{g}} \otimes V_{\boldsymbol{h}}\big) \otimes \mathcal{X}^{-1}$, and define the $\boldsymbol{f}$*-unbalanced local condition* $\mathrm{H}^1_{\boldsymbol{f}}(\mathbf{Q}_p,\mathbf{V}^{\dagger})$ by

$$\mathrm{H}^1_{\boldsymbol{f}}(\mathbf{Q}_p,\mathbf{V}^{\dagger}) := \mathrm{im}\big(\mathrm{H}^1(\mathbf{Q}_p,\mathscr{F}^{\boldsymbol{f}}_p(\mathbf{V}^{\dagger})) \to \mathrm{H}^1(\mathbf{Q}_p,\mathbf{V}^{\dagger})\big).$$

It is easy to see that the maps appearing in these definitions are injective, and in the following we shall use this to identify $\mathrm{H}^1_?(\mathbf{Q}_p,\mathbf{V}^{\dagger})$ with $\mathrm{H}^1(\mathbf{Q}_p,\mathscr{F}^?_p(\mathbf{V}^{\dagger}))$ for $? \in \{\mathrm{bal},\boldsymbol{f}\}$.

Definition 4.2.2. Let $? \in \{\mathrm{bal},\boldsymbol{f}\}$, and define the Selmer group $\mathrm{Sel}^?(\mathbf{Q},\mathbf{V}^{\dagger})$ by

$$\mathrm{Sel}^?(\mathbf{Q},\mathbf{V}^{\dagger}) := \ker\Big\{\mathrm{H}^1(\mathbf{Q},\mathbf{V}^{\dagger}) \to \frac{\mathrm{H}^1(\mathbf{Q}_p,\mathbf{V}^{\dagger})}{\mathrm{H}^1_?(\mathbf{Q}_p,\mathbf{V}^{\dagger})} \times \prod_{v\neq p}\mathrm{H}^1(\mathbf{Q}_v^{\mathrm{nr}},\mathbf{V}^{\dagger})\Big\}.$$

We call $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbf{V}^{\dagger})$ (resp. $\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q},\mathbf{V}^{\dagger})$) the *balanced* (resp. $\boldsymbol{f}$*-unbalanced*) Selmer group.

Let $\mathbf{A}^\dagger = \mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{V}^\dagger, \mu_{p^\infty})$ and for $? \in \{\mathrm{bal}, \boldsymbol{f}\}$ define $\mathrm{H}^1_?(\mathbf{Q}_p, \mathbf{A}^\dagger) \subset \mathrm{H}^1(\mathbf{Q}_p, \mathbf{A}^\dagger)$ to be the orthogonal complement of $\mathrm{H}^1_?(\mathbf{Q}_p, \mathbf{V}^\dagger)$ under the local Tate duality

$$\mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}^\dagger) \times \mathrm{H}^1(\mathbf{Q}_p, \mathbf{A}^\dagger) \to \mathbf{Q}_p/\mathbf{Z}_p.$$

Similarly as above, we then define the balanced and $\boldsymbol{f}$-unbalanced Selmer groups with coefficients in $\mathbf{A}^\dagger$ by

$$\mathrm{Sel}^?(\mathbf{Q}, \mathbf{A}^\dagger) := \ker\left\{\mathrm{H}^1(\mathbf{Q}, \mathbf{A}^\dagger) \to \frac{\mathrm{H}^1(\mathbf{Q}_p, \mathbf{A}^\dagger)}{\mathrm{H}^1_?(\mathbf{Q}_p, \mathbf{A}^\dagger)} \times \prod_{v \neq p} \mathrm{H}^1(\mathbf{Q}_v^{\mathrm{nr}}, \mathbf{A}^\dagger)\right\},$$

and let $X^?(\mathbf{Q}, \mathbf{A}^\dagger) = \mathrm{Hom}_{\mathbf{Z}_p}(\mathrm{Sel}^?(\mathbf{Q}, \mathbf{A}^\dagger), \mathbf{Q}_p/\mathbf{Z}_p)$ denote the Pontryagin dual of $\mathrm{Sel}^?(\mathbf{Q}, \mathbf{A}^\dagger)$.

4.3. **Diagonal classes.** We continue to denote by $(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ a triple of primitive Hida families as in §4.1.1 satisfying (4.1), and put $N = \mathrm{lcm}(N_f, N_g, N_h)$. Let

$$\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in \mathrm{H}^1(\mathbf{Q}, \mathbf{V}^\dagger(N)) \tag{4.5}$$

be the big diagonal class constructed in [BSV22, §8.1], where $\mathbf{V}^\dagger(N)$ is a free $\mathcal{R}$-module isomorphic to finitely many copies of $\mathbf{V}^\dagger$. (Note that this is essentially the class $\widetilde{\boldsymbol{\kappa}}_m^{(1)}$ in (2.13) with $m = 1$.) The definition of the Selmer groups in §4.2 extends immediately to $\mathbf{V}^\dagger(N)$, and by Corollary 8.2 in *loc. cit.* one knows that $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}) \in \mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}^\dagger(N))$.

Put

$$\mathscr{F}_p^3(\mathbf{V}^\dagger) = V_{\boldsymbol{f}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}^+ \otimes \mathcal{X}^{-1} \subset \mathbf{V}^\dagger.$$

Then clearly $\mathscr{F}_p^3(\mathbf{V}^\dagger) \subset \mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger)$, with quotient given by

$$\mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger)/\mathscr{F}_p^3(\mathbf{V}^\dagger) \cong \mathbf{V}_{\boldsymbol{f}}^{\boldsymbol{gh}} \oplus \mathbf{V}_{\boldsymbol{g}}^{\boldsymbol{fh}} \oplus \mathbf{V}_{\boldsymbol{h}}^{\boldsymbol{fg}}, \tag{4.6}$$

where

$$\begin{aligned} \mathbf{V}_{\boldsymbol{f}}^{\boldsymbol{gh}} &= V_{\boldsymbol{f}}^- \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}^+ \otimes \mathcal{X}^{-1}, \\ \mathbf{V}_{\boldsymbol{g}}^{\boldsymbol{fh}} &= V_{\boldsymbol{f}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}}^- \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}^+ \otimes \mathcal{X}^{-1}, \\ \mathbf{V}_{\boldsymbol{h}}^{\boldsymbol{fg}} &= V_{\boldsymbol{f}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{g}}^+ \hat{\otimes}_{\mathcal{O}} V_{\boldsymbol{h}}^- \otimes \mathcal{X}^{-1}. \end{aligned} \tag{4.7}$$

4.3.1. *Reciprocity law.* Assume that the congruence ideal $C(\boldsymbol{f}) \subset \mathbb{I}_f$ is principal, generated by the nonzero $\eta_{\boldsymbol{f}} \in \mathbb{I}_f$. As explained in [BSV22, §7.3], one can deduce from results in [KLZ17] the construction of an injective three-variable p-adic regulator map with pseudo-null cokernel

$$\mathrm{Log}^{\eta_{\boldsymbol{f}}} : \mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}_{\boldsymbol{f}}^{\boldsymbol{gh}}) \to \mathcal{R} \tag{4.8}$$

characterised by the property that for all $\mathfrak{Z} \in \mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}_{\boldsymbol{f}}^{\boldsymbol{gh}})$ and all points $\underline{Q} = (Q_0, Q_1, Q_2) \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}$ of weight (k_0, k_1, k_2) with $\epsilon_{Q_i} = 1$ $(i = 0, 1, 2)$ we have

$$\begin{aligned} \frac{\mathrm{Log}^{\eta_{\boldsymbol{f}}}(\mathfrak{Z})(\underline{Q})}{\eta_{\boldsymbol{f}_{Q_0}}} &= (p-1)\alpha_{Q_0}\left(1 - \frac{\beta_{Q_0}\alpha_{Q_1}\alpha_{Q_2}}{p^{c_{\underline{Q}}}}\right)\left(1 - \frac{\alpha_{Q_0}\beta_{Q_1}\beta_{Q_2}}{p^{c_{\underline{Q}}}}\right)^{-1} \\ &\times \begin{cases} \frac{(-1)^{c_{\underline{Q}}-k_0}}{(c_{\underline{Q}}-k_0)!} \cdot \left\langle \log_p(\mathfrak{Z}_{\underline{Q}}), \eta_{\boldsymbol{f}_{Q_0}} \otimes \omega_{\boldsymbol{g}_{Q_1}} \otimes \omega_{\boldsymbol{h}_{Q_2}} \right\rangle_{\mathrm{dR}}, & \text{if } \underline{Q} \in \mathfrak{X}_{\mathcal{R}}^{\mathrm{bal}}, \\ (k_0 - c_{\underline{Q}} - 1)! \cdot \left\langle \exp_p^*(\mathfrak{Z}_{\underline{Q}}), \eta_{\boldsymbol{f}_{Q_0}} \otimes \omega_{\boldsymbol{g}_{Q_1}} \otimes \omega_{\boldsymbol{h}_{Q_2}} \right\rangle_{\mathrm{dR}}, & \text{if } \underline{Q} \in \mathfrak{X}_{\mathcal{R}}^{\boldsymbol{f}}. \end{cases} \end{aligned}$$

Here, $c_{\underline{Q}} = (k_0 + k_1 + k_2 - 2)/2$ is as in Theorem 4.1.1, α_{Q_0} denotes the specialisation of $a_p(\boldsymbol{f})$ at Q_0, we put $\beta_{Q_0} = \chi_f'(p)p^{k_0-1}\alpha_{Q_0}^{-1}$, and $(\alpha_{Q_1}, \beta_{Q_1})$ (resp. $(\alpha_{Q_2}, \beta_{Q_2})$) are defined likewise with $\boldsymbol{g}$ (resp. $\boldsymbol{h}$) in place of $\boldsymbol{f}$.

Denote by $\mathrm{res}_p(\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}))_{\boldsymbol{f}}$ the image of $\kappa(\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h})$ under natural map

$$\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q}, \mathbf{V}^\dagger) \xrightarrow{\mathrm{res}_p} \mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger)) \to \mathrm{H}^1(\mathbf{Q}_p, \mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger)/\mathscr{F}_p^3(\mathbf{V}^\dagger)) \to \mathrm{H}^1(\mathbf{Q}_p, \mathbf{V}_{\boldsymbol{f}}^{\boldsymbol{gh}}) \tag{4.9}$$

arising from the restriction at p and the projection onto the first direct summand in (4.6).

Theorem 4.3.1. *Let $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ be a triple of primitive Hida families as in Theorem 4.1.1. Then*

$$\mathrm{Log}^{\eta_{\boldsymbol{f}}}(\mathrm{res}_p(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))_{\boldsymbol{f}}) = \mathscr{L}_p^{\boldsymbol{f},\eta_{\boldsymbol{f}}}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}).$$

Proof. This is Theorem A in [BSV22] (see also [DR22, Thm. 10]). □

Remark 4.3.2. The map $\mathrm{Log}^{\eta_{\boldsymbol{f}}}$ in (4.8) depends on a choice of level-N test vectors $(\breve{\boldsymbol{f}},\breve{\boldsymbol{g}},\breve{\boldsymbol{h}})$, and implictly in the above statements we took the triple $(\breve{\boldsymbol{f}}^\star,\breve{\boldsymbol{g}}^\star,\breve{\boldsymbol{h}}^\star)$ constructed in [Hsi21]. For any triple $(\breve{\boldsymbol{f}},\breve{\boldsymbol{g}},\breve{\boldsymbol{h}})$, one deduces from [KLZ17] the existence of a p-adic regulator map

$$\mathrm{Log}_{(\breve{\boldsymbol{f}},\breve{\boldsymbol{g}},\breve{\boldsymbol{h}})} : \mathrm{H}^1(\mathbf{Q}_p,\mathbf{V}_{\boldsymbol{f}}^{\boldsymbol{gh}}) \rightarrow \mathrm{Frac}(\mathbb{I}_f)\hat{\otimes}_{\mathcal{O}}\mathbb{I}_g\hat{\otimes}_{\mathcal{O}}\mathbb{I}_h$$

characterised by a similar interpolation property, and the explictly reciprocity law of Theorem 4.3.1 applies more generally to give

$$\mathrm{Log}_{(\breve{\boldsymbol{f}},\breve{\boldsymbol{g}},\breve{\boldsymbol{h}})}(\mathrm{res}_p(\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}))_{\boldsymbol{f}}) = \mathscr{L}_p^{\boldsymbol{f}}(\breve{\boldsymbol{f}},\breve{\boldsymbol{g}},\breve{\boldsymbol{h}}),$$

where $\mathscr{L}_p^{\boldsymbol{f}}(\breve{\boldsymbol{f}},\breve{\boldsymbol{g}},\breve{\boldsymbol{h}})$ is as in Remark 4.1.2.

4.3.2. *Iwasawa–Greenberg main conjectures.* Let $(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ be a triple of primitive Hida families as in Theorem 4.1.1, and assume that the associated ring $\mathcal{R}$ is regular. As explained in [ACR21, §7.3], the following result can be seen as the equivalence between two different formulation of the Iwasawa main conjecture in the style of Greenberg [Gre94] for the p-adic deformation $\mathbf{V}^\dagger$.

Proposition 4.3.3. *The following statements* (I) *and* (II) *are equivalent:*

(I) $\mathscr{L}_p^{\boldsymbol{f},\eta_{\boldsymbol{f}}}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ *is nonzero, the modules* $\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q},\mathbf{V}^\dagger)$ *and* $X^{\boldsymbol{f}}(\mathbf{Q},\mathbf{A}^\dagger)$ *are both $\mathcal{R}$-torsion, and*

$$\mathrm{char}_{\mathcal{R}}\big(X^{\boldsymbol{f}}(\mathbf{Q},\mathbf{A}^\dagger)\big) = \big(\mathscr{L}_p^{\boldsymbol{f}}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})^2\big)$$

in $\mathcal{R}\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$.

(II) $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ *is not $\mathcal{R}$-torsion, the modules* $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbf{V}^\dagger)$ *and* $X^{\mathrm{bal}}(\mathbf{Q},\mathbf{A}^\dagger)$ *have both $\mathcal{R}$-rank one, and*

$$\mathrm{char}_{\mathcal{R}}\big(X^{\mathrm{bal}}(\mathbf{Q},\mathbf{A}^\dagger)_{\mathrm{tors}}\big) = \mathrm{char}_{\mathcal{R}}\left(\frac{\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbf{V}^\dagger)}{\mathcal{R}\cdot\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})}\right)^2$$

in $\mathcal{R}\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$*, where the subscript* tors *denotes the $\mathcal{R}$-torsion submodule.*

Proof. This follows from Theorem 4.3.1 and global duality in the same way as [ACR21, Thm. 7.15]. See [Lai22] for the details in the stated level of generality. □

4.4. **CM Hida families.** We conclude this section with the explicit construction of certain CM Hida families, following the exposition in [Hsi21, §8.1]. Let K be an imaginary quadratic field of discriminant $-D_K<0$, and suppose that $p=\mathfrak{p}\bar{\mathfrak{p}}$ splits in K, with $\mathfrak{p}$ the prime of K above p induced by our fixed embedding $\imath_p:\overline{\mathbf{Q}}\rightarrow\overline{\mathbf{Q}}_p$.

Let K_∞ be the $\mathbf{Z}_p^2$-extension of K, and denote by $K_{\mathfrak{p}^\infty}$ the maximal subfield of K_∞ unramified outside $\mathfrak{p}$. Put

$$\Gamma_\infty := \mathrm{Gal}(K_\infty/K)\simeq\mathbf{Z}_p^2, \qquad \Gamma_{\mathfrak{p}} := \mathrm{Gal}(K_{\mathfrak{p}^\infty}/K)\simeq\mathbf{Z}_p.$$

For every ideal $\mathfrak{c}\subset\mathcal{O}_K$ we denote by $K[\mathfrak{c}]$ the ray class field of K of conductor $\mathfrak{c}$ (so in particular $K_{\mathfrak{p}^\infty}$ is the maximal $\mathbf{Z}_p$-extension of K inside $K[\mathfrak{p}^\infty]$). Denote by $\mathrm{Art}_{\mathfrak{p}}$ the restriction of the Artin map to $K_{\mathfrak{p}}^\times$, with geometric normalisation. Then $\mathrm{Art}_{\mathfrak{p}}$ induces an embedding $1+p\mathbf{Z}_p\rightarrow\Gamma_{\mathfrak{p}}$, where we identified $\mathbf{Z}_p^\times$ and $\mathcal{O}_{K_{\mathfrak{p}}}^\times$ via $\imath_p$. Write $I_{\mathfrak{p}}^{\mathrm{w}}=\mathrm{Art}_{\mathfrak{p}}(1+p\mathbf{Z}_p)|_{K_{\mathfrak{p}^\infty}}$ and put $[\Gamma_{\mathfrak{p}}:I_{\mathfrak{p}}^{\mathrm{w}}]=p^b$.

Fix a topological generator $\gamma_{\mathfrak{p}}\in\Gamma_{\mathfrak{p}}$ with $\gamma_{\mathfrak{p}}^{p^b}=\mathrm{Art}_{\mathfrak{p}}(1+p)|_{K_{\mathfrak{p}^\infty}}$, and for each variable S let $\Psi_S:\Gamma_\infty\rightarrow\mathcal{O}[\![S]\!]^\times$ be the universal character given by

$$\Psi_S(\sigma) = (1+S)^{l(\sigma)},$$

where $l(\sigma) \in \mathbf{Z}_p$ is such that $\sigma|_{K_{\mathfrak{p}^\infty}} = \gamma_{\mathfrak{p}}^{l(\sigma)}$. Upon possibly enlarging $\mathcal{O}$, assume that it contains an element $\mathbf{v}$ with $\mathbf{v}^{p^b} = 1 + p$. Assume that $\mathfrak{c}$ is prime-to-p, and for any finite order character $\xi : G_K \to \mathcal{O}^\times$ of conductor dividing $\mathfrak{c}$ put

$$\boldsymbol{\theta}_\xi(S)(q) = \sum_{(\mathfrak{a},\mathfrak{p}\mathfrak{c})=1} \xi(\sigma_{\mathfrak{a}}) \Psi^{-1}_{\mathbf{v}^{-1}(1+S)-1}(\sigma_{\mathfrak{a}}) q^{N_{K/\mathbf{Q}}(\mathfrak{a})} \in \mathcal{O}[\![S]\!][\![q]\!],$$

where $\sigma_{\mathfrak{a}} \in \mathrm{Gal}(K[\mathfrak{c}\mathfrak{p}^\infty]/K)$ is the Artin symbol of $\mathfrak{a}$. Then $\boldsymbol{\theta}_\xi(S)$ is a Hida family defined over $\mathcal{O}[\![S]\!]$ of tame level $N_{K/\mathbf{Q}}(\mathfrak{c}) D_K$ and tame character $(\xi \circ \mathscr{V})\epsilon_K \omega^{-1}$, where $\mathscr{V} : G_{\mathbf{Q}}^{\mathrm{ab}} \to G_K^{\mathrm{ab}}$ is the transfer map and ϵ_K is the quadratic character corresponding to $K/\mathbf{Q}$.

5. Definite case

In this section we deduce our applications to the Bloch–Kato conjecture and the Iwasawa main conjecture for anticyclotomic twists of f/K in the case where $\epsilon(f/K) = +1$.

5.1. **Anticyclotomic p-adic L-functions.** Let $f \in S_k(\Gamma_0(pN_f))$ be a p-ordinary p-stabilised newform of weight $k = 2r \geq 2$, and tame level N_f defined over $\mathcal{O}$, and denote by $\alpha = \alpha_p(f) \in \mathcal{O}^\times$ the U_p-eigenvalue of f. Assume that f is p-old, and let $f^\circ \in S_k(\Gamma_0(N_f))$ be the newform associated with f. Write

$$N_f = N^+ N^-$$

with N^+ (resp. N^-) divisible only by primes which are split (resp. inert) in K, and fix an ideal $\mathfrak{N}^+ \subset \mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{N}^+ \simeq \mathbf{Z}/N^+\mathbf{Z}$.

Let Γ^- be the Galois group of the anticyclotomic $\mathbf{Z}_p$-extension K_∞/K. By definition, the map $\sigma \mapsto l(\sigma^{1-\mathbf{c}}|_{K_{\mathfrak{p}^\infty}})$ factor through Γ^-, and we let γ_- be the topological generator of Γ^- mapping to 1 under the resulting isomorphism $\Gamma^- \simeq \mathbf{Z}_p$. As usual, we identity the anticyclotomic Iwasawa algebra

$$\Lambda_K^- := \mathcal{O}[\![\Gamma^-]\!]$$

with the one-variable power series ring $\mathcal{O}[\![W]\!]$ via $\gamma_- \mapsto 1 + W$. For any prime-to-p ideal $\mathfrak{a}$ of K, let $\sigma_{\mathfrak{a}}$ be the image of $\mathfrak{a}$ in the Galois group of the ray class field $K[p^\infty]/K$ under the Artin reciprocity map.

Theorem 5.1.1. *Let χ be a ring class character of conductor $c\mathcal{O}_K$ with values in $\mathcal{O}$, and suppose:*

(i) $(pN_f, cD_K) = 1$,

(ii) *N^- is the squarefree product of an odd number of primes.*

Then there exists a unique element $\Theta_p^{\mathrm{BD}}(f/K,\chi)(W) \in \mathcal{O}[\![W]\!]$ such that for every character ϕ of Γ^- of infinity type $(j,-j)$ with $0 \leq j < r$ and conductor p^n, we have

$$\Theta_p^{\mathrm{BD}}(f/K,\chi)(\phi(\gamma_-) - 1)^2 = \frac{p^{(2r-1)n}}{\alpha_p(f)^{2n}} \cdot \Gamma(r)^2 \cdot \mathcal{E}_p(f,\chi\phi)^2 \cdot \frac{L(f^\circ,\chi\phi,r)}{(2\pi)^{2r} \cdot \Omega_{f^\circ,N^-}} \cdot u_K^2 \sqrt{D_K} \chi\phi(\sigma_{\mathfrak{N}^+}) \cdot \varepsilon_p,$$

where:

- $\mathcal{E}_p(f,\chi\phi) = \begin{cases} (1 - \alpha_p(f)^{-1} p^{r-1} \chi\phi(\mathfrak{p}))(1 - \alpha_p(f) p^{r-1} \chi\phi(\bar{\mathfrak{p}})) & \text{if } n = 0, \\ 1 & \text{if } n > 0, \end{cases}$
- $\Omega_{f^\circ,N^-} = 2^{2r} \cdot \|f^\circ\|^2_{\Gamma_0(N_f)} \cdot \eta^{-1}_{f,N^-}$ *is the* Gross period *of f° (see* [Hsi21, p. 524]*),*
- $u_K = |\mathcal{O}_K^\times|/2$, *and $\varepsilon_p \in \{\pm 1\}$ is the local root number of f° at p.*

Proof. This is Theorem A in [CH18b] (see also [Hun17, Thm. A]), extending and refining a construction in [BD96] in weight 2. □

5.2. **Factorisation of triple product p-adic L-functions.** Let $f \in S_{2r}(pN_f)$ be a p-stabilised newform as in §5.1, and suppose the residual representation $\bar{\rho}_f$ satisfies:

$$\bar{\rho}_f \text{ is absolutely irreducible and } p\text{-distinguished.} \tag{5.1}$$

By Hida theory, f is the specialisation of a unique primitive Hida family $\boldsymbol{f} \in S^o(N_f, \mathbb{I})$ at an arithmetic point $Q_0 \in \mathfrak{X}_{\mathbb{I}}^+$ of weight $2r$. Let $\mathfrak{f}_1, \mathfrak{f}_2 \subset \mathcal{O}_K$ be ideals coprime to pN_f, and let ξ_1, ξ_2 be ray class characters of K of conductors dividing $\mathfrak{f}_1, \mathfrak{f}_2$, respectively. Let χ_{ξ_i} be the central character of ξ_i $(i = 1, 2)$. We assume that

$$\chi_{\xi_1}\chi_{\xi_2} = 1, \tag{5.2}$$

and let

$$\boldsymbol{g}_1 = \boldsymbol{\theta}_{\xi_1}(S_1) \in \mathcal{O}[\![S_1]\!][\![q]\!], \quad \boldsymbol{g}_2 = \boldsymbol{\theta}_{\xi_2}(S_2) \in \mathcal{O}[\![S_2]\!][\![q]\!] \tag{5.3}$$

be the CM Hida families attached to ξ_1 and ξ_2, respectively.

The triple $(\boldsymbol{f}, \boldsymbol{g}_1, \boldsymbol{g}_2)$ satisfies conditions (4.1) and (4.4) and the associated $\boldsymbol{f}$-unbalanced triple product p-adic L-function $\mathscr{L}_p^{\boldsymbol{f},\eta_{\boldsymbol{f}}}(\boldsymbol{f}, \boldsymbol{g}_1, \boldsymbol{g}_2)$ is an element in $\mathcal{R} = \mathbb{I}\hat{\otimes}_{\mathcal{O}}\mathcal{O}[\![S_1]\!]\hat{\otimes}_{\mathcal{O}}\mathcal{O}[\![S_2]\!] \simeq \mathbb{I}[\![S_1, S_2]\!]$; in the following we let

$$\mathscr{L}_p^{\boldsymbol{f},\eta_{\boldsymbol{f}}}(f, \boldsymbol{g}_1, \boldsymbol{g}_2) \in \mathcal{O}[\![S_1, S_2]\!] \tag{5.4}$$

denote its image under the natural map $\mathbb{I}[\![S_1, S_2]\!] \to \mathcal{O}[\![S_1, S_2]\!]$ defined by Q_0. Denote by $\mathbf{c}$ the non-trivial automorphism of $K/\mathbf{Q}$, and for a Hecke character ψ put $\psi^{\mathbf{c}}(\sigma) := \psi(\mathbf{c}\sigma\mathbf{c})$.

Proposition 5.2.1. *Assume (5.1), (5.2), and that N^- is the squarefree product of an odd number of primes. Set*

$$W_1 = \mathbf{v}^{-1}(1+S_1)^{1/2}(1+S_2)^{1/2} - 1, \quad W_2 = (1+S_1)^{1/2}(1+S_2)^{-1/2} - 1.$$

Then

$$\mathscr{L}_p^{\boldsymbol{f},\eta_{\boldsymbol{f}}}(f, \boldsymbol{g}_1, \boldsymbol{g}_2)(S_1, S_2) = \pm\mathbf{w} \cdot \Theta_p^{\mathrm{BD}}(f/K, \xi_1\xi_2)(W_1) \cdot \Theta_p^{\mathrm{BD}}(f/K, \xi_1\xi_2^{\mathbf{c}})(W_2) \cdot \frac{\eta_{f^\circ}}{\eta_{f^\circ, N^-}},$$

where $\mathbf{w}$ is a unit in $\mathcal{O}[\![S_1, S_2]\!]$.

Proof. This is an immediate extension of Proposition 8.1 in [Hsi21], where the case $\xi_2 = \xi_1^{-1}$ is treated. ☐

5.3. **Selmer group decompositions.** As in §5.2, suppose $\boldsymbol{f}$ is the Hida family passing through a p-ordinary p-stabilised newform $f \in S_k(pN_f)$ of weight $k = 2r \geq 2$ (so $\boldsymbol{f}_{Q_0} = f$), and

$$(\boldsymbol{g}, \boldsymbol{h}) = (\boldsymbol{g}_1, \boldsymbol{g}_2) = (\boldsymbol{\theta}_{\xi_1}(S_1), \boldsymbol{\theta}_{\xi_2}(S_2))$$

are CM Hida families as in (5.3).

Write $\mathbf{V}_{Q_0}^\dagger$ for the specialisation of $\mathbf{V}^\dagger$ at Q_0. Let $V_f^\vee$ be the Galois representation associated to f, and recall that $\det(V_f^\vee) = \varepsilon_{\mathrm{cyc}}^{2r-1}$ in our conventions. Setting $T_i = \mathbf{v}^{-1}(1+S_i) - 1$ $(i = 1, 2)$, we have $\det(V_{\boldsymbol{g}_{T_1}} \otimes V_{\boldsymbol{h}_{T_2}}) = \Psi_{T_1}\Psi_{T_2} \circ \mathscr{V}$, and so

$$\begin{aligned} \mathbf{V}_{Q_0}^\dagger &\simeq T_f^\vee \otimes (\mathrm{Ind}_K^{\mathbf{Q}}\xi_1^{-1}\Psi_{T_1}) \otimes (\mathrm{Ind}_K^{\mathbf{Q}}\xi_2^{-1}\Psi_{T_2}) \otimes \varepsilon_{\mathrm{cyc}}^{1-r}(\Psi_{T_1}^{-1/2}\Psi_{T_2}^{-1/2} \circ \mathscr{V}) \\ &\simeq \big(T_f^\vee(1-r) \otimes \mathrm{Ind}_K^{\mathbf{Q}}\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\big) \oplus \big(T_f^\vee(1-r) \otimes \mathrm{Ind}_K^{\mathbf{Q}}\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\big), \end{aligned} \tag{5.5}$$

where $T_f^\vee$ is a $G_{\mathbf{Q}}$-stable $\mathcal{O}$-lattice inside $V_f^\vee$, and we put

$$W_1 = \mathbf{v}^{-1}(1+S_1)^{1/2}(1+S_2)^{1/2} - 1, \quad W_2 = (1+S_1)^{1/2}(1+S_2)^{-1/2} - 1$$

as in Proposition 5.2.1. In particular, we get

$$\mathrm{H}^1(\mathbf{Q}, \mathbf{V}_{Q_0}^\dagger) \simeq \mathrm{H}^1(K, T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}) \oplus \mathrm{H}^1(K, T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}) \tag{5.6}$$

by Shapiro's lemma.

Proposition 5.3.1. *Under (5.6), the balanced Selmer group* $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbf{V}^\dagger_{Q_0})$ *decomposes as*

$$\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbf{V}^\dagger_{Q_0})\simeq \mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K,T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}})\oplus\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K,T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}),$$

and the $\boldsymbol{f}$*-unbalanced Selmer group* $\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q},\mathbf{V}^\dagger_{Q_0})$ *decomposes as*

$$\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q},\mathbf{V}^\dagger_{Q_0})\simeq \mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K,T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}})\oplus\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K,T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}).$$

Proof. From (5.5) we see that the balanced local condition is given by

$$\begin{aligned}\mathscr{F}_p^{\mathrm{bal}}(\mathbf{V}^\dagger_{Q_0})\simeq\big(T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\big)&\oplus\big(T_f^{\vee,+}(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\big)\\&\oplus\big(T_f^{\vee,+}(1-r)\otimes\xi_1\xi_2^{\mathbf{c}}\Psi_{W_2}^{\mathbf{c}-1}\big).\end{aligned}\tag{5.7}$$

Put $\widetilde{\mathbf{V}}^\dagger_{Q_0}=(T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}})\oplus(T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}})$, so by (5.5) we have

$$\mathrm{H}^1(\mathbf{Q},\mathbf{V}^\dagger_{Q_0})\simeq\mathrm{H}^1(K,\widetilde{\mathbf{V}}^\dagger_{Q_0}),\tag{5.8}$$

and from (5.7) we obtain

$$\begin{aligned}\mathscr{F}_{\mathfrak{p}}^{\mathrm{bal}}(\widetilde{\mathbf{V}}^\dagger_{Q_0})&\simeq\big(T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\big)\oplus\big(T_f^{\vee,+}(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\big),\\ \mathscr{F}_{\bar{\mathfrak{p}}}^{\mathrm{bal}}&\simeq\{0\}\qquad\qquad\qquad\qquad\qquad\oplus T_f^{\vee,+}(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}},\end{aligned}$$

and this yields the claimed description of $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbf{V}^\dagger_{Q_0})$. On the other hand, we similarly find that the $\boldsymbol{f}$-balanced local condition is given by

$$\begin{aligned}\mathscr{F}_{\mathfrak{p}}^{\boldsymbol{f}}(\widetilde{\mathbf{V}}^\dagger_{Q_0})&\simeq\big(T_f^{\vee,+}(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\big)\oplus\big(T_f^{\vee,+}(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\big),\\ \mathscr{F}_{\bar{\mathfrak{p}}}^{\boldsymbol{f}}(\widetilde{\mathbf{V}}^\dagger_{Q_0})&\simeq\big(T_f^{\vee,+}(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\big)\oplus\big(T_f^{\vee,+}(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\big),\end{aligned}$$

from where the claimed description of $\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q},\mathbf{V}^\dagger_{Q_0})$ follows. □

As a consequence we also obtain the following decomposition for the Selmer groups with coefficients in $\mathbf{A}^\dagger_{Q_0}=\mathrm{Hom}_{\mathbf{Z}_p}(\mathbf{V}^\dagger_{Q_0},\mu_{p^\infty})$, mirroring in the case of $\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q},\mathbf{A}^\dagger_{Q_0})$ the factorisation of p-adic L-functions in Proposition 5.2.1.

Corollary 5.3.2. *The balanced Selmer group* $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbf{A}^\dagger_{Q_0})$ *decomposes as*

$$\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbf{A}^\dagger_{Q_0})\simeq\mathrm{Sel}_{\mathrm{str},\mathrm{rel}}(K,A_f(r)\otimes\xi_1\xi_2\Psi_{W_1}^{\mathbf{c}-1})\oplus\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K,A_f(r)\otimes\xi_1\xi_2^{\mathbf{c}}\Psi_{W_2}^{\mathbf{c}-1}),$$

where $A_f(r)=\mathrm{Hom}_{\mathbf{Z}_p}(T_f^\vee(1-r),\mu_{p^\infty})$*; and the* $\boldsymbol{f}$*-unbalanced Selmer group* $\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q},\mathbf{A}^\dagger_{x_1})$ *decomposes as*

$$\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q},\mathbf{A}^\dagger_{Q_0})\simeq\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K,A_f(r)\otimes\xi_1\xi_2\Psi_{W_1}^{\mathbf{c}-1})\oplus\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K,A_f(r)\otimes\xi_1\xi_2^{\mathbf{c}}\Psi_{W_2}^{\mathbf{c}-1}).$$

Proof. This is immediate from Proposition 5.3.1 and local Tate duality. □

5.4. **Explicit reciprocity law.** With the same setting as in §5.3, put

$$\mathbb{V}^\dagger=\mathbf{V}^\dagger_{Q_0}\otimes_{\mathcal{O}[\![S_1,S_2]\!]}\mathcal{O}[\![S_1,S_2]\!]/(S_1-S_2)$$

(so in the quotient the weights of the CM Hida families $\boldsymbol{g}$ and $\boldsymbol{h}$ move in tandem), and denote by

$$\kappa(f,\underline{\boldsymbol{gh}})\in\mathrm{H}^1(\mathbf{Q},\mathbb{V}^\dagger)\tag{5.9}$$

the resulting restriction of the three-variable big diagonal class $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ in (4.5). (Here we are implicitly choosing level-N test vectors $(\breve{\boldsymbol{f}},\breve{\boldsymbol{g}},\breve{\boldsymbol{h}})$, where $N=\mathrm{lcm}(N_f,N_{K/\mathbf{Q}}(\mathfrak{f}_1\mathfrak{f}_2)D_K)$ to project the classes from $\mathbb{V}^\dagger(N)$ to $\mathbb{V}^\dagger$.) Similarly, we denote by $\mathscr{L}_p^{\boldsymbol{f},\eta_{\boldsymbol{f}}}(f,\underline{\boldsymbol{gh}})$ the image of (5.4) in the quotient $\mathcal{O}[\![S_1,S_2]\!]/(S_1-S_2)$.

Since $\kappa(f,\underline{\boldsymbol{gh}})\in\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbb{V}^\dagger)$ as a consequence of [BSV22, Cor. 8.2], we can write

$$\kappa(f,\underline{\boldsymbol{gh}})=(\kappa_1(f,\underline{\boldsymbol{gh}}),\kappa_2(f,\underline{\boldsymbol{gh}}))\tag{5.10}$$

according to the decomposition from Proposition 5.3.1; in particular, we have

$$\kappa_1(f,\underline{\boldsymbol{gh}})\in\mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K,T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}),\tag{5.11}$$

where $W_1 = \mathbf{v}^{-1}(1+S_1)-1$.

Similarly as in §4.1.1, denote by $\mathfrak{X}^{\rm cls}_{\mathcal{O}[\![W_1]\!]}$ the set of ring homomorphisms $Q \in \mathrm{Spec}(\mathcal{O}[\![W_1]\!])(\overline{\mathbf{Q}}_p)$ with $Q(W_1) = \zeta_Q(1+p)^{k_Q-1}-1$ for some $\zeta_Q \in \mu_{p^\infty}$ and $k_Q \in \mathbf{Z}_{\geq 1}$, and for any $\mathcal{O}[\![W_1]\!]$-module M denote by M_Q the corresponding specialisation. Write $T_f^{\vee,-} := T_f^\vee/T_f^{\vee,+}$, where the $G_{\mathbf{Q}_p}$-action is given by the unramified character sending an arithmetic Frobenius to $\alpha_p(f)$. Then it is easy to see that for any $Q \in \mathfrak{X}^{\rm cls}_{\mathcal{O}[\![W_1]\!]}$ the Bloch–Kato dual exponential and logarithm maps give rise to $L(\zeta_Q)$-isomorphisms

$$\begin{aligned} \exp_{\mathfrak{p}}^* &: \mathrm{H}^1(K_{\mathfrak{p}}, T_f^{\vee,-}(1-r)\otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}})_Q \to L(\zeta_Q), \quad \text{if } 1\leq k_Q\leq r, \\ \log_{\mathfrak{p}} &: \mathrm{H}^1(K_{\mathfrak{p}}, T_f^{\vee,-}(1-r)\otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}})_Q \to L(\zeta_Q), \quad \text{if } k_Q > r, \end{aligned} \tag{5.12}$$

where L is the field of fractions of $\mathcal{O}$.

Denote by $p_f^- : T_f^\vee(1-r) \to T_f^{\vee,-}(1-r)$ the natural projection.

Theorem 5.4.1. *There is an injective $\mathcal{O}[\![W_1]\!]$-module homomorphism with pseudo-null cokernel*

$$\mathscr{L}_{\mathfrak{p}}^{\eta_f} : \mathrm{H}^1(K_{\mathfrak{p}}, T_f^{\vee,-}(1-r)\otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}) \to \mathcal{O}[\![W_1]\!]$$

such that for any $\mathfrak{Z} \in \mathrm{H}^1(K_{\mathfrak{p}}, T_f^{\vee,-}(1-r)\otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}})$ and $Q \in \mathfrak{X}^{\rm cls}_{\mathcal{O}[\![W_1]\!]}$ we have

$$\mathscr{L}_{\mathfrak{p}}^{\eta_f}(\mathfrak{Z})_Q = \begin{cases} c_Q\cdot \exp_{\mathfrak{p}}^*(\mathfrak{Z}_Q) & \text{if } 1\leq k_Q\leq r, \\ c_Q\cdot \log_{\mathfrak{p}}(\mathfrak{Z}_Q) & \text{if } k_Q > r, \end{cases}$$

where c_Q is an explicit nonzero constant. Moreover, we have the explicit reciprocity law

$$\mathscr{L}_{\mathfrak{p}}^{\eta_f}\big(p_f^-(\mathrm{res}_{\mathfrak{p}}(\kappa_1(f,\underline{\boldsymbol{gh}})))\big)(W_1) = \mathscr{L}_p^{\boldsymbol{f},\eta_{\boldsymbol{f}}}(f,\underline{\boldsymbol{gh}})(S_1),$$

where $S_1 = \mathbf{v}(1+W_1)-1$.

Proof. In terms of (5.5), we find that $\mathscr{F}_p^3(\mathbb{V}^\dagger) = T_f^{\vee,+}(1-r)\otimes\xi_1\xi_2\Psi_{W_1}^{1-\mathbf{c}}$, which together with (5.7) gives the decomposition

$$\begin{aligned} \mathscr{F}_p^{\rm bal}(\mathbb{V}^\dagger)/\mathscr{F}_p^3(\mathbb{V}^\dagger) \simeq{}& \big(T_f^{\vee,-}(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\big) \\ &\oplus \big(T_f^{\vee,+}(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\big)_{W_2} \oplus \big(T_f^{\vee,+}(1-r)\otimes\xi_1\xi_2^{\mathbf{c}}\Psi_{W_2}^{\mathbf{c}-1}\big)_{W_2}, \end{aligned}$$

with the terms in the direct sum corresponding to $\mathbb{V}_f^{\boldsymbol{gh}}$, $\mathbb{V}_{\boldsymbol{h}}^{\boldsymbol{fg}}$, and $\mathbb{V}_{\boldsymbol{g}}^{\boldsymbol{fh}}$ in (4.6), respectively. Here the subscript W_2 denotes the quotient by W_2, noting that $\mathcal{O}[\![S_1,S_2]\!]/(S_1-S_2)\simeq\mathcal{O}[\![W_1,W_2]\!]/(W_2)$.

Thus we find that under the first isomorphism of Proposition 5.3.1, the composite map in (4.9) corresponds to the projection onto $\mathrm{Sel}_{\rm rel,str}(K, T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}})$ composed with the natural map

$$\begin{aligned} \mathrm{Sel}_{\rm rel,str}(K, T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}) &\xrightarrow{\mathrm{res}_{\mathfrak{p}}} \mathrm{H}^1(K_{\mathfrak{p}}, T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}) \\ &\xrightarrow{p_f^-} \mathrm{H}^1(K_{\mathfrak{p}}, T_f^{\vee,-}(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}), \end{aligned}$$

and so under the corresponding isomorphisms we have

$$\mathrm{res}_p(\kappa(f,\underline{\boldsymbol{gh}}))_f = p_f^-(\mathrm{res}_{\mathfrak{p}}(\kappa_1(f,\underline{\boldsymbol{gh}}))) \in \mathrm{H}^1(\mathbf{Q}_p, \mathbb{V}_f^{\boldsymbol{gh}}) \simeq \mathrm{H}^1(K_{\mathfrak{p}}, T_f^{\vee,-}(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}).$$

Finally, the construction of $\mathscr{L}_{\mathfrak{p}}^{\eta_f}$ is deduced from a specialization of the three-variable p-adic regulator map $\mathrm{Log}^{\eta_{\boldsymbol{f}}}$ in §4.3 by the same argument as in [ACR21, Prop. 7.3], and the associated explicit reciprocity law then follows from Theorem 4.3.1. ∎

Remark 5.4.2. Without the need to assume condition (5.1) on $\bar\rho_f$, for any choice of level-N test vectors $(\breve f, \breve{\boldsymbol{g}}, \breve{\boldsymbol{h}})$ the same argument as in the proof of Theorem 5.4.1 gives an equality

$$\mathscr{L}_{\mathfrak{p},(\breve f,\underline{\breve{\boldsymbol{g}}\breve{\boldsymbol{h}}})}\big(p_f^-(\mathrm{res}_{\mathfrak{p}}(\kappa_1(f,\underline{\boldsymbol{gh}})))\big)(W_1) = \mathscr{L}_p^{\boldsymbol{f}}(\breve f,\underline{\breve{\boldsymbol{g}}\breve{\boldsymbol{h}}})(S_1),$$

where $\mathscr{L}_{\mathfrak{p},(\breve f,\underline{\breve{\boldsymbol{g}}\breve{\boldsymbol{h}}})}$ and $\mathscr{L}_p^{\boldsymbol{f}}(\breve f,\underline{\breve{\boldsymbol{g}}\breve{\boldsymbol{h}}})$ are specilisation of the map $\mathrm{Log}_{(\breve{\boldsymbol{f}},\breve{\boldsymbol{g}},\breve{\boldsymbol{h}})}$ and the p-adic L-function $\mathscr{L}_p^{\boldsymbol{f}}(\breve{\boldsymbol{f}},\breve{\boldsymbol{g}},\breve{\boldsymbol{h}})$ in Remark 4.3.2 and Remark 4.1.2, respectively.

5.5. **On the Bloch–Kato conjecture in rank** 0**.** In this section we deduce our first applications to the Bloch–Kato conjecture in analytic rank zero for the twisted G_K-representation

$$V_{f,\chi} := V_f^\vee(1-r)\otimes\chi^{-1}.$$

Denote by K_c the ring class field of K of conductor c. If χ is a Hecke character of conductor $c\mathcal{O}_K$, then its p-adic avatar is a locally algebraic character of $\mathrm{Gal}(K_{cp^\infty}/K)$. The Galois group $\Gamma^- = \mathrm{Gal}(K_\infty^-/K)$ of the anticyclotomic $\mathbf{Z}_p$-extension of K is the maximal $\mathbf{Z}_p$-free quotient of $\mathrm{Gal}(K_{cp^\infty}/K)$. Fix a (non-canonical) splitting

$$\mathrm{Gal}(K_{cp^\infty}/K) \simeq \Delta_c\times\Gamma^-, \tag{5.13}$$

where Δ_c is the torsion subgroup of $\mathrm{Gal}(K_{cp^\infty}/K)$. Note that every character of Δ_c can be viewed as the p-adic avatar of a ring class character of K of conductor dividing $cp^s\mathcal{O}_K$ for sufficiently large s. If χ is as above, we then write $\chi = \chi_t\cdot\chi_w$ according to the decomposition (5.13).

Theorem 5.5.1. *Let $f\in S_k(\Gamma_0(pN_f))$ be a p-ordinary p-stabilised newform of weigh $k=2r\geq 2$ which is old at p, and let χ be anticyclotomic Hecke character of conductor $c\mathcal{O}_K$ and infinity type $(-j,j)$ for some $j\geq 0$. Assume that:*

- *N^- is a square-free product of an odd number of primes;*
- *$\bar\rho_f$ is absolutely irreducible;*
- *$(pN_f, cD_K)=1$;*
- *$p>k-2$;*
- *$p\nmid h_K$, the class number of K;*
- *χ_t has conductor prime-to-p.*

Then

$$L(f/K,\chi,r)\neq 0 \quad\Longrightarrow\quad \mathrm{Sel}_{\mathrm{BK}}(K,V_{f,\chi})=0,$$

and hence the Bloch–Kato conjecture for $V_{f,\chi}$ holds in analytic rank zero.

Proof. We begin by noting that under our hypotheses the modular form f is not of CM-type (since $N^->1$) and the sign in the functional equation of $L(f/K,\chi,s)$ is -1 for $j\geq r$, so without loss of generality below we assume that $0\leq j<r$.

Write $\chi_t=\alpha/\alpha^{\mathbf{c}}$ with α a ray class character of K of conductor $\mathfrak{f}\subset\mathcal{O}_K$ prime-to-p (note that this is possible by e.g. [DR17, Lem. 6.9] or [Hid06b, Lem. 5.31] and our assumption on χ_t). Now, we fix a prime $\ell\neq p$ split in K, and for an auxiliary ring class character β (to be further specified below) of ℓ-power conductor we consider the setting of §5.2 with the CM Hida families

$$\boldsymbol{g}=\boldsymbol{\theta}_{\xi_1}(S_1),\quad \boldsymbol{h}=\boldsymbol{\theta}_{\xi_2}(S_2),$$

where

$$\xi_1:=\beta\alpha,\quad \xi_2:=\beta^{-1}\alpha^{-\mathbf{c}}.$$

Then the decomposition (5.5) of the associated $\mathbf{V}^\dagger_{Q_0}$ specialised to $S_1=S_2$ yields

$$\mathbb{V}^\dagger\simeq\big(T_f^\vee(1-r)\otimes\mathrm{Ind}_K^{\mathbf{Q}}\chi_t^{-1}\Psi_{W_1}^{1-\mathbf{c}}\big)\oplus\big(T_f^\vee(1-r)\otimes\mathrm{Ind}_K^{\mathbf{Q}}\beta^{-2}, \tag{5.14}$$

where $W_1=\mathbf{v}^{-1}(1+S_1)-1$. Denoting by Q the specialization $W\mapsto\zeta(1+p)^j-1$ ($\zeta\in\mu_{p^\infty}$) corresponding to χ_w, it follows that

$$L(\mathbb{V}_Q^\dagger,0)=L(f/K,\chi,r)\cdot L(f/K,\beta^2,r).$$

By [CH18b, Thm. D], the character β may be chose so that $\Theta_p^{\mathrm{BD}}(f/K,\beta^2)(0)$ is nonzero, and for such choice of β by Theorem 5.4.1 (see also Remark 5.4.2) we then have

$$L(f/K,\chi,r)\neq 0\quad\Longrightarrow\quad \mathrm{res}_{\mathfrak{p}}(\kappa_1(f,\underline{\boldsymbol{gh}})_Q)\neq 0.$$

By construction, the class $\kappa_1(f,\underline{\boldsymbol{gh}})_Q\in\mathrm{Sel}_{\mathrm{rel,str}}(K,V_{f,\chi})$ is the base class of the anticyclotomic Euler system

$$\{z_{f,\psi_1,\psi_2,\mathfrak{m}}\in\mathrm{Sel}_{\mathrm{rel,str}}(K[m],T_{f,\chi}):\ \mathfrak{m}\in\mathcal{N}\}$$

of (3.3), with (ψ_1, ψ_2) the Hecke characters associated with $(\boldsymbol{g}_{S_1}, \boldsymbol{h}_{S_2})$. Hence by Theorem 3.3.1 we deduce that the Selmer group $\mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K, V_{f,\chi})$ is one-dimensional, spanned by

$$z_{f,\chi} = \mathrm{cor}_{K[1]/K}(z_{f,\psi_1,\psi_2,(1)}) = \kappa_1(f, \underline{\boldsymbol{gh}})_Q.$$

Since also $\mathrm{res}_{\mathfrak{p}}(z_{f,\chi}) \neq 0$, the vanishing of $\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K, V_{f,\chi})$ then follows by a standard argument using Poitou–Tate duality (see e.g. [CH18a, Thm. 7.9]). Since by Lemma 3.1.2 for $0 \leq j < r$ the latter group is the same as $\mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi})$, this yields the result. ☐

5.6. **On the Iwasawa main conjecture.** Our next application is a divisibility in the anticyclotomic Iwasawa main conjecture for modular forms in the definite setting. For an eigenform f of weight $k = 2r \geq 2$ and trivial nebentypus and χ an anticyclotomic character, put

$$A_{f,\chi} = \mathrm{Hom}_{\mathbf{Z}_p}(T_f^\vee(1-r) \otimes \chi^{-1}, \mu_{p^\infty}).$$

Theorem 5.6.1. *Let the hypotheses be as in Theorem 5.5.1, and assume in addition that:*

- $\bar{\rho}_f$ *is p-distinguished;*
- f *has big image (see 3.3.2).*

Then $\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K_\infty^-, A_{f,\chi})$ *is cotorsion over* Λ_K^-*, and we have the divisibility*

$$\mathrm{char}_{\Lambda_K^-}\left(\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K_\infty^-, A_{f,\chi})^\vee\right) \supset \left(\Theta_p^{\mathrm{BD}}(f/K, \chi)\right)$$

in $\Lambda_K^- \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.

Proof. Proceeding as in the proof of Theorem 5.5.1 we obtain the decomposition (5.14), which by Proposition 5.2.1 translates into the factorisation

$$\mathscr{L}_p^{\boldsymbol{f},\eta_f}(f, \underline{\boldsymbol{gh}})(S_1) = \pm \mathbf{w} \cdot \Theta_p^{\mathrm{BD}}(f/K, \chi_t)(W_1) \cdot \Theta_p^{\mathrm{BD}}(f/K, \beta^2)(0) \cdot \frac{\eta_{f^\circ}}{\eta_{f^\circ, N^-}}, \tag{5.15}$$

where $W_1 = \mathbf{v}^{-1}(1 + S_1) = 1$, and from Proposition 5.3.1 we have

$$\mathrm{Sel}^{\boldsymbol{f}}(\mathbf{Q}, \mathbb{A}^\dagger) \simeq \mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K_\infty^-, A_f(r) \otimes \chi) \oplus \mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K, A_f(r) \otimes \beta^2), \tag{5.16}$$

where $\mathbb{A}^\dagger = \mathrm{Hom}_{\mathbf{Z}_p}(\mathbb{V}^\dagger, \mu_{p^\infty})$.

By Theorem 5.1.1 and [CH18b, Thm. D], the auxiliary ring class character β may be chosen so that $\Theta_p^{\mathrm{BD}}(f/K, \beta^2)(0) \neq 0$. Moreover, by Vatsal's nonvanishing results [Vat03] and their extension to higher weights by Chida–Hsieh [CH18b], the p-adic L-function $\Theta_p^{\mathrm{BD}}(f/K, \chi_t)(W_1)$ is nonzero. Hence from (5.15) and Theorem 5.4.1 it follows that the class

$$\kappa_1(f, \underline{\boldsymbol{gh}}) \in \mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K, T_f^\vee(1-r) \otimes \xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}})$$

is non-torsion over $\mathcal{O}[\![W_1]\!]$. Since by construction $\kappa_1(f, \underline{\boldsymbol{gh}})$ is the base class of the Λ_K^--adic anticyclotomic Euler system

$$\left\{\mathbf{z}_{f,\psi_1,\psi_2,\mathfrak{m}} \in \mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K[mp^\infty], T_{f,\chi}) : \ \mathfrak{m} \in \mathcal{N}\right\}$$

in (3.4), with (ψ_1, ψ_2) the Hecke characters corresponding to $(\boldsymbol{g}_{S_1}, \boldsymbol{h}_{S_2})$, the result follows immediately from Theorem 3.3.3 applied to

$$\mathbf{z}_{f,\chi} := \mathrm{cor}_{K[1]/K}(\mathbf{z}_{f,\psi_1,\psi_2,(1)}) = \kappa_1(f, \underline{\boldsymbol{gh}}), \tag{5.17}$$

the equivalence in Proposition 4.3.3, and the Selmer group decomposition (5.16), using that by the rank 0 cases of the Bloch–Kato conjecture established in Theorem 5.5.1, the nonvanishing of $L(f/K, \beta^2, r)$ implies that $\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K, A_f(r) \otimes \beta^2)$ is finite. ☐

Remark 5.6.2. A divisibility in the anticyclotomic Iwsawa Main Conjecture for $V_{f,\chi}$ was first obtained by Bertolini–Darmon [BD05] in weight $k = 2$ and by Chida–Hsieh [CH15] in higher weights using Heegner points and level-raising congruences. Our proof of Theorem 5.6.1 is completely different from theirs, and in particular it dispenses with any "level-raising" hypothesis.

5.7. **On the Bloch–Kato conjecture in rank 1.** The arguments in the proof of Theorem 5.6.1 give the following result towards the Bloch–Kato conjecture in rank 1.

Theorem 5.7.1. *Let the hypotheses be as in Theorem 5.5.1, and assume in addition that:*

- $\bar{\rho}_f$ *is p-distinguished;*
- f *has big image.*

If $j \geq r$ (which implies $L(f/K, \chi, r) = 0$), then

$$\dim_{L_{\mathfrak{P}}} \mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi}) \geq 1.$$

Moreover, there exists a class $z_{f,\chi} \in \mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi})$ such that

$$z_{f,\chi} \neq 0 \quad \Longrightarrow \quad \dim_{L_{\mathfrak{P}}} \mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi}) = 1.$$

Proof. The proof of Theorem 5.6.1 shows the non-torsionness of the class $\mathbf{z}_{f,\chi} \in \mathrm{Sel}_{\mathrm{rel,str}}(K_\infty^-, T_{f,\chi})$, which is the base of a Λ_K^--adic anticyclotomic Euler systems as in (5.17) for the relaxed-strict Selmer group. By Theorem 3.3.3, it follows that $\mathrm{Sel}_{\mathrm{rel,str}}(K_\infty^-, T_{f,\chi})$ has Λ_K^--rank 1. Since by (a straighforward variation of) Mazur's control theorem the natural map

$$\mathrm{Sel}_{\mathrm{rel,str}}(K_\infty^-, T_{f,\chi})/(\gamma_- - 1)\mathrm{Sel}_{\mathrm{rel,str}}(K_\infty^-, T_{f,\chi}) \rightarrow \mathrm{Sel}_{\mathrm{rel,str}}(K, T_{f,\chi}) \tag{5.18}$$

is injective with finite cokernel, we conclude that $\mathrm{Sel}_{\mathrm{rel,str}}(K, T_{f,\chi})$ has positive $\mathcal{O}$-rank, and by Lemma 3.1.2 the first part of the theorem follows.

On the other hand, letting $z_{f,\chi} \in \mathrm{Sel}_{\mathrm{rel,str}}(K, T_{f,\chi})$ be the image of $\mathbf{z}_{f,\chi}$ under the projection (5.18), the last claim in the result follows from Theorem 3.3.1. ☐

Remark 5.7.2. From the Euler system of Beilinson–Flach elements constructed by Lei–Loeffler–Zerbes and Kings–Loeffler–Zerbes [LLZ14, KLZ17] attached to the Rankin–Selberg convolution of f and a suitable θ-series, one can produce a class $\mathrm{BF}_{f,\chi} \in \mathrm{H}^1(K, V_{f,\chi})$. As shown in [LLZ15] and [BL18], this class forms the basis of an anticyclotomic Euler system for $V_{f,\chi}$, but not for the correct local conditions at p. Indeed, with notations as in the proof of Theorem 5.5.1, it follows from the explicit reciprocity law of [KLZ17] that, for $j \geq r$, the class $\mathrm{BK}_{f,\chi}$ lands in $\mathrm{Sel}_{\mathrm{rel,str}}(K, V_{f,\chi}) = \mathrm{Sel}_{\mathrm{BK}}(K, V_{f,\chi})$ precisely when the p-adic L-value $\Theta_p^{\mathrm{BD}}(f/K, \chi_t)(\zeta(1+p)^j - 1)$ vanishes (see [Cas17, Thm. 2.4], [BL18, Thm. 3.11]). However, since $j \geq r$ is *outside* the range of interpolation of $\Theta_p^{\mathrm{BD}}(f/K, \chi_t)$, such vanishing is not a consequence of the forced vanishing of $L(f/K, \chi, r)$, and Theorem 5.7.1 seems to fall outside the scope of these classes. (On the other hand, it also seems to fall outside the scope of Heegner cycles, since the squarefree integer N^- is assumed to have an odd number of prime factors, so Heegner cycles are not directly available, and the level-raising techniques of Bertolini–Darmon [BD05] are only known to yield results towards the Bloch–Kato conjecture in rank 0, see e.g. [LV10].)

6. Indefinite case

In this section we deduce our applications to the Bloch–Kato conjecture and the Iwasawa main conjecture for anticyclotomic twists of f/K when $\epsilon(f/K) = -1$.

Since the nonvanishing results we shall need from [Hsi14] are currently only available in the literature under the classical Heegner hypothesis, in the following we shall restrict to this case, but note that with the required extension of [Hsi14] at hand (see [Bur17, Mag] for progress in this direction), our results directly extend to the general indefinite case.

6.1. **Anticyclotomic p-adic L-functions.** Let $f \in S_k(\Gamma_0(pN_f))$ be a p-ordinary p-stabilised newform as in §5.1 (in particular, f is old at p) where $k = 2r \geq 2$, and let K be an imaginary quadratic field of discriminant $-D_K < 0$ in which $p = \mathfrak{p}\bar{\mathfrak{p}}$ splits. Assume that K satisfies the classical *Heegner hypothesis*:

$$\text{every prime } \ell | N_f \text{ splits in } K, \tag{6.1}$$

and fix an ideal $\mathfrak{N} \subset \mathcal{O}_K$ with $\mathcal{O}_K/\mathfrak{N} \simeq \mathbf{Z}/N_f\mathbf{Z}$.

Recall that Γ^- denotes the Galois group of the anticyclotomic $\mathbf{Z}_p$-extension of K, and $\Lambda_K^- = \mathcal{O}[\![\Gamma^-]\!]$ is the associated Iwasawa algebra. Let Ω_p and Ω_K be CM periods attached to K as in [CH18a, §2.5], and put

$$\Lambda_K^{-,\mathrm{ur}} = \Lambda_K^- \hat{\otimes} \mathbf{Z}_p^{\mathrm{ur}},$$

where $\mathbf{Z}_p^{\mathrm{ur}}$ is the completion of the ring of integers of the maximal unramified extension of $\mathbf{Q}_p$. Similarly as before, we shall often identify $\Lambda_K^{-,\mathrm{ur}}$ with the one-variable power series ring $\mathbf{Z}_p^{\mathrm{ur}}[\![W]\!]$ via $\gamma_- \mapsto 1+W$ for a fixed topological generator $\gamma_- \in \Gamma^-$.

Theorem 6.1.1. *Let χ be an $\mathcal{O}$-valued ring class character of conductor $c\mathcal{O}_K$ with $(pN_f, cD_K) = 1$. Then there exists a unique element $\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K,\chi) \in \mathbf{Z}_p^{\mathrm{ur}}[\![W]\!]$ such that every character ϕ of Γ^- of infinity type $(j,-j)$ with $j \geq r$ and conductor p^n, we have*

$$\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K,\chi)(\phi(\gamma_-)-1)^2 = \frac{\Omega_p^{4j}}{\Omega_K^{4j}} \cdot \frac{\Gamma(r+j)\Gamma(j+1-r)\phi(\mathfrak{N}^{-1})}{4(2\pi)^{2j+1}\sqrt{D_K}^{2j-1}} \cdot e_{\mathfrak{p}}(f,\chi\phi) \cdot L(f/K,\chi\phi,r),$$

where

$$e_{\mathfrak{p}}(f,\chi\phi) = \begin{cases} \left(1 - a_p\chi\phi(\bar{\mathfrak{p}})p^{-r} + \chi\phi(\bar{\mathfrak{p}})^2 p^{-1}\right)^2 & \text{if } n = 0, \\ \varepsilon(\frac{1}{2}, \chi_{\mathfrak{p}}\phi_{\mathfrak{p}})^{-2} & \text{else,} \end{cases}$$

with $\varepsilon(\frac{1}{2},\chi_{\mathfrak{p}}\phi_{\mathfrak{p}})$ the local ε-factor in [CH18a, p. 570] *attached to the component at $\mathfrak{p}$ of $\chi\phi$. Moreover, $\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K,\chi)$ is a nonzero element of $\Lambda_K^{-,\mathrm{ur}}$.*

Proof. This is a reformulation of results contained in [CH18a, §3]. In particular, since $(N_f, D_K) = 1$ by hypothesis (6.1), the nonvanishing of $\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K,\chi)$ follows from [CH18a, Thm. 3.9]. $\square$

Remark 6.1.2. The CM period $\Omega_K \in \mathbf{C}^\times$ in Theorem 6.1.1 agrees with that in [BDP13, (5.1.16)], but is *different* from the period Ω_∞ defined in [dS87, p. 66] and [HT93, (4.4b)]. In fact, one has

$$\Omega_\infty = 2\pi i \cdot \Omega_K.$$

In terms of Ω_∞, the interpolation formula in Theorem 6.1.1 reads

$$\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K,\chi)(\phi(\gamma_-)-1)^2 = \frac{\Omega_p^{4j}}{\Omega_\infty^{4j}} \cdot \frac{\Gamma(r+j)\Gamma(j+1-r)\phi(\mathfrak{N}^{-1})}{4(2\pi)^{1-2j}\sqrt{D_K}^{2j-1}} \cdot e_{\mathfrak{p}}(f,\chi\phi) \cdot L(f/K,\chi\phi,r).$$

This is the form of the interpolation that we shall use later.

6.2. **Factorisation of triple product p-adic L-function.** As in §5.2, we now let $\boldsymbol{f} \in S^o(N_f,\mathbb{I})$ be the primitive Hida family associated to f; so $\boldsymbol{f}$ specialises to f at an arithmetic point $Q_0 \in \mathfrak{X}_{\mathbb{I}}^+$ of weight $2r$, and consider a pair of CM Hida families

$$(\boldsymbol{g},\boldsymbol{h}) = (\boldsymbol{g}_1,\boldsymbol{g}_2) = (\boldsymbol{\theta}_{\xi_1}(S_1),\boldsymbol{\theta}_{\xi_2}(S_2)) \in \mathcal{O}[\![S_1]\!][\![q]\!] \times \mathcal{O}[\![S_2]\!][\![q]\!] \tag{6.2}$$

similar to (5.3).

The triple product p-adic L-function of relevance in this section is the $\boldsymbol{g}$-unbalanced $\mathscr{L}_p^{\boldsymbol{g},\eta_{\boldsymbol{g}}}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$, which is an element in $\mathcal{R} = \mathbb{I}\hat{\otimes}_{\mathcal{O}}\mathcal{O}[\![S_1]\!]\hat{\otimes}_{\mathcal{O}}\mathcal{O}[\![S_2]\!] \simeq \mathbb{I}[\![S_1,S_2]\!]$. In the following we let

$$\mathscr{L}_p^{\boldsymbol{g},\eta_{\boldsymbol{g}}}(f,\boldsymbol{g},\boldsymbol{h}) \in \mathcal{O}[\![S_1,S_2]\!]$$

be the image of $\mathscr{L}_p^{\boldsymbol{g},\eta_{\boldsymbol{g}}}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ under the map $\mathbb{I}[\![S_1,S_2]\!] \to \mathcal{O}[\![S_1,S_2]\!]$ given by $Q_0 : \mathbb{I} \to \mathcal{O}$.

6.2.1. *Anticyclotomic Katz p-adic L-function.* Before we can state and prove the main result of this section, we need to recall the interpolation property of the Katz p-adic L-functions [Kat78], following the exposition in [dS87].

Fix an ideal $\mathfrak{c} \subset \mathcal{O}_K$ prime-to-p stable under the action of complex conjugation, and denote by $Z(\mathfrak{c})$ the ray class group of K of conductor $\mathfrak{c}p^\infty$ (so $Z(\mathfrak{c}) \simeq \mathrm{Gal}(K[\mathfrak{c}p^\infty]/K)$).

Theorem 6.2.1. *There exists an element $\mathcal{L}^{\mathrm{Katz}}_{\mathfrak{p},\mathfrak{c}} \in \mathcal{O}[\![Z(\mathfrak{c})]\!]\hat{\otimes}\mathbf{Z}_p^{\mathrm{ur}}$ such that for every character ξ of $Z(\mathfrak{c})$ of infinity type (k,j) with $k > -j \geq 0$ satisfies*

$$\mathcal{L}^{\mathrm{Katz}}_{\mathfrak{p},\mathfrak{c}}(\xi) = \frac{\Omega_p^{k-j}}{\Omega_\infty^{k-j}} \cdot \Gamma(k) \cdot \left(\frac{\sqrt{D_K}}{2\pi}\right)^j \cdot (1-\xi^{-1}(\mathfrak{p})p^{-1})(1-\xi(\bar{\mathfrak{p}})) \cdot L(\xi,0).$$

Moreover, we have the functional equation

$$\mathcal{L}^{\mathrm{Katz}}_{\mathfrak{p},\mathfrak{c}}(\xi) = \mathcal{L}^{\mathrm{Katz}}_{\mathfrak{p},\mathfrak{c}}(\xi^{-\mathbf{c}}\mathbf{N}^{-1}),$$

where the equality is up to a p-adic unit.

Proof. Our $\mathcal{L}^{\mathrm{Katz}}_{\mathfrak{p},\mathfrak{c}}$ corresponds to the measure denoted $\mu(\mathfrak{c}\bar{\mathfrak{p}}^\infty)$ in [dS87, Thm. II.4.14]. The stated functional equation is given in [dS87, Thm. II.6.4]. □

Let $\Gamma_\mathfrak{c}$ be the maximal torsion-free subgroup of $Z(\mathfrak{c})$, and fix a (non-canonical) splitting

$$Z(\mathfrak{c}) \simeq \Delta_\mathfrak{c} \times \Gamma_\mathfrak{c}$$

with $\Delta_\mathfrak{c}$ a finite group and $\Gamma_\mathfrak{c} \simeq \mathbf{Z}_p^2$. For $\mathfrak{c}'|\mathfrak{c}$ the natural projection $Z(\mathfrak{c}) \twoheadrightarrow Z(\mathfrak{c}')$ takes $\Delta_\mathfrak{c}$ to $\Delta_{\mathfrak{c}'}$, inducing an isomorphism $\Gamma_\mathfrak{c} \xrightarrow{\sim} \Gamma_{\mathfrak{c}'}$. Thus in the following we shall identify $\Gamma_\mathfrak{c}$ with $\Gamma_K := \Gamma_{(1)}$. Since $p > 2$, the action of complex conjugation $\mathbf{c}$ splits

$$\Gamma_K \simeq \Gamma^+ \times \Gamma^-$$

with $\Gamma^\pm \simeq \mathbf{Z}_p$, and where $\mathbf{c}$ sends $\gamma \in \Gamma^\pm$ to $\gamma^{\pm 1}$. Then of course Γ^- is identified with the Galois group $\mathrm{Gal}(K_\infty^-/K)$ of the anticyclotomic $\mathbf{Z}_p$-extension of K.

Suppose η is a Hecke character of K of conductor dividing $\mathfrak{c}p^\infty$. Viewing η as a character on $Z(\mathfrak{c}) \simeq \Delta_\mathfrak{c} \times \Gamma_K$, we put $\bar{\eta} := \eta|_{\Delta_\mathfrak{c}}$, and denote by $\mathcal{L}^{\mathrm{Katz},-}_{\mathfrak{p},\bar{\eta}}$ the image of $\mathcal{L}^{\mathrm{Katz}}_{\mathfrak{p},\mathfrak{c}}$ under the composite map

$$\mathcal{O}[\![Z(\mathfrak{c})]\!]\hat{\otimes}\mathbf{Z}_p^{\mathrm{ur}} \to \mathcal{O}[\![\Gamma_K]\!]\hat{\otimes}\mathbf{Z}_p^{\mathrm{ur}} \to \Lambda_K^{-,\mathrm{ur}},$$

where the first arrow is the natural projection defined by $\bar{\eta}$, and the second is given by $\gamma \mapsto \gamma^{\mathbf{c}-1}$ for $\gamma \in \Gamma_K$. Put also $\bar{\eta}^- := \bar{\eta}^{\mathbf{c}-1}$.

Lemma 6.2.2. *Let ξ be a ray class character of K such that $\bar{\xi}^-$ has conductor $\mathfrak{c}$ prime-to-p. Assume that:*

(i) *$\mathfrak{c}$ is only divisible by primes that are split in K;*
(ii) *$\Delta_\mathfrak{c}$ has order prime-to-p;*
(iii) *$\bar{\xi}^-$ has order at least 3.*

Then the congruence ideal of the CM Hida family $\boldsymbol{\theta}_\xi(S)$ is generated by

$$\frac{h_K}{w_K} \cdot \mathcal{L}^{\mathrm{Katz},-}_{\mathfrak{p},\bar{\xi}^-},$$

where $h_K = |\mathrm{Pic}(\mathcal{O}_K)|$ and $w_K = |\mathcal{O}_K^\times|$.

Proof. As explained in [ACR22, Prop. 4.6], this is a consequence of the proof of the anticyclotomic Iwasawa main conjecture for Hecke characters by Hida–Tilouine [HT93, HT94] and Hida [Hid06a]. □

6.2.2. *The factorisation result.* We shall work with the following integral normalisation of the triple product p-adic L-function of Theorem 4.1.1.

Definition 6.2.3. Put

$$\mathscr{L}_p^{\boldsymbol{g}}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}) := \mathscr{L}_p^{\boldsymbol{g},\eta_{\boldsymbol{g}}^\star}(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h}),$$

where $\eta_{\boldsymbol{g}}^\star = \frac{h_K}{w_K} \cdot \mathcal{L}^{\mathrm{Katz},-}_{\mathfrak{p},\bar{\xi}^-}$ is the generator of $C(\boldsymbol{g})$ given by Lemma 6.2.2.

Note that ξ_1 can be replaced by a twist $\xi_1 \cdot \phi \circ \mathbf{N}$ for a Dirichlet character ϕ without changing $\bar{\xi}_1^-$, and thus in the following we may assume that ξ_1 satisfies the following minimality hypotheses:

(6.3) the conductor of ξ_1 is minimal among Dirichlet twists.

The following is an analogue of Proposition 5.2.1 in the indefinite setting. Note that a variant of this result first appeared in the work of Darmon–Lauder–Rotger (see [DLR15, Thm. 3.9]), but unfortunately the formulation of their result is not well-suited for our Iwasawa-theoretic purposes in this paper (see also the more recent [BCS23, §8] for a factorisation result closer to ours).

Proposition 6.2.4. *Assume that ξ_1 satisfies the conditions in Lemma 6.2.2. Put*

$$W_1 = \mathbf{v}^{-1}(1+S_1)^{1/2}(1+S_2)^{1/2} - 1, \quad W_2 = (1+S_1)^{1/2}(1+S_2)^{-1/2} - 1.$$

Then

$$\mathscr{L}_p^{\boldsymbol{g}}(f,\boldsymbol{g},\boldsymbol{h})(S_1,S_2) = \pm\mathbf{w}\cdot\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K,\xi_1\xi_2)(W_1)\cdot\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K,\xi_1\xi_2^{\mathbf{c}})(W_2),$$

where $\mathbf{w}$ is a unit.

Proof. Let k_1, k_2 be integers with $k_1 \equiv k_2 \pmod 2$ and $k_1 \geq k_2 + 2r$. Set $S_i = \mathbf{v}(1+p)^{k_i-1} - 1$ $(i = 1,2)$, so the corresponding specialisations of W_i are given by

$$W_1 = (1+p)^{(k_1+k_2-2)/2} - 1, \quad W_2 = (1+p)^{(k_1-k_2)/2},$$

and denote by $\mathbf{V}^\dagger_{\underline{Q}}$ the specialisation of $\mathbf{V}^\dagger$ at $\underline{Q} = (Q_0, S_1, S_2)$. Putting $T_i = \mathbf{v}^{-1}(1+S_i) - 1$ for the ease of notation, we have

$$\det(T_f^\vee \otimes V_{\boldsymbol{g}_{T_1}} \otimes V_{\boldsymbol{g}_{T_2}}) = \varepsilon_{\mathrm{cyc}}^{2r-1}\cdot(\xi_1\xi_2\Psi_{T_1}\Psi_{T_2}\circ\mathscr{V}) = \varepsilon_{\mathrm{cyc}}^{2r-1}\cdot(\Psi_{T_1}\Psi_{T_2}\circ\mathscr{V}),$$

using that the central characters of ξ_1 and ξ_2 are inverses of each other for the second equality, and so

$$\begin{aligned}\mathbf{V}^\dagger_{\underline{Q}} &= T_f^\vee\otimes(\mathrm{Ind}_K^{\mathbf{Q}}\xi_1^{-1}\Psi_{T_1})\otimes(\mathrm{Ind}_K^{\mathbf{Q}}\xi_2^{-1}\Psi_{T_2})\otimes\varepsilon_{\mathrm{cyc}}^{1-r}(\Psi_{T_1}^{-1/2}\Psi_{T_2}^{-1/2}\circ\mathscr{V}) \\ &\simeq \big(T_f^\vee(1-r)\otimes\mathrm{Ind}_K^{\mathbf{Q}}\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\big)\oplus\big(T_f^\vee(1-r)\otimes\mathrm{Ind}_K^{\mathbf{Q}}\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\big).\end{aligned} \tag{6.4}$$

Thus we find that the completed L-value appearing in the interpolation formula of Theorem 4.1.1 is given by

$$\begin{aligned}\Gamma_{\mathbf{V}^\dagger_{\underline{Q}}}(0)\cdot L(\mathbf{V}^\dagger_{\underline{Q}},0) &= \frac{\Gamma\big(\frac{k_1+k_2}{2}+r-1\big)\Gamma\big(\frac{k_1-k_2}{2}-r+1\big)\Gamma\big(\frac{k_1+k_2}{2}-r\big)\Gamma\big(\frac{k_1-k_2}{2}+r\big)}{2^4\cdot(2\pi)^{2k_1}}, \\ &\quad\times L(f/K,\xi_1\xi_2\Psi_{W_1}^{\mathbf{c}-1},r)\cdot L(f/K,\xi_1\xi_2^{\mathbf{c}}\Psi_{W_2}^{\mathbf{c}-1},r),\end{aligned} \tag{6.5}$$

and similarly the modified Euler factor decomposes as

$$\begin{aligned}\mathcal{E}_p(\mathscr{F}_p^{\boldsymbol{g}}(\mathbf{V}^\dagger_{\underline{Q}})) &= \big(1 - a_p(\xi_1\xi_2\Psi_{W_1}^{\mathbf{c}-1})(\bar{\mathfrak{p}})p^{-r} + (\xi_1\xi_2\Psi_{W_2}^{\mathbf{c}-1})(\bar{\mathfrak{p}})^2p^{-1}\big)^2 \\ &\quad\times\big(1 - a_p(\xi_1\xi_2^{\mathbf{c}}\Psi_{W_2}^{\mathbf{c}-1})(\bar{\mathfrak{p}})p^{-r} + (\xi_1\xi_2^{\mathbf{c}}\Psi_{W_2}^{\mathbf{c}-1})(\bar{\mathfrak{p}})^2p^{-1}\big)^2.\end{aligned} \tag{6.6}$$

Moreover, letting χ_g' be the prime-to-p part of the nebentypus character of $\boldsymbol{g}_{T_1}$, we have

$$\Big(1 - \frac{\chi_g'(p)p^{k_1-1}}{\xi_1^{-1}\Psi_{T_1}(\bar{\mathfrak{p}})^2}\Big)\Big(1 - \frac{\chi_g'(p)p^{k_1-2}}{\xi_1^{-1}\Psi_{T_1}(\bar{\mathfrak{p}})^2}\Big) = \big(1 - \xi_1^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}(\mathfrak{p})\big)\big(1 - \xi_1^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}(\mathfrak{p})p^{-1}\big),$$

and therefore the canonical period $\Omega_{\boldsymbol{g}_{T_1}}$ in Theorem 4.1.1 associated with $\eta_{\boldsymbol{g}}^\star$ is given by

$$\Omega_{\boldsymbol{g}_{T_1}} = (-2\sqrt{-1})^{k_1+1}\cdot\frac{\|\boldsymbol{g}_{T_1}^\circ\|^2_{\Gamma_0(C)}}{\eta^\star_{\boldsymbol{g}_{T_1}}}\cdot\big(1 - \xi_1^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}(\mathfrak{p})\big)\big(1 - \xi_1^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}(\mathfrak{p})p^{-1}\big), \tag{6.7}$$

where $C = N_{K/\mathbf{Q}}(\mathfrak{f}_1)D_K$ and we note that here Σ_{exc} consists of the primes $q \mid C$ inert in K.

On the other hand, since g_{T_1} has weight k_1, from Hida's formula for the adjoint L-value [HT93, Thm. 7.1] (using that ξ_1 satisfies the minimality condition (6.3)) and Dirichlet's class number formula we obtain

$$\|g_{T_1}^\circ\|^2_{\Gamma_0(C)} = \Gamma(k_1)\cdot\frac{D_K^2}{2^{2k_1}\pi^{k_1+1}}\cdot\frac{2\pi h_K}{w_K\sqrt{D_K}}\cdot L(\xi_1^{1-\mathbf{c}}\Psi_{T_1}^{\mathbf{c}-1},1)\cdot\prod_{q\in\Sigma_{\mathrm{exc}}}(1+q^{-1}).$$

The character $\xi_1^{1-\mathbf{c}}\Psi_{T_1}^{\mathbf{c}-1}\mathbf{N}^{-1}$ has infinity type $(k_1, 2-k_1)$, and so for $k_1 \geq 2$ it lies in the range of interpolation of $\mathcal{L}_{\mathfrak{p},\mathfrak{f}}^{\mathrm{Katz}}$. Noting that $L(\xi_1^{1-\mathbf{c}}\Psi_{T_1}^{\mathbf{c}-1}\mathbf{N}^{-1},0) = L(\xi_1^{1-\mathbf{c}}\Psi_{T_1}^{\mathbf{c}-1},1)$, from the above formula for this value and in Theorem 6.2.1 we thus obtain

$$\tag{6.8}\begin{aligned}\mathcal{L}_{\mathfrak{p},\mathfrak{f}}^{\mathrm{Katz}}(\xi_1^{1-\mathbf{c}}\Psi_{T_1}^{\mathbf{c}-1}\mathbf{N}^{-1}) = \left(\frac{\Omega_p}{\Omega_\infty}\right)^{2k_1-2} &\cdot \frac{\pi^{2k_1-2}\cdot 2^{3k_1-3}}{\sqrt{D_K}^{k_1+1}}\\ &\times \left(1-\xi^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}(\mathfrak{p})\right)\left(1-\xi_1^{\mathbf{c}-1}\Psi_{T_1}^{1-\mathbf{c}}(\mathfrak{p})p^{-1}\right)\cdot \|\boldsymbol{g}_{T_1}^{\circ}\|^2_{\Gamma_0(C)}\cdot\frac{w_K}{h_K}.\end{aligned}$$

Moreover, by the functional equation for Katz's p-adic L-function and the definition of $\eta_{\boldsymbol{g}}^{\star}$ we have the relation

$$\frac{h_K}{w_K}\cdot\mathcal{L}_{\mathfrak{p},\mathfrak{f}}^{\mathrm{Katz}}(\xi_1^{1-\mathbf{c}}\Psi_{T_1}^{\mathbf{c}-1}\mathbf{N}^{-1}) \sim_p \eta_{\boldsymbol{g}_{T_1}}^{\star},$$

where $\sim_p$ denotes equality up to a p-adic unit, and so from (6.7) and (6.8) we arrive at

$$\tag{6.9}\frac{1}{\Omega_{\boldsymbol{g}_{T_1}}} \sim_p \left(\frac{\Omega_p}{\Omega_\infty}\right)^{2k_1-2}\cdot\frac{(2\pi)^{2k_1-2}}{\sqrt{-D_K}^{k_1+1}}.$$

Finally, note that the characters $\xi_1\xi_2\Psi_{W_1}^{\mathbf{c}-1}$ and $\xi_1\xi_2^{\mathbf{c}}\Psi_{W_2}^{\mathbf{c}-1}$ in the right-hand side of (6.5) are both anticyclotomic, and of infinity type $((k_1+k_2)/2-1, -(k_1+k_2)/2+1)$ and $((k_1-k_2)/2, (k_2-k_1)/2)$, respectively, and so for $k_1 \geq k_2+2r$ they are in the range of interpolation for $\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K,\xi_1\xi_2)$ and $\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K,\xi_1\xi_2^{\mathbf{c}})$, respectively. Thus substituting (6.5), (6.6), and (6.9) into the interpolation formula for $\mathscr{L}_p^{\boldsymbol{g}}(f,\boldsymbol{g},\boldsymbol{h})$ in Theorem 4.1.1 and comparing with Theorem 6.1.1 we finally arrive at

$$\mathscr{L}_p^{\boldsymbol{g}}(f,\boldsymbol{g},\boldsymbol{h})(S_1,S_2)^2 \sim_p \frac{-1}{D_K^{k_1+1}}\cdot\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K,\xi_1\xi_2)(W_1)^2\cdot\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K,\xi_1\xi_2^{\mathbf{c}})(W_2)^2,$$

and this yiels the proof of the result. □

6.3. **Selmer group decomposition.** We keep the setting in §6.2, so in particular $f \in S_{2r}(pN_f)$ is the specialisation of $\boldsymbol{f}$ at $Q_0 \in \mathfrak{X}_{\mathbb{I}}^{+}$, and write $\mathbf{V}_{Q_0}^{\dagger}$ for the corresponding specialisation of $\mathbf{V}^{\dagger}$.

As in the proof of Proposition 6.2.4, setting $T_i = \mathbf{v}^{-1}(1+S_i)-1$ $(i=1,2)$ we have

$$\tag{6.10}\begin{aligned}\mathbf{V}_{Q_0}^{\dagger} &= T_f^{\vee}\otimes(\mathrm{Ind}_K^{\mathbf{Q}}\xi_1^{-1}\Psi_{T_1})\otimes(\mathrm{Ind}_K^{\mathbf{Q}}\xi_2^{-1}\Psi_{T_2})\otimes\varepsilon_{\mathrm{cyc}}^{1-r}(\Psi_{T_1}^{-1/2}\Psi_{T_2}^{-1/2}\circ\mathscr{V})\\ &\simeq \left(T_f^{\vee}(1-r)\otimes\mathrm{Ind}_K^{\mathbf{Q}}\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\right)\oplus\left(T_f^{\vee}(1-r)\otimes\mathrm{Ind}_K^{\mathbf{Q}}\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\right),\end{aligned}$$

where $T_f^{\vee}$ is a $G_{\mathbf{Q}}$-stable $\mathcal{O}$-lattice inside $V_f^{\vee}$, and we put

$$W_1 = \mathbf{v}^{-1}(1+S_1)^{1/2}(1+S_2)^{1/2}-1,\quad W_2 = (1+S_1)^{1/2}(1+S_2)^{-1/2}-1.$$

In particular, we get

$$\tag{6.11}\mathrm{H}^1(\mathbf{Q},\mathbf{V}_{Q_0}^{\dagger}) \simeq \mathrm{H}^1(K,T_f^{\vee}(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}})\oplus\mathrm{H}^1(K,T_f^{\vee}(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}})$$

by Shapiro's lemma.

Proposition 6.3.1. *Under (6.11), the balanced Selmer group* $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbf{V}_{Q_0}^{\dagger})$ *decomposes as*

$$\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbf{V}_{Q_0}^{\dagger}) \simeq \mathrm{Sel}_{\mathrm{rel,str}}(K,T_f^{\vee}(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}})\oplus\mathrm{Sel}_{\mathrm{ord,ord}}(K,T_f^{\vee}(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}),$$

and the $\boldsymbol{g}$-unbalanced Selmer group $\mathrm{Sel}^{\boldsymbol{g}}(\mathbf{Q},\mathbf{V}_{Q_0}^{\dagger})$ *decomposes as*

$$\mathrm{Sel}^{\boldsymbol{g}}(\mathbf{Q},\mathbf{V}_{Q_0}^{\dagger}) \simeq \mathrm{Sel}_{\mathrm{rel,str}}(K,T_f^{\vee}(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}})\oplus\mathrm{Sel}_{\mathrm{rel,str}}(K,T_f^{\vee}(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}).$$

Proof. The proof of the decomposition for $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbf{V}_{Q_0}^{\dagger})$ is the same as in Proposition 5.3.1, so we focus on $\mathrm{Sel}^{\boldsymbol{g}}(\mathbf{Q},\mathbf{V}_{Q_0}^{\dagger})$. Put

$$\widetilde{\mathbf{V}}_{Q_0}^{\dagger} = \left(T_f^{\vee}(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\right)\oplus\left(T_f^{\vee}(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\right),$$

so by Shapiro's lemma we have

$$\mathrm{H}^1(\mathbf{Q},\mathbf{V}_{Q_0}^\dagger)\simeq\mathrm{H}^1(K,\widetilde{\mathbf{V}}_{Q_0}^\dagger).$$

A direct computation shows that the $\boldsymbol{g}$-unbalanced local condition is given by

$$\begin{aligned}\mathscr{F}_p^{\boldsymbol{g}}(\mathbf{V}_{Q_0}^\dagger)&=T_f^\vee\otimes\xi_1^{-1}\Psi_{T_1}\otimes\big(\xi_2^{-1}\Psi_{T_2}\oplus\xi_2^{-\mathbf{c}}\Psi_{T_2}^{\mathbf{c}}\big)\otimes\varepsilon_{\mathrm{cyc}}^{1-r}(\Psi_{T_1}^{-1/2}\Psi_{T_2}^{-1/2}\circ\mathscr{V})\\&=\big(T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_{W_1}^{1-\mathbf{c}}\big)\oplus\big(T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_{W_2}^{1-\mathbf{c}}\big).\end{aligned}$$

Therefore, we have

$$\mathscr{F}_{\mathfrak{p}}(\widetilde{\mathbf{V}}_{Q_0}^\dagger)=\widetilde{\mathbf{V}}_{Q_0}^\dagger,\quad\mathscr{F}_{\bar{\mathfrak{p}}}(\widetilde{\mathbf{V}}_{Q_0}^\dagger)=0,$$

and this yields the stated decomposition for $\mathrm{Sel}^{\boldsymbol{g}}(\mathbf{Q},\mathbf{V}_{Q_0}^\dagger)$. □

Corollary 6.3.2. *The balanced Selmer group* $\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbf{A}_{Q_0}^\dagger)$ *decomposes as*

$$\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbf{A}_{Q_0}^\dagger)\simeq\mathrm{Sel}_{\mathrm{str,rel}}(K,A_f(r)\otimes\xi_1\xi_2\Psi_{W_1}^{\mathbf{c}-1})\oplus\mathrm{Sel}_{\mathrm{ord,ord}}(K,A_f(r)\otimes\xi_1\xi_2^{\mathbf{c}}\Psi_{W_2}^{\mathbf{c}-1}),$$

where $A_f(r)=\mathrm{Hom}_{\mathbf{Z}_p}(T_f^\vee(1-r),\mu_{p^\infty})$*; and the* $\boldsymbol{g}$*-unbalanced Selmer group* $\mathrm{Sel}^{\boldsymbol{g}}(\mathbf{Q},\mathbf{A}_{Q_0}^\dagger)$ *decomposes as*

$$\mathrm{Sel}^{\boldsymbol{g}}(\mathbf{Q},\mathbf{A}_{Q_0}^\dagger)\simeq\mathrm{Sel}_{\mathrm{str,rel}}(K,A_f(r)\otimes\xi_1\xi_2\Psi_{W_1}^{\mathbf{c}-1})\oplus\mathrm{Sel}_{\mathrm{str,rel}}(K,A_f(r)\otimes\xi_1\xi_2^{\mathbf{c}}\Psi_{W_2}^{\mathbf{c}-1}).$$

Proof. As in Corollary 5.3.2, this is immediate from Proposition 6.3.1 and local Tate duality. □

6.4. **Explicit reciprocity law.** In the setting of §6.3, we now put

$$\mathbb{V}^\dagger=\mathbf{V}_{Q_0}^\dagger\otimes_{\mathcal{O}[\![S_1,S_2]\!]}\mathcal{O}[\![S_1,S_2]\!]/(\mathbf{v}^2(1+S_2)^{-1}-1),$$

and let

$$\kappa(f,\boldsymbol{g},\boldsymbol{h}_{\mathbf{v}^2-1})\in\mathrm{H}^1(\mathbf{Q},\mathbb{V}^\dagger)\tag{6.12}$$

be the resulting restriction of the three-variable big diagonal class $\kappa(\boldsymbol{f},\boldsymbol{g},\boldsymbol{h})$ in (4.5), where $\boldsymbol{h}_{\mathbf{v}^2-1}$ is the weight one theta series obtained by specialising $\boldsymbol{h}=\boldsymbol{\theta}_{\xi_2}(S_2)$ at $S_2=\mathbf{v}^2-1$.

Since $\kappa(f,\boldsymbol{g},\boldsymbol{h}_{\mathbf{v}^2-1})\in\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbb{V}^\dagger)$, we can write

$$\kappa(f,\boldsymbol{g},\boldsymbol{h}_{\mathbf{v}^2-1})=(\kappa_1(f,\boldsymbol{g},\boldsymbol{h}_{\mathbf{v}^2-1}),\kappa_2(f,\boldsymbol{g},\boldsymbol{h}_{\mathbf{v}^2-1}))\tag{6.13}$$

according to the decomposition from Proposition 6.3.1; in particular, we have

$$\kappa_2(f,\boldsymbol{g},\boldsymbol{h}_{\mathbf{v}^2-1})\in\mathrm{Sel}_{\mathrm{ord,ord}}(K,T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_W^{1-\mathbf{c}}),$$

where $W=(1+S_1)^{1/2}-1$. Similarly as in §5.4, for any $\mathcal{O}[\![W]\!]$-module M we denote by M_Q its specialisation at $Q\in\mathfrak{X}_{\mathcal{O}[\![W]\!]}^{\mathrm{cls}}$. Then, if Q has weight $k_Q\geq 1$, it is easy to see that the Bloch–Kato dual exponential and logarithm maps give rise to $L(\zeta_Q)$-isomorphisms

$$\begin{aligned}\exp_{\bar{\mathfrak{p}}}^*&:\mathrm{H}^1(K_{\bar{\mathfrak{p}}},T_f^{\vee,+}(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_W^{1-\mathbf{c}})_Q\to L(\zeta_Q),\\\mathrm{loc}_{\bar{\mathfrak{p}}}&:\mathrm{H}^1(K_{\bar{\mathfrak{p}}},T_f^{\vee,+}(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_W^{1-\mathbf{c}})_Q\to L(\zeta_Q),\end{aligned}\tag{6.14}$$

according to whether $k_Q>2r$ or $1\leq k_Q\leq 2r$, respectively.

Theorem 6.4.1. *Let* $\eta_{\boldsymbol{g}}$ *be a generator of the congruence ideal of* $\boldsymbol{g}$*. There is an injective* $\mathcal{O}[\![W]\!]$*-module homomorphism with pseudo-null cokernel*

$$\mathscr{L}_{\bar{\mathfrak{p}}}^{\eta_{\boldsymbol{g}}}:\mathrm{H}^1(K_{\bar{\mathfrak{p}}},T_f^{\vee,+}(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_W^{1-\mathbf{c}})\to\mathcal{O}[\![W]\!]$$

such that for any $\mathfrak{Z}\in\mathrm{H}^1(K_{\bar{\mathfrak{p}}},T_f^{\vee,+}(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_W^{1-\mathbf{c}})$ *and* $Q\in\mathfrak{X}_{\mathcal{O}[\![W]\!]}^{\mathrm{cls}}$ *we have*

$$\mathscr{L}_{\bar{\mathfrak{p}}}^{\eta_{\boldsymbol{g}}}(\mathfrak{Z})_Q=\begin{cases}c_Q\cdot\exp_{\bar{\mathfrak{p}}}^*(\mathfrak{Z}_Q)&\text{if } k_Q>2r,\\c_Q\cdot\log_{\bar{\mathfrak{p}}}(\mathfrak{Z}_Q)&\text{if } 1\leq k_Q\leq 2r,\end{cases}$$

where c_Q *is an explicit nonzero constant. Moreover, we have the* explicit reciprocity law

$$\mathscr{L}_{\bar{\mathfrak{p}}}^{\eta_{\boldsymbol{g}}^*}\big(\mathrm{res}_{\bar{\mathfrak{p}}}(\kappa_2(f,\boldsymbol{g},\boldsymbol{h}_{\mathbf{v}^2-1}))\big)=\mathscr{L}_{\mathfrak{p}}^{\boldsymbol{g}}(f,\boldsymbol{g},\boldsymbol{h})((1+W)^2-1,\mathbf{v}^2-1).$$

Proof. In terms of the direct sum decomposition in (6.10) we have

$$\mathscr{F}_p^{\mathrm{bal}}(\mathbb{V}^\dagger) \simeq \big(T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_W^{1-\mathbf{c}}\big)\oplus\big(T_f^{\vee,+}(1-r)\otimes(\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_W^{1-\mathbf{c}}+\xi_1\xi_2^{\mathbf{c}}\Psi_W^{\mathbf{c}-1})\big)$$

as $G_{\mathbf{Q}_p}$-representations, while a direct computation shows that $\mathscr{F}_p^3(\mathbb{V}^\dagger) = T_f^{\vee,+}(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_W^{1-\mathbf{c}}$. From these we obtain

$$\begin{aligned}\mathscr{F}_p^{\mathrm{bal}}(\mathbb{V}^\dagger)/\mathscr{F}_p^3(\mathbb{V}^\dagger) \simeq \big(T_f^{\vee,-}(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_W^{1-\mathbf{c}}\big)&\oplus\big(T_f^{\vee,+}(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_W^{1-\mathbf{c}}\big)\\ &\oplus\big(T_f^{\vee,+}(1-r)\otimes\xi_1\xi_2^{\mathbf{c}}\Psi_W^{\mathbf{c}-1}\big),\end{aligned}$$

with the terms in the direct sum corresponding to $\mathbb{V}_f^{\boldsymbol{gh}}$, $\mathbb{V}_h^{\boldsymbol{fg}}$, and $\mathbb{V}_{\boldsymbol{g}}^{fh}$ in (4.6), respectively, where $h = \boldsymbol{h}_{\mathbf{v}^2-1}$. Thus we find that the composite map in (4.9)

$$\mathrm{Sel}^{\mathrm{bal}}(\mathbf{Q},\mathbb{V}^\dagger)\to\mathrm{H}^1(\mathbf{Q}_p,\mathbb{V}_{\boldsymbol{g}}^{fh})$$

corresponds under the isomorphism of Proposition 6.3.1 to the projection onto $\mathrm{Sel}_{\mathrm{ord,ord}}(K,T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_W^{1-\mathbf{c}})$ composed with the restriction map

$$\mathrm{Sel}_{\mathrm{ord,ord}}(K,T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_W^{1-\mathbf{c}})\xrightarrow{\mathrm{res}_{\bar{\mathfrak{p}}}}\mathrm{H}^1(K_{\bar{\mathfrak{p}}},T_f^{\vee,+}(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_W^{1-\mathbf{c}}).$$

In particular, under the corresponding identifications it follows that

$$\begin{aligned}\mathrm{res}_p(\kappa(f,\boldsymbol{g},\boldsymbol{h}_{\mathbf{v}^2-1}))_{\boldsymbol{g}} &= \mathrm{res}_{\bar{\mathfrak{p}}}(\kappa_2(f,\boldsymbol{g},\boldsymbol{h}_{\mathbf{v}^2-1}))\\ &\in\mathrm{H}^1(\mathbf{Q}_p,V_{\boldsymbol{g}}^{fh})\simeq\mathrm{H}^1(K_{\bar{\mathfrak{p}}},T_f^{\vee,+}(1-r)\otimes\xi_1^{-1}\xi_2^{-\mathbf{c}}\Psi_W^{1-\mathbf{c}}).\end{aligned}$$

Finally, the construction of $\mathscr{L}_{\bar{\mathfrak{p}}}^{\eta_{\boldsymbol{g}}}$ is deduced from a specialisation of the three-variable p-adic regulator map $\mathrm{Log}^{\eta_{\boldsymbol{g}}}$ in §4.3 (see [ACR21, Prop. 7.3]), and the associated explicit reciprocity law then follows from Theorem 4.3.1. ☐

6.5. **On the Bloch–Kato conjecture in rank** 0**.** As another application of the Euler system construction in this paper, we can now deduce a result towards the Bloch–Kato conjecture for

$$V_{f,\chi} = V_f^\vee(1-r)\otimes\chi^{-1}$$

analogous to Theorem 5.5.1 but in the indefinite setting. Note that a similar result was obtained in [CH18a, Thm. B] using the generalised Heegner cycles of Bertolini–Darmon–Prasanna.

Theorem 6.5.1. *Let $f\in S_k(\Gamma_0(pN_f))$ be a p-ordinary p-stabilised newform of weigh $k=2r\geq 2$ which is old at p, and let χ be anticyclotomic Hecke character of conductor $c\mathcal{O}_K$ and infinity type $(-j,j)$ for some $j\geq 0$. Assume that:*

- *every prime $\ell\mid N_f$ splits in K;*
- *$(pN_f, cD_K)=1$;*
- *$p\nmid h_K$, the class number of K;*
- *$\bar{\rho}_f$ is absolutely irreducible and p-distinguished;*
- *χ_t has conductor prime-to-p.*

Assume in addition that f is not of CM type. Then

$$L(f/K,\chi,r)\neq 0 \quad\Longrightarrow\quad \mathrm{Sel}_{\mathrm{BK}}(K,V_{f,\chi})=0,$$

and hence the Bloch–Kato conjecture for $V_{f,\chi}$ holds in analytic rank zero.

Proof. This follows by an argument similar to the proof of Theorem 5.5.1 after some modifications. Write $\chi_t=\alpha/\alpha^{\mathbf{c}}$ for a ray class character α as in the proof of that result, but now put

$$\xi_1 := \beta\alpha,\quad \xi_2 := \beta\alpha^{-1}, \tag{6.15}$$

with β an auxiliary ring class character of K (to be further specified below) of ℓ-power conductor for a prime $\ell\neq p$ split in K. Consider the setting of §6.2 with the CM Hida families

$$\boldsymbol{g}=\boldsymbol{\theta}_{\xi_1}(S_1),\quad \boldsymbol{h}=\boldsymbol{\theta}_{\xi_2}(S_2).$$

If $(-j,j)$ denotes the infinity type of χ, then χ is the specialisation of $\xi_1\xi_2^{\mathbf{c}}\Psi_W^{\mathbf{c}-1}$ at $W=\zeta(1+p)^j$ for some $\zeta\in\mu_{p^\infty}$, and letting $\mathbf{V}^\dagger_{Q_0,\chi}$ be the corresponding specialisation of $\mathbf{V}^\dagger_{Q_0}$ at $(S_1,S_2)=(\zeta^2(1+p)^{2j}-1,\mathbf{v}^2-1)$ from the decomposition (6.10) we obtain the factorisation

$$L(\mathbf{V}^\dagger_{Q_0,\chi},0)=L(f/K,\beta^2\chi,r)\cdot L(f/K,\chi,r). \tag{6.16}$$

By our assumption (6.1) on N_f, the sign in the functional equation of $L(f/K,\chi,s)$ is -1 for $0\leq j<r$, so without loss of generality we assume that $j\geq r$. Then the above (Q_0,S_1,S_2) is in the range of interpolation of $\mathscr{L}_p^{\boldsymbol{g}}(f,\boldsymbol{g},\boldsymbol{h})$, and by [Hsi14, Thm. C] we may choose β as above so that $L(f/K,\beta^2\chi,r)\neq 0$. Thus from (6.16) and Theorem 4.1.1 (see also Remark 4.1.2), with such choice of β we conclude that

$$L(f/K,\chi,r)\neq 0\quad\Longrightarrow\quad \mathscr{L}_p^{\boldsymbol{g}}(\breve{f},\breve{\boldsymbol{g}},\breve{\boldsymbol{h}})(\zeta^2(1+p)^{2j}-1,\mathbf{v}^2-1)\neq 0$$

for a suitable choice of test vectors $(\breve{f},\breve{\boldsymbol{g}},\breve{\boldsymbol{h}})$. Denoting by Q the specialisation $W\mapsto\zeta^2(1+p)^{2j}-1$, from Theorem 6.4.1 and Remark 4.3.2 we conclude that

$$L(f/K,\chi,r)\neq 0\quad\Longrightarrow\quad \mathrm{res}_{\bar{\mathfrak{p}}}(\kappa_2(f,\boldsymbol{g},\boldsymbol{h}_{\mathbf{v}^2-1})_Q)\neq 0.$$

Since by construction $\kappa_2(f,\boldsymbol{g},\boldsymbol{h}_{\mathbf{v}^2-1})_Q\in\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K,V_{f,\chi})$ is the base class of the anticyclotomic Euler system

$$\left\{{}^{\mathbf{c}}z_{f,\psi_1,\psi_2,\mathfrak{m}}\in\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K[m],T_{f,\psi_1,\psi_2^{\mathbf{c}}})\,:\,\mathfrak{m}\in\mathcal{N}\right\}$$

as in the proof of Theorem 3.3.1, with (ψ_1,ψ_2) the Hecke characters corresponding to $(\boldsymbol{g}_{S_1},\boldsymbol{h}_{S_2})$, by Theorem 3.3.1 we deduce that $\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K,V_{f,\chi})$ is one-dimensional, spanned by

$${}^{\mathbf{c}}z_{f,\chi}:=\mathrm{cor}_{K[1]/K}({}^{\mathbf{c}}z_{f,\psi_1,\psi_2,(1)})=\kappa_2(f,\boldsymbol{g},\boldsymbol{h}_{\mathbf{v}^2-1})_Q.$$

The vanishing of the Bloch–Kato Selmer group $\mathrm{Sel}_{\mathrm{BK}}(K,V_{f,\chi})=\mathrm{Sel}_{\mathrm{rel},\mathrm{str}}(K,V_{f,\chi})$ (see Lemma 3.1.2) now follows from this and the nonvanishing of $\mathrm{res}_{\bar{\mathfrak{p}}}({}^{\mathbf{c}}z_{f,\chi})$ using global duality (cf. [CH18a, Thm. 7.9]). $\square$

6.6. **On the Iwasawa main conjecture.** We conclude by giving an application to the Iwasawa–Greenberg main conjecture for $\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K,\chi)$. As before, we put

$$A_{f,\chi}=\mathrm{Hom}_{\mathbf{Z}_p}(T_f^\vee(1-r)\otimes\chi^{-1},\mu_{p^\infty}).$$

Theorem 6.6.1. *Let the hypotheses be an in Theorem 6.5.1, and assume in addition that f has big image. Then $\mathrm{Sel}_{\mathrm{str},\mathrm{rel}}(K,A_{f,\chi})$ is cotorsion over Λ_K^-, and we have the divisibility*

$$\mathrm{char}_{\Lambda_K^-}\left(\mathrm{Sel}_{\mathrm{str},\mathrm{rel}}(K,A_{f,\chi})^\vee\right)\supset\left(\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K,\chi)\right)$$

in $\Lambda_K^{-,\mathrm{ur}}\otimes_{\mathbf{Z}_p}\mathbf{Q}_p$.

Proof. Arguing as in the proof of Theorem 6.5.1 we get the factorisation

$$\mathscr{L}_p^{\boldsymbol{g}}(f,\boldsymbol{g},\boldsymbol{h})((1+W)^2-1,\mathbf{v}^2-1)=\pm\mathbf{w}\cdot\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K,\beta^2)(W)\cdot\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K,\chi)(W)$$

where the auxiliary ring class character β can be chosen, by virtue of [Hsi14, Thm. C], so that the factor $\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K,\beta^2)$ is a unit in $\Lambda_K^{-,\mathrm{ur}}$ and conditions (i)–(iii) in Lemma 6.2.2 hold. By Theorem 6.1.1 the second factor $\mathscr{L}_{\mathfrak{p}}^{\mathrm{BDP}}(f/K,\chi)$ is nonzero, from Theorem 6.4.1 we conclude that the resulting class

$$\kappa_2(f,\boldsymbol{g},\boldsymbol{h}_{\mathbf{v}^2}-1)\in\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K,T_f^\vee(1-r)\otimes\xi_1^{-1}\xi_2^{-1}\Psi_W^{1-\mathbf{c}})$$

is non-torsion. Since by construction this class is the base of a Λ_K^--adic anticyclotomic Euler system

$$\left\{{}^{\mathbf{c}}\mathbf{z}_{f,\psi_1,\psi_2,\mathfrak{m}}\in\mathrm{Sel}_{\mathrm{ord},\mathrm{ord}}(K[mp^\infty],T_{f,\psi_1,\psi_2^{\mathbf{c}}})\,:\,\mathfrak{m}\in\mathcal{N}\right\}$$

as in the proof of Theorem 3.3.3, the result follows from Theorem 3.3.3, Proposition 4.3.3, Proposition 6.3.1, and Theorem 6.5.1. $\square$

Remark 6.6.2. Note that Theorem 6.6.1 also yields a proof of a divisibility towards the Perrin-Riou main conjecture for generalised Heegner cycles formulated in [LV19] (see [BCK21, Thm. 5.2] for the argument), removing some of the hypotheses in the main result of [LV19].

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