A Particle Approximation of the Jeans-Vlasov Equation

SL Math Particle Interactive Systems

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Outline

Introduction

Problem Statement

Let $F : \mathbb{R}^d \to \mathbb{R}^d$ be an **interaction force** with a singularity at zero.
Consider the **Jeans-Vlasov equation** / collisionless Boltzmann equation

$$(PDE) \begin{cases} \partial_t f + v \cdot \nabla_x f + E(t,x) \cdot \nabla_v f = 0 \\ E(t,x) = \int\limits_{\mathbb{R}^d} \rho(t,y) F(x-y) dy \\ \rho(t,x) = \int\limits_{\mathbb{R}^d} f(t,x,v) dv. \end{cases}$$

An exact solution to this PDE has not been discovered (yet).
Our goal is to develop a particle method to approximate the solution of this PDE.

The Particle Method

Idea: We start with N particles $\{X_i(0)\}_{i=1}^N \subseteq \mathbb{R}^d$ with velocities $\{V_i(0)\}_{i=1}^N \subseteq \mathbb{R}^d$.

We create a modified interaction force $F_N : \mathbb{R}^d \to \mathbb{R}^d$ which is bounded at zero.

We evolve our particles according to the ODE

$$(ODE) \begin{cases} \dot{X}_i(t) = V_i(t) \\ \dot{V}_i(t) = \frac{1}{N} \sum_{j=1}^{N} F_N(X_i - X_j). \end{cases}$$

If we define $\mu_N(t,x) = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i(t)}(x)$ we aim to show that

$$\lim_{N \to \infty} \mu_N(t,x) = f(t,x),$$

where f is the exact solution to the Jeans-Vlasov equation under an appropriate initialization.

Theorem and Outline of Proof

Limiting Assumptions on F

F satisfies the (S^α) condition if there exists $C > 0$ such that for all $x \in \mathbb{R}^d \backslash \{0\}$,

$$|F(x)| \leq \frac{C}{|x|^\alpha} \text{ and } |\nabla F(x)| \leq \frac{C}{|x|^{\alpha+1}}.$$

F is **weakly singular** if it satisfies the (S^α) condition for some $\alpha < 1$.

A family of forces $\{F_N\} \cup \{F\}$ is **strongly singular with cutoff** if F satisfies the (S^α) condition for some $\alpha < d - 1$, and if for each N, there exists m such that

- $\forall |x| \geq N^{-m}$, $F_N(x) = F(x)$.
- $\forall |x| \leq N^{-m}$, $|F_N(x)| \leq N^{m\alpha}$.

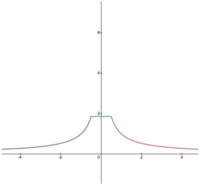
Example: Strong Singularity

Consider the case $d = 2$. Fix $0 < \kappa < 1$. Define $F(x) = \frac{x}{|x|^{2-\kappa}}$, and

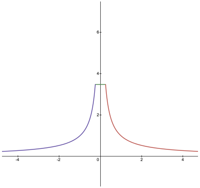
$$F_N(x) = \begin{cases} F(x) & |x| \geq N^{-1} \\ \frac{x}{|x| N^{\kappa - 1}} & |x| \leq N^{-1} \end{cases}.$$

We claim that our forces are strongly singular with cutoff. Set $\alpha = 1 - \kappa$, $m = 1$.

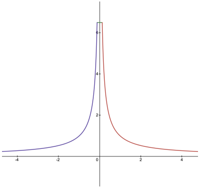
1. F satisfies the S^{α} condition.
2. For $|x| \geq N^{-m}$, $F_N(x) = F(x)$.
3. For $|x| \leq N^{-m}$, $|F_N(x)| = N^{1-\kappa} = N^{m\alpha}$.



$N = 2$



$N = 4$



$N = 8$

Theorem

Theorem

Assume $d \geq 2$ and that $\{F_N\}$ satisfies a (S_m^α) condition for some $1 \leq \alpha < d-1$, with a cutoff order satisfying

$$m < m^* := \frac{1}{2d} \min\left(\frac{d-2}{\alpha-1}, \frac{2d-1}{\alpha}\right).$$

Choose any $\gamma \in \left(\frac{m}{m^*}, 1\right)$.

Assume $f^0 \in \mathcal{L}^\infty(\mathbb{R}^{2d})$ has compact support and total mass one, and denote by f the unique, bounded, and compactly supported solution of the Jeans-Vlasov equation with initial condition f^0 on the interval $[0, T^*)$.

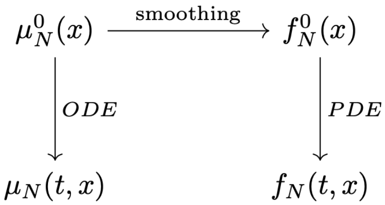
Under certain light conditions on the initialization of our particles, for any $T < T^*$, there exists constants C_0 and C_1 such that for all $N \geq e^{C^1 T}$ and for all $t \in [0, T]$,

$$W_1(\mu_N(t), f(t)) \leq e^{C_0 t}\left(W_1(\mu_N^0, f_N^0) + 2N^{-\frac{\gamma}{2d}}\right).$$

Proof: Notation

We define $f_N^0(x)$ as the “smoothing" of our initial empirical distribution $\mu_N^0(x)$. Then, we define $f_N(t, x, v)$ as the solution to the Jeans-Vlasov equation with initial condition

$$f(0, x, v) = f_N^0(x, v).$$



Sketch of Proof

By the triangle inequality,

$$W_1(\mu_N(t), f(t)) \leq W_1(\mu_N(t), f_N(t)) + W_1(f_N(t), f(t)).$$

A standard stability estimate allows us to bound the second term:

$$W_1(f_N(t), f(t)) \leq e^{Ct} W_1(f_N^0, f^0).$$

Since $W_1 \leq W_\infty$, we bound the first term by

$$W_\infty(\mu_N(t), f_N(t)).$$

Why W_∞? The infinite distance is the only MKW distance with which we can handle a localized singularity in the force and Dirac masses in the empirical distribution.

Sketch of Proof

Goal: Bound $W_\infty(\mu_N(t), f_N(t))$.
To bound this quantity, we compare:

1. $\tilde{E}_N(t,i) := \frac{1}{N} \sum_{j=1}^{N} \int_{t-\epsilon}^{t} F(X_i(s) - X_j(s))\, ds.$
2. $\tilde{E}_\infty(t,x,v) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{t-\epsilon}^{t} F(x_s - y) f(s,y,w)\, ds\, dy\, dw$, where x_s denotes the position at time s of the point starting at (t,x,v) when following the characteristics defined by f_N.

These quantities describe the force generated by N particles, vs. the force generated by a continuous distribution. If we can show that these quantities are similar, then

$$\mu_N(t,x) \approx f_N(t,x).$$

Numerical Implementation

Our setup

We continue using the interaction forces described earlier:

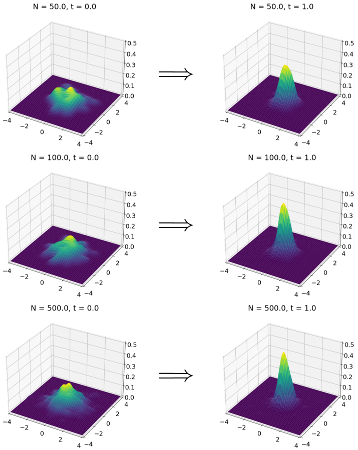
$$F(x) = \frac{x}{|x|^{2-\kappa}},$$

$$F_N(x) = \begin{cases} F(x) & |x| \geq N^{-1} \\ \frac{x}{|x| N^{\kappa-1}} & |x| \leq N^{-1} \end{cases}.$$

These forces approximate the **Coulombian force**, which models

1. Ions or electrons evolving without collisions.
2. Gravitational interactions between stars and galaxies.

Results



We start by generating a **reference** particle solution with many, many particles (in this case $N = 960$), which we hope models the exact solution well.

Then, for smaller N, we calculate the L_1 distance between the empirical solution and the reference solution.

N	Error (L_1)
160	0.31890
320	0.29904
640	0.24443

Future Work

Ideas for future work:

1. Prove the $d = 2$ case.
2. Work with different interaction forces in our numerical simulations.
3. Use the W_1 norm in our numerical simulations to evaluate the error (since this is the norm used in our “big theorem").

Conclusion

References

- Hauray, Maxime and Pierre-Emmanuel Jabin. “Particle approximation of Vlasov equations with singular forces: Propagation of chaos." Annales scientifiques de l’École normale supérieure, 2014.
- Santambrogio, Filippo. *Optimal Transport for Applied Mathematicians.* Birkhäuer, 2015.
- Villani, Cédric. *Topics in Optimal Transport.* Graduate Studies in Mathematics, 1973.

Infinite Wasserstein Distance

We define a **transference plan** as a product measure π on $\mathbb{R}^d \times \mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} d\pi(x,\cdot) = d\mu(x), \int_{\mathbb{R}^d} d\pi(\cdot,x) = d\nu(x).$$

Let $\Pi(\mu,\nu)$ equal the set of all transference plans.
For $\pi \in \Pi(\mu,\nu)$, we define

$$||x-y||_{L^\infty(\pi)} = \max\{|x-y| : (x,y) \in \text{spt}(\pi)\}.$$

We define the **infinite Wasserstein distance** W_∞ by

$$W_\infty(\mu,\nu) = \inf\{||x-y||_{L^\infty(\pi)} : \pi \in \Pi(\mu,\nu)\}.$$

Remarks

- $\lim_{p\to\infty} W_p(\mu,\nu) = W_\infty(\mu,\nu)$.
- $W_1 \leq W_\infty$.

Infinite Wasserstein Distance

Need to restrict to space of measures with bounded support.

- Otherwise, $W_\infty(\mu, \nu)$ may be infinite.

If μ is absolutely continuous w.r.t Lebesgue measure,

$$W_\infty(\mu, \nu) = \inf\{||T(x) - x||_{L^\infty(\mu)} : T\#\mu = \nu\}.$$

From Santambrogio:

> *[The W_∞ distance] measures the minimal maximal displacement that should be done to move particles from one distribution to the other.*

In other words, for each transference plan, we measure the greatest distance we have to transport a particle. Then, we take the infimum of each of these distance.

Initialization of Particles

From Hauray et. al:

> *In many physical settings, the initial positions and velocities are selected randomly and typically independently (or almost independently)...*
>
> *Moreover, we emphasize that the two problems with initial particles on a mesh, or with initial particles not equally distributed seem to be very different.*

In our numerical simulations, particles and velocities are distributed according to the normal distribution with mean one, standard deviation zero.

Sketch of Proof

To compare $\tilde{E}_N(t, i)$ and $\tilde{E}_\infty(t, x, v)$, we consider three domains:

1. Contribution of particles j and points y “far enough" from X_i and x in the physical space.
2. Contribution of particles j and points y ϵ-close in the physical space to X_i and x, but with sufficiently different velocities.
3. Contribution of particles ϵ-close in $\mathbb{R}^{2d}$ in position and velocity.