Set up: A is hyperbolic, $\mathbb{R}^n = E_s \oplus E_u$ is the decomposition of $\mathbb{R}^n$ into the A invariant spaces such that $A|_{E_s}$ has eigenvalues of negative real part and $A|_{E_u}$ has eigenvalues of positive real part. P_s, P_u are the corresponding projections on E_s, E_u respectively. The time t map for the ivp

$$\dot{x} = Ax + f(x) \tag{1.1}$$

is ψ_t; for any solution of (1.1) $x(t) = \psi_t(x(0))$. Here $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$\|f(x)\| \le C \quad \text{and} \quad \|f(x) - f(y)\| \le L\|x - y\| \quad \text{and} \quad f(0) = 0. \tag{1.2}$$

Theorem 1.1. *Let (1.2) hold. Then there is a number $0 < L_0(A)$ such that if $L < L_0(A)$ then there is unique bounded and continuous $p : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$e^{tA}x + p(e^{tA}x) = \psi_t(x + p(x)) \quad \text{for} \quad x \in \mathbb{R}^n, t \in \mathbb{R}. \tag{1.3}$$

Moreover, $x \to x + p(x)$ is a homeomorphism of $\mathbb{R}^n$.

To begin, let us first note that if p is bounded, then (1.3) guarantees that $I + p$ is a homeomorphism. Indeed, any countinuous bounded perturbation of the identity is onto $\mathbb{R}^n$, so we only need to show that $x + p(x) = y + p(y)$ implies $x = y$. However, from (1.3) we find then that $e^{tA}(x - y) = -p(e^{tA}x) + p(e^{tA}y)$ and the right-hand side is bounded independently of t. Since A is hyperbolic, this forces $x = y$.

It remains to establish the existence and uniqueness of p. We need the standard observation:

Proposition 1.2. *Let A be hyperbolic and $g : \mathbb{R} \to \mathbb{R}^n$ be continuous and bounded. Then a solution $x(t)$ of*

$$\dot{x} = Ax + g(t)$$

on $\mathbb{R}$ is bounded for $0 \le t$ (respectively, $t \le 0$) if and only if

$$\begin{gathered} P_u x(0) + \int_0^\infty e^{-\tau A} P_u g(\tau)\, d\tau = 0 \\ \text{(respectively } P_s x(0) - \int_{-\infty}^0 e^{-\tau A} P_s g(\tau)\, d\tau = 0). \end{gathered} \tag{1.4}$$

To continue the proof of Theorem 1.1 we differentiate (1.3) to find

$$\begin{aligned} Ae^{tA}x + \frac{d}{dt}p(e^{tA}x) &= A\psi_t(x+p(x)) + f(\psi_t(x+p(x))) \\ &= A(e^{tA}x + p(e^{tA}x)) + f(e^{tA}x + p(e^{tA}x)) \end{aligned}$$

so

$$\frac{d}{dt}p\left(e^{tA}x\right) = Ap\left(e^{tA}x\right) + f(e^{tA}x + p(e^{tA}x)). \tag{1.5}$$

Clearly the argument is reversible, and (1.5) implies (1.3).

Since f and p in (1.5) are bounded, we conclude from Proposition 1.2 that necessarily

$$\begin{aligned} P_u p(x) + \int_0^\infty e^{-\tau A} P_u f(e^{\tau A}x + p(e^{\tau A}x))\, d\tau = 0, \\ P_s p(x) - \int_{-\infty}^0 e^{-\tau A} P_s f(e^{\tau A}x + p(e^{\tau A}x))\, d\tau = 0. \end{aligned} \tag{1.6}$$

This fixed point problem for p trivially submits to the contraction mapping theorem if the Lipschitz constant for f is sufficiently small (depending on A). There is a unique bounded and continuous fixed point.

Assuming (1.6) holds, we deduce the integral form of (1.5), namely

$$p\left(e^{tA}x\right) = e^{tA}\left(p(x) + \int_0^t e^{-\tau A} f(e^{\tau A}x + p(e^{\tau A}x))\, d\tau\right).$$

This arises by replacing x by $e^{tA}x$ in each of the relations in (1.6) and adding.

Happy New Year!