**Homework 8 Selected Solutions**

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1e) $R = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x \leq y\}$

**Answer.** $R$ is not a function because $(1, 2) \in R$ and $(1, 3) \in R$ but $2 \neq 3$. [I didn't take any points off if you didn't give an explanation, but you need to know how to show that a relation is not a function by giving a very *specific* counterexample. Notice also that $x$ and $y$ do not range over all real numbers in some of the problems.]

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2f) $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : y = \frac{x^2-4}{x-2}\}$

**Answer.** Let $f$ be the function defined above.

$$\text{Dom}(f) = \mathbb{Z} \backslash \{2\} = \{x \in \mathbb{Z} \mid x \neq 2\}$$

$$\text{Rng}(f) = \mathbb{Z} \backslash \{4\} = \{y \in \mathbb{Z} \mid y \neq 4\}.$$

Other possible codomains: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, etc (anything that contains $\text{Rng}(f)$).

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14b) Show that $f : \mathbb{Z}_4 \to \mathbb{Z}_6$ given by $f(\overline{x}) = [2x + 1]$ is not well-defined.

**Solution.** We have $\overline{0} = \overline{4}$ in $\mathbb{Z}_4$. However,

$$f(\overline{0}) = [2(0) + 1] = [1] \qquad \text{and} \qquad f(\overline{4}) = [2(4) + 1] = [3],$$

and $[1] \neq [3]$ because 6 does not divide $(1 - 3)$. Hence, $f$ is not well-defined.

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15e) Show that $f : \mathbb{Z}_6 \to \mathbb{Z}_6$ given by $f(\overline{x}) = \overline{x^2 + 3}$ is well-defined.

**Solution.** Assume that $\overline{x} = \overline{y}$ in $\mathbb{Z}_6$, where $x, y \in \mathbb{Z}$. Then $x \equiv y \pmod 6$, so $6 \mid (x - y)$. Since $x + y \in \mathbb{Z}$, we have $6 \mid (x - y)(x + y)$ also. But

$$(x - y)(x + y) = x^2 - y^2 = (x^2 - 3) - (y^2 - 3),$$

so $[x^2 - 3] = [y^2 - 3]$. This shows that $f(\overline{x}) = f(\overline{y})$, whence $f$ is well-defined.

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2i) Find $f \circ g$ and $g \circ f$ (I will only do $f \circ g$) for

$$f(x) = \begin{cases} x+1 & \text{if } x \le 0 \\ 2x & \text{if } x > 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 2x & \text{if } x \le -1 \\ -x & \text{if } x > -1. \end{cases}$$

**Solution.** First, by definition of $g$ we have

$$(f \circ g)(x) = \begin{cases} f(2x) & \text{if } x \le -1 \\ f(-x) & \text{if } x > -1. \end{cases}$$

Notice that if $x \le -1$, then $2x \le -2$ so in particular $2x \le 0$, whence

$$(f \circ g)(x) = f(2x) = (2x) + 1.$$

Next, observe that $-x \le 0 \iff x \ge 0$, or equivalently $-x > 0 \iff x < 0$. Hence, if $x > -1$, then we have

$$(f \circ g)(x) = f(-x) = \begin{cases} 2(-x) & \text{if } -1 < x < 0 & (-x > 0 \text{ in this case}) \\ (-x) + 1 & \text{if } x \ge 0 & (-x \le 0 \text{ in this case}). \end{cases}$$

In summary, we have

$$(f \circ g)(x) = \begin{cases} 2x+1 & \text{if } x \le -1 \\ -2x & \text{if } -1 < x < 0 \\ 1 - x & \text{if } x \ge 0. \end{cases}$$

[$f \circ g$ and $(f \circ g)(x)$ are not the same thing. The former is the function $f \circ g$, which is a relation and in particular a set. But the latter is the function $f \circ g$ evaluated at $x$, which is an element in the codomain.]

5d) Define $f : \mathbb{Z}_8 \to \mathbb{Z}_4$, $g : \mathbb{Z}_4 \to \mathbb{Z}_8$, and $h : \mathbb{Z}_8 \to \mathbb{Z}_8$

$$f(\overline{x}) = [x+2], \quad g([x]) = \overline{2x}, \quad h(\overline{x}) = \overline{2x+4}.$$

By comparing images, verify $(g \circ f)(x) = h(x)$ for all $x \in \mathbb{Z}_8$.

**Solution.** There are 8 elements in $\mathbb{Z}_8$, namely $\overline{0}, \overline{1}, ..., \overline{7}$. We have

$$\begin{aligned}
(g \circ f)(\overline{0}) &= g([2]) = \overline{4} & h(\overline{0}) &= [4] \\
(g \circ f)(\overline{1}) &= g([3]) = \overline{6} & h(\overline{1}) &= [6] \\
(g \circ f)(\overline{2}) &= g([0]) = \overline{0} & h(\overline{2}) &= [0] \\
(g \circ f)(\overline{3}) &= g([1]) = \overline{2} & h(\overline{3}) &= [2] \\
(g \circ f)(\overline{4}) &= g([2]) = \overline{4} & h(\overline{4}) &= [4] \\
(g \circ f)(\overline{5}) &= g([3]) = \overline{6} & h(\overline{5}) &= [6] \\
(g \circ f)(\overline{6}) &= g([0]) = \overline{0} & h(\overline{6}) &= [0] \\
(g \circ f)(\overline{7}) &= g([1]) = \overline{2} & h(\overline{7}) &= [2].
\end{aligned}$$

Hence, indeed $(g \circ f)(x) = h(x)$ for all $x \in \mathbb{Z}_8$.

[Many of you did $(g \circ f)(\overline{x}) = g([x+2]) = \overline{2x+4} = h(\overline{x})$, which isn't wrong, but the problem asks you to verify this by *comparing images*, i.e by computing $(g \circ f)(x)$ and $h(x)$ and checking that they are equal for every $x \in \mathbb{Z}_8$. Also, say $x \in \mathbb{Z}$, the symbols

$$x \in \mathbb{Z}, \qquad \overline{x} \in \mathbb{Z}_8, \qquad [x] \in \mathbb{Z}_4$$

represent completely different objects (they lie in different sets). It is important to make the distinction. Moreover, if $x \in \mathbb{Z}_8$ say, then $f(x) = [x+2]$ does not make sense because $x$ is not longer an integer.]