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Read with suspicion. Please let me know about wrong things you find below. Thanks.

Solutions of Some Problems

Problem 9 (label "cmc") part (c): This question is answered in the affirmative if we show that $\{f_j\}$ is Cauchy in measure together with the convergence of a subsequence $f_{j_l} \to f$ a.e. implies that $f_j \to f$ in measure. Since

$$\{|f - f_k| > \epsilon\} \subset \{|f_k - f_{j_l}| > \epsilon/2\} \cup \{|f_{j_l} - f| > \epsilon/2\}$$

and the measure of the first set can be made small by taking k, l large, it suffices to show that the measure of the second set can be made small by taking l large. This reduces the issue to showing that Cauchy in measure and convergence a.e. implies convergence in measure.

Thus let $\{f_j\}$ be Cauchy in measure and $f_j \to f$ a.e. Recall that $f_j \to f$ a.e by itself is *not*, in general, enough to imply that $f_j \to f$ in measure. Throw out the null set of x's for which $f_j(x) \not\to f(x)$, so that we may assume $f_j \to f$ everywhere. Then $|f(x) - f_k(x)| > \epsilon$ implies that $|f_j(x) - f_k(x)| > \epsilon$ as soon as j is large enough (with "large enough" depending on x and ϵ). That is,

$$\{|f - f_k| > \epsilon\} \subset \bigcup_{J=1}^{\infty} \bigcap_{j=J}^{\infty} \{|f_j - f_k| > \epsilon\}\,.$$

Since $\cap_{j=J}^{\infty} \{|f_j - f_k| > \epsilon\}$ gets bigger as J increases, this implies that

$$\mu(\{|f - f_k| > \epsilon\}) \le \lim_{J\to\infty} \mu\left(\cap_{j=J}^{\infty} \{|f_j - f_k| > \epsilon\}\right) \le \limsup_{J\to\infty} \mu\left(\{|f_J - f_k| > \epsilon\}\right).$$

Since $\{f_j\}$ is Cauchy in measure, the right-hand side is less than ϵ if k is sufficiently large. (This solution was suggested by the psets of Tomas and Kamille).

Problem 17 (label "nolo"): Consider the mapping $f \mapsto s_N(f) := \sum_{-N}^{N} \hat{f}(n)e^{int}$, which takes f into the N^{th} partial sum of its Fourier series. Show that it is not true that for all f in $L^1(-\pi, \pi)$ we have $\|f - s_N(f)\|_{L^1(-\pi,\pi)} \to 0$ as $N \to \infty$. Hint: In outline, but not in detail, the proof mimics the proof that Fourier series of a continuous function do not converge pointwise to the function. You need to show that $\|S_N\|$ is unbounded in N, and this is related to $\|D_N\|_{L^\infty(-\pi,\pi)}$.

Solution

We first remark that final part of the "Hint" is not directly used below; our advice was misleading, apologies.

Let $D_N(t) = \sin((N + 1/2)t)/\sin(t/2)$. It suffices to show that if

show (0.1)
$$T_N f(t) := \int_{-\pi}^{\pi} D_N(t - s) f(s)\, ds = 2\pi S_N(f)(t)$$

then $\|T_N\|$ (as a linear mapping from $L^1(-\pi,\pi)$ into itself) is unbounded in N. By the uniform boundedness principle, this implies that $\{S_N(f)\}$ is unbounded in $L^1(-\pi,\pi)$ for a dense G_δ bunch of f's in $L^1(-\pi,\pi)$, and for those f's, $\|S_N(f)-f\|_{L^1(-\pi,\pi)} \to 0$ as $N\to\infty$ is impossible.

Let us assume, to the contrary, that $\|T_N\| \le M$ for all N. Then if $f \in L^1(-\pi,\pi)$, $g\in L^\infty(-\pi,\pi)$, we would have, using Fubini and then that D_N is even,

dualize

$$
\begin{aligned}
\Big|\int_{\pi}^{\pi} T_N(f)(t)g(t)\,dt\Big| &= \Big|\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} D_N(t-s)f(s)\,ds\right)g(t)\,dt\Big| \\
&= \Big|\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} D_N(t-s)g(t)\,dt\right)f(s)\,ds\Big| \\
&= \Big|\int_{-\pi}^{\pi} T_N(g)(s)f(s)\,ds\Big| \\
&\le \|T_N(f)\|_{L^1(-\pi,\pi)}\|g\|_{L^\infty(-\pi,\pi)} \\
&\le M\|f\|_{L^1(-\pi,\pi)}\|g\|_{L^\infty(-\pi,\pi)}
\end{aligned}
\tag{0.2}
$$

The first inequality above is a bound on the far left-hand side of the chain of relations, and the second inequality is from the assumption $\|T_N\|\le M$ for all N.

Choosing $f=\chi_E$ after the third equality sign, where $E\subset(-\pi,\pi)$, this tells us that

$$\Big|\int_E T_N(g)(s)\,ds\Big| \le M|E|\|g\|_{L^\infty(-\pi,\pi)}$$

where $|E|$ is the Lebesgue measure of E, and this implies that

$$\|T_N(g)\|_{L^\infty(-\pi,\pi)} \le M\|g\|_{L^\infty(-\pi,\pi)}$$

However, we already shown that this cannot hold, even for $g\in C_p([-\pi,\pi])$. See the arguments following equation (6.44) in the notes (label "ubfs").

Problem 19 (label "nolo") part (iv) The problem here was to show that $T_\epsilon f := \rho_\epsilon * f$ does not converge to the identity in the normed space $\mathcal{L}(L^p(\mathbb{R}^n), L^p(\mathbb{R}^n))$, or

$$\limsup_{\epsilon\downarrow 0}\|T_\epsilon - I\| \ne 0.$$

We'll work in $L^1(\mathbb{R}^n)$. Assume the contrary. Then there is a $\kappa(\epsilon)>0$ satisfying $\limsup_{\epsilon\downarrow 0}\kappa(\epsilon)=0$ such that

dst

$$\|\rho_\epsilon * f - f\|_1 \le \kappa(\epsilon)\|f\|_1. \tag{0.3}$$

Put $f=g_\delta(x)$ where $\delta>0$ and

$$g(x) := \frac{1}{\omega_n}\chi_{B(0,1)}(x),\quad g_\delta(x) := \frac{1}{\omega_n\delta^n}\chi_{B(0,1)}\left(x/\delta\right) = \frac{1}{\omega_n\delta^n}\chi_{B(0,\delta)}\left(x\right);$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. That is, g_δ is related to g as ρ_ϵ is related to ρ. Then $\|g_\delta\|_1=1$ and (0.3) tells us that

$$\|\rho_\epsilon * g_\delta - g_\delta\|_1 \le \kappa(\epsilon)$$

Next, for fixed $\epsilon > 0$, $\lim_{\delta \downarrow 0} \|\rho_\epsilon * g_\delta - \rho_\epsilon\|_1 = 0$ by the theorem on approximate identities. Putting this together with (0.3), we have

comboest

$$\begin{aligned} \|\rho_\epsilon - g_\delta\|_1 &\leq \|\rho_\epsilon * g_\delta - g_\delta\|_1 + \|\rho_\epsilon * g_\delta - \rho_\epsilon\|_1 \\ &\leq \kappa(\epsilon) + \theta(\delta, \epsilon) \end{aligned} \tag{0.4}$$

where $\lim_{\delta \downarrow 0} \theta(\delta, \epsilon) = 0$ for fixed $\epsilon > 0$. However,

$$\|\rho_\epsilon - g_\delta\|_1 = \int_{\mathbb{R}^n} \left| \frac{1}{\epsilon^n} \rho\left(\frac{x}{\epsilon}\right) - \frac{1}{\delta^n \omega_n} \chi_{B(0,\delta)}(x) \right| dx.$$

Changing variables by $y = x/\epsilon$ leads to

$$\|\rho_\epsilon - g_\delta\|_1 = \int_{\mathbb{R}^n} \left|\rho(y) - \frac{\epsilon^n}{\delta^n \omega_n} \chi_{B(0,\delta/\epsilon)}(y)\right| dx.$$

This and (0.4) yields

$$\int_{|y|>\delta/\epsilon} \rho(y)\, dy \leq \|\rho_\epsilon - g_\delta\|_1 \leq \kappa(\epsilon) + \theta(\delta, \epsilon).$$

This cannot be, for it implies that

$$1 = \int_{\mathbb{R}^n} \rho\, dy = \lim_{\delta \downarrow 0} \int_{|y|>\delta/\epsilon} \rho\, dy \leq \lim_{\delta \downarrow 0} (\kappa(\epsilon) + \theta(\delta, \epsilon)) = \kappa(\epsilon),$$

and $\kappa(\epsilon) < 1$ if ϵ is sufficiently small.

Problem 21 (label "bm") Show that if $f : \mathbb{R}^n \to \mathbb{R}$ is Lebesgue measurable, then there is a Borel measurable function f_0 such that $f = f_0$ a.e. You may quote - do so in detail by giving page numbers or statements - results from KF or your lecture notes from last quarter, but not other sources. Close any gaps to accommodate "infinite measures."

Solution It suffices to assume that $f \geq 0$. Indeed, splitting f into its positive and negative parts, the result may be applied to each of them. If $f \geq 0$, we showed in class that there is an increasing sequence of simple functions $\{s_j\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} s_j(x) = f(x)$ everywhere. If we can show the result for simple functions, then there are Borel measurable functions b_j such that $s_j(x) = b_j(x)$ except on a Lebesgue null set N_j. It then follows that

$$f(x) = \lim_{j\to\infty} s_j(x) = \limsup_{j\to\infty} b_j(x) \quad \text{except on} \quad N := \cup_{j=1}^{\infty} N_j,$$

and N is a Lebesgue null set. Since measurability is preserved under "lim sup", $\limsup_{j\to\infty} b_j$ is Borel measurable, and we are done.

Thus the issue is reduced to treating simple functions, that is finite linear combinations of functions of the form χ_E where E is Lebesgue measurable. Clearly it suffices to treat a single χ_E. From last term, you know that if $|E|$ (the Lebesgue measure of E) is finite, then there is an F_σ set $E_0 \subset E$ such that $|E \setminus E_0| = 0$. In the general case, let $E_m = E \cap \{x : |x| \leq m\}$ and $E_{m,0}$ be an F_σ satisfying $E_{m,0} \subset E_m$ and $|E_m \setminus E_{m,0}| = 0$. Then

$$E \setminus \cup_{m=1}^{\infty} E_{m,0} = \cup_{j=1}^{\infty} (E_m \setminus E_{m,0})$$

shows that $E_0 := \cup_{m=1}^{\infty} E_{m,0}$ is an F_σ set - hence Borel measurable - contained in E such that $\chi_E = \chi_{E_0}$ ae (wrt Lebesgue measure).

More Hints

Exercise 33 (label "spmeas"). The problem is to show that if $(X, \mathcal{M})$ is a measurable space, then the set of complex measures on it is a Banach space under the "total variation" norm

$$\|\mu\| = |\mu|(X).$$

The linear structure of the space of complex measures is entirely natural: $(a\mu_1 + b\mu_2)(E) := a\mu_1(E) + b\mu_2(E)$. It is easy to check that the norm proposed is a norm. If $\{\mu_j\}$ is a Cauchy sequence in this norm, then for any $E \in \mathcal{M}$,

$$|\mu_j(E) - \mu_k(E)| \leq |\mu_j - \mu_k|(E) \leq \|\mu_j - \mu_k\|.$$

It follows that μ may be defined on $\mathcal{M}$ by

$$\mu(E) := \lim_{j\to\infty} \mu_j(E).$$

We need to show that μ is a measure. Since each μ_j is a measure, if $E_1, E_2, \ldots, E_m$ is a finite sequence of pairwise disjoint sets in $\mathcal{M}$,

$$\mu\left(\cup_{l=1}^m E_l\right) = \lim_{j\to\infty} \mu_j\left(\cup_{l=1}^m E_l\right) = \lim_{j\to\infty} \sum_{l=1}^m \mu_j(E_l) = \sum_{l=1}^m \lim_{j\to\infty} \mu_j(E_l) = \sum_{l=1}^m \mu(E_l),$$

because we may commute limits and finite sums. Thus μ is finitely additive.

We want, however, to have the analogous result for infinite partitions $\{E_l\}_{l=1}^{\infty}$ of a set E. One cannot willy-nilly commute infinite sums and limits. However, the result just remarked gives us

$$\mu(E) = \sum_{l=1}^m \mu(E_l) + \mu\left(\cup_{l=m+1}^{\infty} E_l\right).$$

Thus the heart of the game is to show that

$$\lim_{m\to\infty} \mu\left(\cup_{l=m+1}^{\infty} E_l\right) = 0.$$