The Central Limit Theorem

Thm: $X_1, \dots, X_n \equiv X$ i.i.d. real RVs with mean μ, variance σ^2 for $\sigma > 0$. Define $S_n = \sum_{i=1}^{n} X_n$, and

$$Z_n = \frac{S_n - n\mu}{\sqrt{n}\,\sigma}.$$

As $n \to \infty$, Z_n converges in distribution to $\mathcal{N}(0,1)$.

Recall: Z_n converges in distribution to Z ($Z_n \xrightarrow{\mathcal{D}} Z$) if $\forall$ bounded, cts. $F : \mathbb{R} \to \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{E}[F(Z_n)] = \mathbb{E}[F(Z)].$$

Today: Techniques for proving the CLT:

1. Reductions
2. Fourier method
3. Moment method

Reductions

Normalization: If CLT holds for one RV X, it also holds for $aX+b$, where $a, b \in \mathbb{R}$, $a \neq 0$.
$\Rightarrow$ Assume $\mu = 0$, $\sigma^2 = 1$.

Truncation: Suppose we have verified the CLT for bounded RVs.
Let X be an unbounded RV, with $\mu = 0$, $\sigma^2 = 1$.
Want to verify X satisfies CLT.

Define $X_{\le N} = X \mathbb{1}_{\{|X| \le N\}}$, $X_{>N} = X \mathbb{1}_{\{|X| > N\}}$ $\Rightarrow$ $S_n = S_{n,\le N} + S_{n,>N}$.
Define $\mu_{\le N} = \mathbb{E}[X_{\le N}]$, $\sigma^2_{\le N} = \mathrm{Var}(X_{\le N})$,
$\mu_{>N} = \mathbb{E}[X_{>N}]$, $\sigma^2_{>N} = \mathrm{Var}(X_{>N})$.

WTS $Z_n = \frac{S_n}{\sqrt{n}} = \overset{①}{\frac{S_{n,\le N_n} - n\mu_{\le N_n}}{\sqrt{n}}} + \overset{②}{\frac{S_{n,>N_n} - n\mu_{>N_n}}{\sqrt{n}}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$.

①: By the CLT for bounded RVs,

$$Z_{n,\le N} := \frac{S_{n,\le N} - n\mu_{\le N}}{\sqrt{n}\,\sigma_{\le N}} \xrightarrow[n\to\infty]{\mathcal{D}} \mathcal{N}(0,1).$$

"Diagonalization argument" $\Rightarrow$ Sequence $\{N_n\}$ s.t. $\lim_{n\to\infty} N_n = \infty$, $Z_{n,\ge N_n}$ still $\xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$.

Dominated Convergence Theorem $\Rightarrow$ $\lim_{n\to\infty} \sigma_{\le N_n} = 1$ ⓐ

$\lim_{n\to\infty} \sigma_{>N_n} = 0$ ⓑ

By ⓐ, $\frac{S_{n,\le N_n} - n\mu_{\le N_n}}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$. ①

②: Claim: $\frac{S_{n,>N_n} - n\mu_{>N_n}}{\sqrt{n}} \xrightarrow{\mathbb{P}} 0$.

Justification: By Chebyshev's,

$$\mathbb{P}\left(\left|\frac{S_{n,>N_n} - n\mu_{>N_n}}{\sqrt{n}}\right| \ge \lambda\right) \le \frac{1}{\lambda^2} \operatorname{Var}\left(\frac{S_{n,>N_n} - n\mu_{>N_n}}{\sqrt{n}}\right)$$

$$= \frac{1}{\lambda^2} \cdot \frac{1}{n} \operatorname{Var}\left(\sum_{i=1}^{n} X_{i,>N_n}\right)$$

$$= \frac{1}{\lambda^2} \cdot \frac{1}{n} \sum_{i=1}^{n} \operatorname{Var}(X_{i,>N_n})$$

$$= \frac{1}{\lambda^2} \sigma_{>N_n}^2.$$

$$\xrightarrow{ⓑ} 0.$$

Since $\frac{S_{n,>N_n} - n\mu_{>N_n}}{\sqrt{n}} \xrightarrow{\mathbb{P}} 0$, $\frac{S_{n,>N_n} - n\mu_{>N_n}}{\sqrt{n}} \xrightarrow{\mathcal{D}} 0$. ②

$\square$

$\Rightarrow$ We may assume $\{X_n\}$ bounded.

Fourier Method

Def: Given a real RV X, the characteristic function $F_X : \mathbb{R} \to \mathbb{C}$ is defined by

$$F_X(t) := \mathbb{E}[e^{itX}].$$

For X, Y independent, $F_{X+Y}(t) = F_X(t)\,F_Y(t)$, $F_{cX}(t) = F_X(\bar{c}t)$.

Ex. Suppose $X \sim \mathcal{N}(0,1)$.

$$F_X(t) = \mathbb{E}[e^{itX}]$$

$$= \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-it)^2} e^{-t^2/2}\, dx$$

$$= e^{-t^2/2}.$$

Thm (Lévy continuity thm, special case): Let V be a finite real or complex vector space, and let $\{X_n\}$ be a sequence of V-valued RVs. Let X be an additional V-valued RV. Then TFAE:

1. F_{X_n} converges to F_X.
2. $X_n \xrightarrow{D} X$.

Proof of CLT: Assume X has mean 0, variance 1.

$$F_X(t) := \mathbb{E}[e^{itX}]$$

$$= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \mathbb{E}[X^k]$$

$$= \overset{k=0}{1} + \overset{k=1}{it(0)} + \overset{k=2}{\frac{-t^2}{2}(1)} + o(|t|^2)$$

$$= 1 - \frac{t^2}{2} + o(|t|^2).$$

$$
\begin{aligned}
F_{Z_n}(t) &= F_{\sum_{i=1}^{n} X_i/\sqrt{n}}(t) \\
&= \prod_{i=1}^{n} F_{X_i}\left(\tfrac{t}{\sqrt{n}}\right) \\
&= \left(F_X\left(\tfrac{t}{\sqrt{n}}\right)\right)^n \\
&= \left(1 - \tfrac{t^2}{2n} + o\left(\left|\tfrac{t}{\sqrt{n}}\right|\right)\right)^n \\
&= \left[1 + \tfrac{-t^2/2}{n} + o\left(\left|\tfrac{t}{\sqrt{n}}\right|\right)\right]^n \\
&\xrightarrow{n\to\infty} e^{-t^2/2} \\
&= F_{\mathcal{N}(0,1)}(t).
\end{aligned}
$$

By the Lévy continuity thm, $Z_n \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$.

$\square$

This method is not ideal, because it relies on:

1. Independence of $\{X_1, \dots, X_n\}$.
2. $e^{A+B} = e^A e^B$.

For random matrix theory, we will be forced to modify one or both of these properties.

<u>The Moment Method</u>

Def: X is subgaussian if $\exists C, c > 0$ s.t. $\forall \lambda > 0$,

$$\mathbb{P}(|X| \geq \lambda) \leq Ce^{-c\lambda^2}.$$

Note: X bounded $\Rightarrow$ X subgaussian.

Thm (Carleman cty. thm): Let $\{X_n\}$ be a sequence of uniformly subgaussian RVs and let X be another subgaussian RV. TFAE:

1. $\forall k \in \mathbb{N}$, $\mathbb{E}[X_n^k] \to \mathbb{E}[X^k]$.

2. $X_n \xrightarrow{D} X$.

Proof CLT: Assume X bounded, $\mu = 0$, $\sigma^2 = 1$.
By the Chernoff bound, Z_n is subgaussian.
If $G \sim \mathcal{N}(0,1)$, WTS $\forall k \in \mathbb{N}$,

$$\mathbb{E}[Z_n^k] \to \mathbb{E}[G^k] = \begin{cases} \frac{k!}{2^{k/2}(k/2)!} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}.$$

The rest of the argument is technical and messy. $\square$