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RESEARCH STATEMENT

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MOTIVATION FOR CURRENT WORK

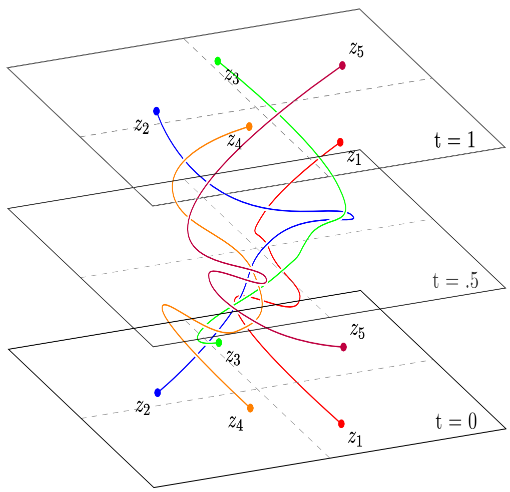


Figure 1: A pure braid path

One very surprising and beautiful connection between topology, geometry and algebra is the braid group arising as the fundamental group of the space of regular orbits of a hyperplane reflection arrangement [5].

To see it, consider the action of the symmetric group S_n on $\mathbb{R}^n$ which acts by permuting coordinates. It is generated by reflections fixing hyperplanes $x_i - x_j = 0$. Denote by X_n the union of all such planes. Topologically, the hyperplane complement $\mathbb{R}^n \backslash X_n$ is not very interesting, being the disjoint union of $n!$ contractable components.

However, if we first complexify $\mathbb{R}^n$ to $\mathbb{C}^n$ (and X_n to $X_n^{\mathbb{C}}$), then $\mathbb{C}^n \backslash X_n^{\mathbb{C}}$ is actually comprised of one path connected component. To compute the fundamental group of the complexified hyperplane complement, we may visualize a point in $A = \mathbb{C}^n \backslash X_n^{\mathbb{C}}$ by projecting each of its coordinates onto a single copy of $\mathbb{C}$.

Notice each of the x_i should be distinct points in $\mathbb{C}$, since we have removed the hyperplanes corresponding to two coordinates being equal. Elements of the fundamental group of A correspond with based loops (continuous maps $f : [0,1] \to A$ with $f(0) = f(1)$). We know that at each $t \in [0,1]$, $f(t)$ can be visualized as an ordered collection of n points in $\mathbb{C}$, none of which are the same.

The restriction of $f(t)$ to a single complex coordinate gives one "strand" in Figure 1. None of the strands intersect (though from our viewpoint one may cross over another). So when we look at the collection of based loops up to homotopy, we get exactly what is called the pure braid group on n strands, PB_n. If we quotient by the action of the symmetric group on the coordinates, allowing where the strands terminate to permute, then we get what is called the braid group B_n.

Further, S_n has presentation:

$$S_n = \langle r_1, r_2, \ldots, r_{n-1} \mid r_i^2 = 1 \ , \ r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1} \ , \ r_i r_j = r_j r_i \rangle \ (|i-j| > 1)$$

which removing the "order relations" gives a presentation for B_n.

$$B_n = \langle r_1, r_2, \ldots, r_{n-1} \mid r_i r_{i+1} r_i = r_{i+1} r_i r_{i+1} \ , \ r_i r_j = r_j r_i \rangle \ (|i-j| > 1)$$

Related groups, called Artin groups, arise in a similar fashion from the spherical and Euclidean Coxeter groups (real reflection groups). Surprisingly, the same correspondance between presentations of Coxeter groups and Artin groups (removing the relations) holds.

A *classifying space* for a group G is a topological space which has fundamental group isomorphic to G and all higher homotopy groups of the space are trivial. In a sense it is a space that carries the information of the group. It turns out that the spaces corresponding to spherical Coxeter (finite real) reflection groups are classifying spaces [4], [2]. Progress has been made using Garside theory towards finding out if the same holds for Euclidean Coxeter groups [8].

The natural next groups to study are complex reflection groups ($\mathbb{C}$RGs). Finite $\mathbb{C}$RGs have been classified and studied extensively. One important result is Steinberg's Fixed-Point Theorem, which states that any linear subspace fixed by a $\mathbb{C}$RG is also generated by reflections, similar to real reflection groups. In addition, finite $\mathbb{C}$RGs give rise to classifying spaces [1].

A list of discrete affine $\mathbb{C}$RGs acting on $\mathbb{C}^n$ was given in [9]. Some progress has been made in understanding the fundamental groups of the corresponding spaces [7].

So for infinite discrete complex reflection groups, the two questions I am interested in are:

1) Is the corresponding hyperplane complement, after quotienting by a group action, a classifying space?

2) Is there a presentation for the reflection group, which when order relations are removed gives a presentation for the fundamental group of the corresponding quotiented hyperplane complement?

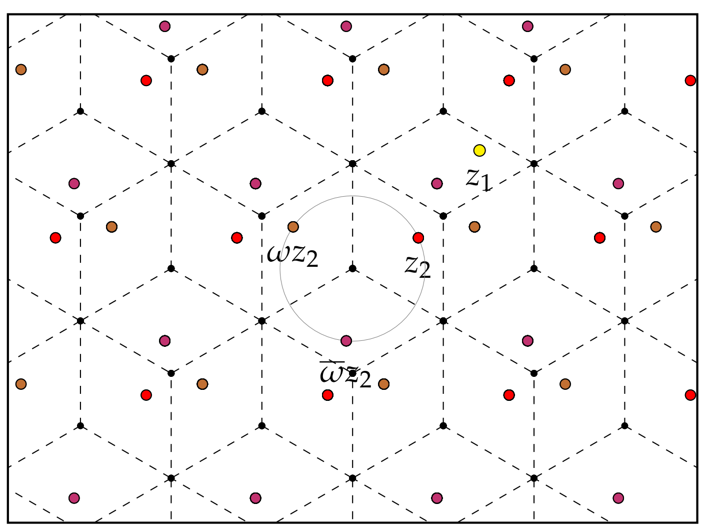
Currently, my coauthor and I have an article in preparation where we use a 24-cell and the quaternions to construct a piecewise Euclidean complex which we believe answers both questions for one such group [3]. In addition, we demonstrate that Steinberg's Fixed-Point Theorem does not hold for affine $\mathbb{C}$RGs.

Using similar techniques we plan to try to answer the questions for other 2-dimensional discrete affine $\mathbb{C}$RGs.

Planned areas for further research

Take a configuration of points in the plane arising from some system of linear complex equations. Allow it to move rigidly (all the points staying in the same position relative to each other). Call this a loom. Consider a collection of looms, all living in the same plane.

One natural question is what is the group formed by motions of these looms which return to where they started and do not crash into each other. Notice that the pure braid group on n strands



is the loom group with n single-point looms. In general, this seems to be a difficult question. However, small, finite examples can be worked out fairly easily. Many Artin groups can be seen as loom groups which have been quotiented by some group of symmetries.

Potential undergraduate research

Understanding the groups arising from such a situation with various types of constraints has a "low floor and high ceiling", making it an ideal topic for undergraduate contribution.

Applied algebraic topology and robotic motion

There are also fascinating connections which apply group of braiding motions to applications in robotics [6]. Falling into the relatively new and exciting field of applied algebraic topology, it is an area I plan to explore more thoroughly.

Mathematics education

I am interested in inquiry-based learning, preservice teacher training, and mathematics curriculum. These interests are further detailed in my teaching philosophy statement.

References

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