Selmer groups and a Cassels-Tate pairing for finite Galois modules

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Part I: Background

Group cohomology

Take G to be a topological group. A G-module is any discrete abelian group M endowed with a continuous action of G.

Given $i \geq 0$, the group $H^i(G, M)$ is the quotient of the group of continuous i-cocycles by the group of continuous i-coboundaries.

For $i = 1$, a continuous 1-cocycle is a continous map $\phi : G \to M$ satisfying

$$\phi(\sigma\tau) = \sigma\phi(\tau) + \phi(\sigma) \quad \text{for all } \sigma, \tau \in G,$$

and 1-coboundaries are cocycles of the form $\sigma \mapsto \sigma m - m$ for some constant m in M.

Group cohomology

If M has trivial G action, we always have

$$H^1(G, M) = \mathrm{Hom}_{\mathrm{cnts}}(G, M).$$

We also find that $H^0(G, M)$ equals the set of m in M invariant under the action of G.
Given an exact sequence $0 \to M_1 \to M \to M_2 \to 0$ of G modules, we have a long exact sequence

$$\begin{aligned} 0 &\to H^0(G, M_1) \to H^0(G, M) \to H^0(G, M_2) \\ &\to H^1(G, M_1) \to H^1(G, M) \to H^1(G, M_2) \\ &\to H^2(G, M_1) \to H^2(G, M) \to H^2(G, M_2) \to \dots \end{aligned}$$

Conventions for fields and places

- F will be a global field: a finite extension of the rationals, or a characteristic p analogue.
- F^s will be a separable closure of F, and we will define

  $$G_F = \mathrm{Gal}(F^s/F).$$

  We will use this notation for other fields as well.
- For each place v of F, we will use the notation F_v for the completion of F at v, and we will fix an embedding of F^s into F_v^s. Writing $G_v = G_{F_v}$, this defines an embedding of G_v in G_F.

Shafarevich-Tate groups

Given a G_F-module M, we define

$$Ш^1(M) = \ker\left(H^1(G_F, M) \to \prod_{v \text{ of } F} H^1(G_v, M)\right).$$

Interpretations:

- $Ш^1(M)$ is the set of global cocycle classes that everywhere locally look like coboundaries.
- $Ш^1(M)$ is the set of étale classes in

  $$H^1_{\text{ét}}(\operatorname{Spec} F, M)$$

  that vanish under the pullback $\operatorname{Spec} F_v \to \operatorname{Spec} F$ for each v.

An application of the long exact sequence

Given an abelian variety A/F, the F^s points of A form a G_F module we will refer to as A.
Choose $n > 0$, and take $A[n]$ to be the submodule of A killed by multiplication by n. The long exact sequence for

$$0 \to A[n] \to A \xrightarrow{\cdot n} A \to 0$$

takes the form

$$H^0(G_F, A) \xrightarrow{\cdot n} H^0(G_F, A) \xrightarrow{\delta} H^1(G_F, A[n]) \to$$
$$H^1(G_F, A) \xrightarrow{\cdot n} H^1(G_F, A)$$

or

$$0 \to A(F)/nA(F) \xrightarrow{\delta} H^1(G_F, A[n]) \to H^1(G_F, A)[n] \to 0.$$

Local conditions and Selmer groups

Fixing a completion F_v, we have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(F)/nA(F) & \xrightarrow{\delta} & H^1(G_F, A[n]) & \longrightarrow & H^1(G_F, A)[n] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A(F_v)/nA(F_v) & \xrightarrow{\delta_v} & H^1(G_v, A[n]) & \longrightarrow & H^1(G_v, A)[n] & \longrightarrow & 0 \end{array}$$

Defining

$$\mathrm{Sel}^n A = \ker\left(H^1(G_F, A[n]) \to \prod_v H^1(G_v, A[n])/\mathrm{im}\ \delta_v\right),$$

we have an exact sequence

$$0 \to A(F)/nA(F) \xrightarrow{\delta} \mathrm{Sel}^n A \to \text{Ш}^1(A)[n] \to 0.$$

Some old conjectures

Conjecture (Shafarevich-Tate)

$Ш^1(A)$ is always finite.

Still open: We at least know that $(Ш^1(A)/Ш^1(A)_{\mathsf{div}})[p^\infty]$ is finite for each prime p.

Conjecture

If A is an elliptic curve, $(Ш^1(A)/Ш^1(A)_{div})[p^\infty]$ has square order.

This was verified by Cassels in the early 1960s.

Cassels' theorem

Theorem (Cassels, '62)

If A is an elliptic curve over a number field, there is an alternating pairing

$$\mathsf{CP}\colon Ш^1(A) \otimes Ш^1(A) \to \mathbb{Q}/\mathbb{Z}$$

with kernel $Ш^1(A)_{\mathsf{div}}$.

Alternating means $\mathsf{CP}(\phi, \phi) = 0$ for all $\phi \in Ш^1(A)$.

Since there is a perfect alternating pairing defined on the finite group $(Ш^1(A)/Ш^1(A)_{\mathsf{div}})[p^\infty]$, it must have square order by basic algebra.

The Cassels-Tate pairing

Theorem (Tate, '63)

Take A/F to be an abelian variety over a global field, and take $A^\vee$ to be the dual variety. Given a prime ℓ not equal to the characteristic of F, there is a bilinear pairing

$$\mathrm{CTP}\colon Ш^1(A)[\ell^\infty] \otimes Ш^1(A^\vee)[\ell^\infty] \to \mathbb{Q}/\mathbb{Z}$$

with kernels $Ш^1(A)[\ell^\infty]_{\mathrm{div}}$ and $Ш^1(A^\vee)[\ell^\infty]_{\mathrm{div}}$.

A pairing on Selmer groups

Given $n, b > 1$ indivisible by the characteristic of F, we have maps $\mathrm{Sel}^n A \to Ш^1(A)$ and $\mathrm{Sel}^b A^\vee \to Ш^1(A^\vee)$.
Composing with the Cassels-Tate pairing then defines a bilinear pairing

$$\mathrm{CTP}\colon \mathrm{Sel}^n A \otimes \mathrm{Sel}^b A^\vee \to \mathbb{Q}/\mathbb{Z}.$$

The left kernel is $b \cdot \mathrm{Sel}^{nb} A$ and the right kernel is $n \cdot \mathrm{Sel}^{nb} A^\vee$.

This pairing can be defined from the exact sequence

$$0 \to A[b] \to A[nb] \to A[n] \to 0$$

together with the local conditions. It gives the obstruction of lifting a Selmer element in $A[n]$ to a Selmer element in $A[nb]$.

Part II: Generalities

Selmer groups

Take F to be a global field, and take M to be a finite G_F-module. We assume that the characteristic of F does not divide the order of M.

For each place v of F, choose a subgroup $\mathcal{L}_v$ of $H^1(G_v, M)$. We assume $\mathcal{L}_v$ is the set of unramified classes at all but finitely many places.

The Selmer group associated to $(M, (\mathcal{L}_v)_v)$ is then defined by

$$\mathrm{Sel}(M, (\mathcal{L}_v)_v) = \ker\left(H^1(G_F, M) \to \prod_{v \text{ of } F} H^1(G_v, M)/\mathcal{L}_v\right).$$

The category of Selmerable modules

Note that $M \mapsto Ш^1(M)$ defines a functor. We want Sel to be a functor too.

Definition

Given F, take SMod_F to be the category

- with objects $(M, (\mathcal{L}_v)_v)$ as before, and
- with morphisms $(M, (\mathcal{L}_v)_v) \to (M', (\mathcal{L}'_v)_v)$ given by any G_F-equivariant homomorphism $f\colon M \to M'$ satisfying

$$f(\mathcal{L}_v) \subseteq \mathcal{L}'_v \quad \text{for all } v \text{ of } F.$$

We will denote this morphism by f.

Given this morphism f, we see that f induces a morphism

$$f\colon \mathrm{Sel}(M, (\mathcal{L}_v)_v) \to \mathrm{Sel}(M', (\mathcal{L}'_v)_v).$$

Sel is a functor from SMod_F to FinAb.

The dual module

Given $(M, (\mathcal{L}_v)_v)$ in SMod_F, define

$$M^\vee = \mathrm{Hom}(M, (F^s)^\times)$$

Local Tate duality gives a bilinear pairing

$$H^1(G_v, M) \otimes H^1(G_v, M^\vee) \to \mathbb{Q}/\mathbb{Z}.$$

Taking $\mathcal{L}_v^\perp$ to be the orthogonal complement to $\mathcal{L}_v$ with respect to this pairing, we define

$$(M, (\mathcal{L}_v)_v)^\vee = \left(M^\vee, \left(\mathcal{L}_v^\perp\right)_v\right).$$

This defines a contravariant functor $\vee \colon \mathrm{SMod}_F \to \mathrm{SMod}_F$, and $\vee \circ \vee$ is naturally isomorphic to the identity functor on SMod_F.

The dual module for abelian varieties.

Take A/F to be an abelian variety, and choose an integer n. We have a canonical isomorphism

$$\iota : A^\vee[n] \xrightarrow{\sim} A[n]^\vee$$

For v a place of F, we have natural connecting maps

$$\begin{aligned} \delta_{A,\,v} &: A(F_v)/nA(F_v) \to H^1(G_v, A[n]) \quad \text{and} \\ \delta_{A^\vee,\,v} &: A^\vee(F_v)/nA^\vee(F_v) \to H^1(G_v, A^\vee[n]). \end{aligned}$$

Then

$$\begin{aligned} \mathrm{Sel}^n A &= \mathrm{Sel}(A[n],\, (\operatorname{im}\delta_{A,\,v})_v) \quad \text{and} \\ \mathrm{Sel}^n A^\vee &= \mathrm{Sel}(A^\vee[n],\, (\operatorname{im}\delta_{A^\vee,\,v})_v), \end{aligned}$$

and the canonical isomorphism above gives an isomorphism

$$\left(A^\vee[n],\ (\operatorname{im}\delta_{A^\vee,\,v})_v\right) \xrightarrow{\iota} \left(A[n],\ (\operatorname{im}\delta_{A,\,v})_v\right)^\vee$$

in SMod_F.

Lifting

Question

Given a morphism $\pi\colon (M, (\mathcal{L}_v)_v) \to (M_2, (\mathcal{L}_{2v})_v)$ in SMod_F corresponding to a surjective G_F homomorphism, and given ϕ in $\mathrm{Sel}\, M_2$, what prevents ϕ from lying in $\pi(\mathrm{Sel}\, M)$?

First issue: the image of ϕ in some $\mathcal{L}_{2v}$ may be outside $\pi(\mathcal{L}_v)$.

Definition

We call a diagram

$$E = \Big[0 \to (M_1, (\mathcal{L}_{1v})_v) \xrightarrow{\iota} (M, (\mathcal{L}_v)_v) \xrightarrow{\pi} (M_2, (\mathcal{L}_{2v})_v) \to 0\Big]$$

in SMod_F *exact* if it gives an exact sequence of G_F-modules and

$$\mathcal{L}_{1v} = \iota^{-1}(\mathcal{L}_v) \quad \text{and} \quad \mathcal{L}_{2v} = \pi(\mathcal{L}_v)$$

for all v.

We sometimes refer to the object $(M, (\mathcal{L}_v)_v)$ as M.

The general Cassels-Tate pairing

Theorem (Tate)

Given an exact sequence

$$E = \left[0 \to M_1 \xrightarrow{\iota} M \xrightarrow{\pi} M_2 \to 0\right]$$

in SMod_F*, there is a natural bilinear pairing*

$$\mathrm{CTP}_E \colon \mathrm{Sel}\, M_2 \otimes \mathrm{Sel}\, M_1^\vee \to \mathbb{Q}/\mathbb{Z}$$

with left kernel $\pi(\mathrm{Sel}\, M)$*.*

This was not the generality Tate was working at, but his construction requires no modification for this case.

Question

What's the right kernel of this pairing?

Dual exact sequence

Given an exact sequence

$$E = \left[0 \to M_1 \xrightarrow{\iota} M \xrightarrow{\pi} M_2 \to 0\right]$$

in SMod_F, the dual diagram

$$E^\vee = \left[0 \to M_2^\vee \xrightarrow{\pi^\vee} M^\vee \xrightarrow{\iota^\vee} M_1^\vee \to 0\right]$$

in SMod_F is also exact.
The Cassels-Tate pairing for $E^\vee$ is then of the form

$$\mathrm{CTP}_{E^\vee} \colon \mathrm{Sel}\, M_1^\vee \otimes \mathrm{Sel}\, M_2^{\vee\vee} \to \mathbb{Q}/\mathbb{Z}.$$

This pairing has left kernel $\iota^\vee(\mathrm{Sel}\, M^\vee)$.

The Cassels-Tate pairing

Theorem (Morgan-S.)

Given exact sequences

$$E \;= \Big[0 \to \; M_1 \;\xrightarrow{\iota}\; M \;\xrightarrow{\pi}\; M_2 \;\to 0\Big] \quad \textit{and}$$
$$E^\vee = \Big[0 \to M_2^\vee \xrightarrow{\pi^\vee} M^\vee \xrightarrow{\iota^\vee} M_1^\vee \to 0\Big]$$

in SMod_F*, we have a natural bilinear pairing*

$$\mathrm{CTP}_E \colon \mathrm{Sel}\, M_2 \otimes \mathrm{Sel}\, M_1^\vee \to \mathbb{Q}/\mathbb{Z}$$

with left and right kernels

$$\pi(\mathrm{Sel}\, M) \quad \textit{and} \quad \iota^\vee(\mathrm{Sel}\, M^\vee),$$

respectively.

The duality identity

Given E and $E^\vee$ as above, we have pairings

$$\begin{aligned} \mathrm{CTP}_E &: \mathrm{Sel}\, M_2 \otimes \mathrm{Sel}\, M_1^\vee \to \mathbb{Q}/\mathbb{Z} \quad \text{and} \\ \mathrm{CTP}_{E^\vee} &: \mathrm{Sel}\, M_1^\vee \otimes \mathrm{Sel}\, M_2^{\vee\vee} \to \mathbb{Q}/\mathbb{Z} \end{aligned}$$

Theorem (Morgan-S.)

Given

$$\phi \in \mathrm{Sel}\, M_2 \cong \mathrm{Sel}\, M_2^{\vee\vee} \quad \textit{and} \quad \psi \in \mathrm{Sel}\, M_1^\vee,$$

we have

$$\mathrm{CTP}_{E^\vee}(\psi, \phi) = \mathrm{CTP}_E(\phi, \psi).$$

As a consequence, the right kernel of CTP_E is the left kernel of $\mathrm{CTP}_{E^\vee}$.

The Cassels-Tate pairing

The exact sequence

$$E = \left[0 \to M_1 \xrightarrow{\iota} M \xrightarrow{\pi} M_2 \to 0\right],$$

in SMod_F functorially yields an exact sequence

$$\operatorname{Sel} M_1 \xrightarrow{\iota} \operatorname{Sel} M \xrightarrow{\pi} \operatorname{Sel} M_2 \xrightarrow{\mathrm{CTP}_E} (\operatorname{Sel} M_1^\vee)^* \xrightarrow{(\iota^\vee)^*} (\operatorname{Sel} M^\vee)^* \xrightarrow{(\pi^\vee)^*} (\operatorname{Sel} M_2^\vee)^*$$

of finite abelian groups.

Part III: Properties

Naturality

Proposition (Morgan-S.)

Given a commutative diagram

$$\begin{array}{ccccccccccc} E_a & = & [\,0 & \longrightarrow & M_{1a} & \xrightarrow{\iota_a} & M_a & \xrightarrow{\pi_a} & M_{2a} & \longrightarrow & 0\,] \\ & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_2} & & \\ E_b & = & [\,0 & \longrightarrow & M_{1b} & \xrightarrow{\iota_b} & M_b & \xrightarrow{\pi_b} & M_{2b} & \longrightarrow & 0\,], \end{array}$$

in SMod_F *with exact rows, and given* ϕ *in* $\mathrm{Sel}\, M_{2a}$ *and* ψ *in* $\mathrm{Sel}\, M_{1b}^\vee$*, we have*

$$\mathrm{CTP}_{E_a}\left(\phi,\, f_1^\vee(\psi)\right) \,=\, \mathrm{CTP}_{E_b}\left(f_2(\phi), \psi\right).$$

Baer sums

In any abelian category, given exact sequences

$$E_a = [0 \to A_1 \to A_a \to A_2 \to 0] \quad \text{and}$$
$$E_b = [0 \to A_1 \to A_b \to A_2 \to 0],$$

there is a natural choice of an exact "sum"

$$E_a + E_b = [0 \to A_1 \to A_{ab} \to A_2 \to 0]$$

for these sequences. This makes the set of extensions of A_2 by A_1 into an abelian group whose operation is known as the *Baer sum*.

Example

The sum of $0 \to \frac{1}{4}\mathbb{Z}/\mathbb{Z} \to \frac{1}{16}\mathbb{Z}/\mathbb{Z} \to \frac{1}{4}\mathbb{Z}/\mathbb{Z} \to 0$ with itself has the form

$$0 \to \frac{1}{4}\mathbb{Z}/\mathbb{Z} \longrightarrow \frac{1}{8}\mathbb{Z}/\mathbb{Z} \oplus \frac{1}{2}\mathbb{Z}/\mathbb{Z} \longrightarrow \frac{1}{4}\mathbb{Z}/\mathbb{Z} \to 0.$$

Baer sums

SMod_F is not an abelian category, since morphisms $(M, (\mathcal{L}_v)_v) \to (M', (\mathcal{L}'_v)_v)$ corresponding to a G_F-isomorphism are monic and epic but perhaps not invertible.

However, it is *quasi-abelian*. Among other things, this means that Baer sums are well defined, and we have the following:

Proposition (Morgan-S.)

Given exact sequences

$$E_a = [0 \to M_1 \to M_a \to M_2 \to 0] \quad \textit{and}$$
$$E_b = [0 \to M_1 \to M_b \to M_2 \to 0]$$

in SMod_F*, we have*

$$\mathrm{CTP}_{E_a+E_b}(\phi, \psi) = \mathrm{CTP}_{E_a}(\phi, \psi) + \mathrm{CTP}_{E_b}(\phi, \psi)$$

for all ϕ *in* $\mathrm{Sel}\, M_2$ *and* ψ *in* $\mathrm{Sel}\, M_1^\vee$*.*

Naturality + Duality identity

Given a commutative diagram

$$\begin{array}{ccccccccccc} E & = & \big[\,0 & \longrightarrow & M_1 & \xrightarrow{\iota} & M & \xrightarrow{\pi} & M_2 & \longrightarrow & 0\,\big] \\ & & & & \big\downarrow{\scriptstyle f_1} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f_2} & & \\ E^\vee & = & \big[\,0 & \longrightarrow & M_2^\vee & \xrightarrow{\pi^\vee} & M^\vee & \xrightarrow{\iota^\vee} & M_1^\vee & \longrightarrow & 0\,\big] \end{array}$$

with exact rows, and given $\phi, \psi \in \operatorname{Sel} M_2$, we have

$$\begin{aligned} \mathrm{CTP}_E(\phi,\, f_2(\psi)) &= \mathrm{CTP}_{E^\vee}(f_2(\psi),\, \phi) && \text{by duality identity} \\ &= \mathrm{CTP}_E(\psi,\, f_1^\vee(\phi)) && \text{by naturality.} \end{aligned}$$

If $f^\vee = f$, then $f_2 = f_1^\vee$, so the pairing

$$\mathrm{CTP}_E(-,\ f_2(-))\colon \operatorname{Sel} M_2 \otimes \operatorname{Sel} M_2 \to \mathbb{Q}/\mathbb{Z}$$

is *symmetric*.

Naturality + Duality identity

Still given the morphism of exact sequences

$$\begin{array}{ccccccccccc} E & = & [\,0 & \longrightarrow & M_1 & \xrightarrow{\iota} & M & \xrightarrow{\pi} & M_2 & \longrightarrow & 0\,] \\ & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_2} & & \\ E^\vee & = & [\,0 & \longrightarrow & M_2^\vee & \xrightarrow{\pi^\vee} & M^\vee & \xrightarrow{\iota^\vee} & M_1^\vee & \longrightarrow & 0\,], \end{array}$$

suppose $f^\vee = -f$. Then $f_2 = -f_1^\vee$, and the pairing

$$\mathrm{CTP}_E(-,\ f_2(-))\colon \operatorname{Sel} M_2 \otimes \operatorname{Sel} M_2 \to \mathbb{Q}/\mathbb{Z}$$

is *antisymmetric*.

Antisymmetry

Suppose A/F is a princiaplly polarized abelian variety over a global field. Given a positive integer n indivisible by the characteristic of F, the Weil pairing

$$A[n^2] \otimes A[n^2] \longrightarrow (F^s)^\times$$

is an alternating perfect pairing. Taking f to be the corresponding map from $A[n^2]$ to $A[n^2]^\vee$, we have a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A[n] & \longrightarrow & A[n^2] & \longrightarrow & A[n] & \longrightarrow & 0 \\ & & \downarrow f_1 & & \downarrow f & & \downarrow f_2 & & \\ 0 & \longrightarrow & A[n]^\vee & \longrightarrow & A[n^2]^\vee & \longrightarrow & A[n]^\vee & \longrightarrow & 0, \end{array}$$

with $f^\vee = -f$.

We then can recover the fact that the original pairing

$$\mathrm{CTP}(-,\ f_2(-))\colon \mathrm{Sel}^n A \otimes \mathrm{Sel}^n A \to \mathbb{Q}/\mathbb{Z}$$

is antisymmetric.

Theta groups

Definition

Given M in SMod_F, a *theta group* for M is a potentially non-abelian group $\mathcal{H}$ acted on continuously by G_F that fits in a G_F-equivariant central extension

$$0 \to (F^s)^\times \to \mathcal{H} \to M \to 0.$$

The commutator pairing on $\mathcal{H}$ defines an alternating pairing

$$M \otimes M \to (F^s)^\times,$$

which by tensor-hom adjunction gives a map $f_{\mathcal{H}} \colon M \to M^\vee$.

Theta groups for abelian varieties

Given a principally polarized abelian variety A/F and a positive integer n indivisible by char F, there is a canonical choice of theta group

$$0 \to (F^s)^\times \to \mathcal{H}_n \xrightarrow{p_n} A[n] \to 0.$$

Take $\mathcal{H}^1 = p_{n^2}^{-1}(A[n])$, and consider the sequence

$$0 \to (F^s)^\times \to \mathcal{H}^1 \to A[n] \to 0.$$

This sequence corresponds to a class

$$\psi_{tht} \in \mathrm{Ext}^1_{G_F}(A[n], (F^s)^\times) = H^1(G_F, A[n]^\vee).$$

The Poonen-Stoll result

Theorem (Poonen-Stoll)

Given A/F and n, the element ψ_{tht} defined above lies in $Ш^\vee(A)[2]$*, and the pairing*

$$\mathrm{CTP}(-,\ f_2(-))\colon \mathrm{Sel}^n A \otimes \mathrm{Sel}^n A \to \mathbb{Q}/\mathbb{Z}$$

satisfies

$$\mathrm{CTP}(\phi, f_2(\phi)) = \mathrm{CTP}(\phi, \psi_{tht})$$

for all $\phi \in \mathrm{Sel}^n A$*.*

The proof uses the geometric definition of the Cassels-Tate pairing.

Generalizing Poonen-Stoll: setup

Suppose we have a theta group

$$0 \to (F^s)^\times \to \mathcal{H} \xrightarrow{p} M \to 0 \qquad (1)$$

and a morphism of exact sequences

$$\begin{array}{ccccccccccc} E & = & \big[\,0 & \longrightarrow & M_1 & \xrightarrow{\iota} & M & \xrightarrow{\pi} & M_2 & \longrightarrow & 0\,\big] \\ & & & & \downarrow f_1 & & \downarrow f=f_{\mathcal{H}} & & \downarrow f_2 & & \\ E^\vee & = & \big[\,0 & \longrightarrow & M_2^\vee & \xrightarrow{\pi^\vee} & M^\vee & \xrightarrow{\iota^\vee} & M_1^\vee & \longrightarrow & 0\,\big] \end{array}$$

in SMod_F. Writing the local conditions for M as $(\mathcal{L}_v)_v$, we assume that the connecting map

$$\delta_v \colon H^1(G_v, M) \to H^2(G_v, (F^s))^\times$$

corresponding to (1) satisfies $\delta_v(\mathcal{L}_v) = 0$.

Write ψ_{tht} for the class in $H^1(G_F, M_1^\vee)$ of

$$0 \to (F^s)^\times \to p^{-1}(\iota(M_1)) \to M_1 \to 0$$

Generalizing Poonen-Stoll

Theorem (Morgan-S.)

Given E, f_2, and ψ_{tht} as above, the cocycle class ψ_{tht} lies in $\mathrm{Sel}\, M_1^\vee$*, and*

$$\mathsf{CTP}_E(\phi, f_2(\phi)) = \mathsf{CTP}(\phi, \psi_{tht}) \in \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}$$

for all ϕ in $\mathrm{Sel}\, M_2$*.*

The proof is a crazy cochain bash, and it recovers the result of Poonen and Stoll.

Part IV: Class groups

Symmetry from roots of unity

Choose a positive integer n, and choose a number field F containing μ_n. For α in $F^\times$, define

$$\chi_{n,\alpha}\colon \mathrm{Gal}(F(\sqrt[n]{\alpha})/F) \to \mu_n$$

by $\chi_{n,\alpha}(\sigma) = \frac{\sigma \sqrt[n]{\alpha}}{\sqrt[n]{\alpha}}$.
Take H_F to be the Hilbert class field of F, and write

$$\mathrm{rec}\colon \mathrm{Cl}\, F \xrightarrow{\sim} \mathrm{Gal}(H_F/F)$$

for the Artin reciprocity map.

Theorem (Lipnowski-Sawin-Tsimerman, Morgan-S.)

Choose d dividing n, and suppose F contains $\mu_{n^2/d}$. Choose ideals I, J of F and units α, β in $F^\times$ subject to the condition

$$(\alpha) = I^n \quad \textit{and} \quad (\beta) = J^n.$$

We assume that $F(\sqrt[n]{\alpha}, \sqrt[n]{\beta})/F$ is unramified everywhere. Then

$$\chi_{n,\alpha}(\mathrm{rec}(J))^d = \chi_{n,\beta}(\mathrm{rec}(I))^d.$$

The dual class group

Take

$$\mathrm{Cl}^*F = \mathrm{Hom}\left(\mathrm{Gal}(H_F/F),\, \mathbb{Q}/\mathbb{Z}\right) \cong \mathrm{Hom}\left(\mathrm{Cl}\ F,\, \mathbb{Q}/\mathbb{Z}\right).$$

There is a natural perfect reciprocity pairing

$$\mathrm{RP}\colon \mathrm{Cl}^*F \otimes \mathrm{Cl}\ F \to \mathbb{Q}/\mathbb{Z}.$$

For any positive integer n, this restricts to a pairing

$$\mathrm{RP}_n\colon \mathrm{Cl}^*F[n] \otimes \mathrm{Cl}\ F[n] \to \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}$$

with kernels $n \cdot \mathrm{Cl}^*F[n^2]$ and $n \cdot \mathrm{Cl}\ F[n^2]$.

Class groups as Selmer groups

We have

$$\mathrm{Cl}^*F[n] = \mathrm{Hom}\left(\mathrm{Gal}(H_F/F),\ \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}\right) = \mathrm{Sel}\left(\tfrac{1}{n}\mathbb{Z}/\mathbb{Z},\ (\mathcal{L}_v)_v\right),$$

with $\mathcal{L}_v$ consisting of the unramified classes at v.
The Selmer group for the dual module sits in an exact sequence

$$0 \to \mathcal{O}_F^\times/(\mathcal{O}_F^\times)^n \to \mathrm{Sel}\,\mu_n \xrightarrow{p_{\mathrm{Cl}}} \mathrm{Cl}\ F[n] \to 0.$$

Taking E_n to be the exact sequence

$$0 \to \tfrac{1}{n}\mathbb{Z}/\mathbb{Z} \to \tfrac{1}{n^2}\mathbb{Z}/\mathbb{Z} \to \tfrac{1}{n}\mathbb{Z}/\mathbb{Z} \to 0$$

in SMod_F with the unramified local conditions, we find

$$\mathrm{CTP}_{E_n}(\phi,\ \psi) = \mathrm{RP}_n(\phi,\ p_{\mathrm{Cl}}(\psi)) \quad \text{for } \phi \in \mathrm{Sel}\,\tfrac{1}{n}\mathbb{Z}/\mathbb{Z},\ \psi \in \mathrm{Sel}\,\mu_n.$$

Duality identity + naturality

Proposition

Fix an isomorphism $f_2\colon \frac{1}{n}\mathbb{Z}/\mathbb{Z} \to \mu_n$*. If* F *contains* μ_{n^2}*, the pairing*

$$\mathrm{CTP}_{E_n}(-,\ f_2(-)) :\ \mathrm{Sel}\,\tfrac{1}{n}\mathbb{Z}/\mathbb{Z} \otimes \mathrm{Sel}\,\tfrac{1}{n}\mathbb{Z}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$$

is symmetric.

In this case, we have a morphism of exact sequences

$$\begin{array}{ccccccccccc} E_n & = & [\,0 & \longrightarrow & \frac{1}{n}\mathbb{Z}/\mathbb{Z} & \longrightarrow & \frac{1}{n^2}\mathbb{Z}/\mathbb{Z} & \longrightarrow & \frac{1}{n}\mathbb{Z}/\mathbb{Z} & \longrightarrow & 0\,] \\ & & & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_2} & & \\ E_n^\vee & = & [\,0 & \longrightarrow & \mu_n & \longrightarrow & \mu_{n^2} & \longrightarrow & \mu_n & \longrightarrow & 0\,] \end{array}$$

where f satisfies $f = f^\vee$.

Duality identity and naturality then give the statement.

A simple case with $d > 1$

Proposition

Take $n = 4$, and fix an isomorphism $f_2\colon \frac{1}{4}\mathbb{Z}/Z \to \mu_4$.
If F contains μ_8, the pairing

$$2 \cdot \mathrm{CTP}_{E_4}(-,\ f_2(-)) :\ \mathrm{Sel}\,\tfrac{1}{4}\mathbb{Z}/\mathbb{Z} \otimes \mathrm{Sel}\,\tfrac{1}{4}\mathbb{Z}/\mathbb{Z} \to \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}$$

is symmetric.

From trilineariy with respect to Baer sum, $2 \cdot \mathrm{CTP}_{E_4}$ can be identified with the pairing associated to the sequence

$$E_4 + E_4 = \left[0 \to \tfrac{1}{4}\mathbb{Z}/\mathbb{Z} \longrightarrow \tfrac{1}{8}\mathbb{Z}/\mathbb{Z} \oplus \tfrac{1}{2}\mathbb{Z}/\mathbb{Z} \longrightarrow \tfrac{1}{4}\mathbb{Z}/\mathbb{Z} \to 0\right],$$

which is symmetrically self-dual over F.

An application of theta groups

Theorem (Morgan-S.)

Suppose F is a CM field with complex conjugation $\kappa : F \to F$ that contains $\mu_{n^2/d}$.
Choose $\alpha \in F^\times$ so $F(\sqrt[n]{\alpha})/F$ is everywhere unramified and totally split at all primes above two, and find the ideal I so

$$(\alpha) = I^n.$$

Then

$$\chi_{n,\alpha}(\mathrm{rec}(\kappa I))^d = 1.$$

Using the previous theorem, it is not hard to show that the left hand side needs to be either $+1$ or -1.

Thank you!