Rick asked me a good question about Theorem 10.4 (label "lpts") on Lebesgue points, reproduced below.

lpts **Theorem 0.1.** *Let $f \in L^1(\mathbb{R}^n)$. Then for almost all $x \in \mathbb{R}^n$*

elpts
$$\lim_{r \downarrow 0} \frac{1}{r^n} \int_{B(x,r)} |f(y) - f(x)|\, dy = 0. \tag{0.1}$$

His question had to do with how this statement interacts with the idea that elements of $L^1(\mathbb{R}^n)$ are equivalence classes of functions which differ from one another only on null sets, and we then cannot talk about the value of such an equivalence class at a single point, like "x."

The full answer illustrates that mathematicians do not always use notation $f \in L^1(\mathbb{R}^n)$ to mean the equivalence class which f represents. Sometimes they have a genuine function in mind. An unambiguous statement is:

nlpts **Theorem 0.2.** *Let f be defined almost everywhere on $\mathbb{R}^n$, Lebesgue measurable, and $\int_{\mathbb{R}^n} |f(y)|\, dy < \infty$. Then (0.1) holds for almost all $x \in \mathbb{R}^n$.*

This is what was proved.

Another way to handle this is to redefine a Lebesgue point of $f \in L^1(\mathbb{R}^n)$ (meaning now the full equivalence class of which f is a representer) by

ndlpts **Definition 0.3.** Let $f \in L^1(\mathbb{R}^n)$. Then x is a Lebesgue point of f if there exists a number L such that

enlpts
$$\lim_{r \downarrow 0} \frac{1}{r^n} \int_{B(x,r)} |f(y) - L|\, dy = 0. \tag{0.2}$$

It is immediate that there is at most one L for which (0.2) holds and that the Lebesgue points and corresponding L's are the same for all functions which agree with f a.e., since the notion depends only on integrals of f, so the notion is well defined on the equivalence classes of $L^1(\mathbb{R}^n)$. Let us define the function L_f by letting $L_f(x) = L$ if x is a Lebesgue point of f and (0.2) holds and regard L_f as undefined elsewhere. *Every* element of the equivalence class of $f \in L^1(\mathbb{R}^n)$ gives rise to the same L_f. By Theorem 0.2, every element of the equivalence class of f agrees with L_f a.e. We can think of L_f as the best representer on this class. Roughly speaking, if $L_f(x)$ is defined and $L_f(x) \neq f(x)$, we made a mistake in assigning the value $f(x)$ to f at x. We should have assigned it the value $L_f(x)$.

For example, if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 0$ if $x \neq 0$ and $f(0) = 1$, then $L_f \equiv 0$. L_f knows that we made something uglier than necessary by letting $f(0) = 1$.

These comments extend to functions on $\mathbb{R}^n$ which are not necessarily L^1, all one needs is integrability over sets of finite measure.