A VISIT WITH THE ∞-LAPLACE EQUATION

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Introduction

In these notes we present an outline of the theory of the archetypal L^∞ variational problem in the calculus of variations. Namely, given an open $U \subset \mathbb{R}^n$ and $b \in C(\partial U)$, find $u \in C(\overline{U})$ which agrees with the boundary function b on ∂U and minimizes

$$\mathcal{F}_\infty(u, U) := \| |Du| \|_{L^\infty(U)} \tag{0.1}$$

among all such functions. Here $|Du|$ is the Euclidean length of the gradient Du of u. We will also be interested in the "Lipschitz constant" functional as well. If K is any subset of $\mathbb{R}^n$ and $u\colon K \to \mathbb{R}$, its least Lipschitz constant is denoted by

$$\mathrm{Lip}(u, K) := \inf \{ L \in \mathbb{R} : |u(x) - u(y)| \le L|x - y| \ \forall\, x, y \in K \}\,. \tag{0.2}$$

Of course, $\inf \emptyset = +\infty$. Likewise, if any definition such as (0.1) is applied to a function for which it does not clearly make sense, then we take the right-hand side to be $+\infty$. One has $\mathcal{F}_\infty(u, U) = \mathrm{Lip}(u, U)$ if U is convex, but equality does not hold in general.

Example 2.1 and Exercise 2 below show that there may be many minimizers of $\mathcal{F}_\infty(\cdot, U)$ or $\mathrm{Lip}(\cdot, U)$ in the class of functions agreeing with a given boundary function b on ∂U. While this sort of nonuniqueness can only take place if the functional involved is not strictly convex, it is more significant here that the functionals are "not local." Let us explain what we mean in contrast with the Dirichlet functional

$$\mathcal{F}_2(u, U) := \frac{1}{2} \int_U |Du|^2 \, dx. \tag{0.3}$$

This functional has the property that if u minimizes it in the class of functions satisfying $u|_{\partial U} = b$ and $V \subset U$, then u minimizes $\mathcal{F}_2(\cdot, V)$ among functions which agree with u on ∂V, properly interpreted. This is what we mean by "local" here. Exercise 2 establishes that both Lip and $\mathcal{F}_\infty$ are not local, as you can also do with a moment's thought.

This lack of locality can be rectified by a notion which directly builds in locality. Given a general nonnegative functional $\mathcal{F}(u, V)$ which makes sense for each open subset V of the domain U of u, one says that $u\colon U \to \mathbb{R}$ is *absolutely minimizing*[2] for $\mathcal{F}$ on U provided that

$$\mathcal{F}(u, V) \le \mathcal{F}(v, V) \text{ for every } v\colon \overline{V} \to \mathbb{R} \text{ such that } u = v \text{ on } \partial U$$

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[2]The author prefers the more descriptive term "locally minimizing," but c'est la vie.

whenever $\overline{V}$ is compactly contained in the domain of u. Of course, we need to supplement this idea with some more precision, depending on $\mathcal{F}$, but we won't worry about that here. Clearly, if u is absolutely minimizing for $\mathcal{F}$ on U, then it is absolutely minimizing for $\mathcal{F}$ on open subsets of U. The absolutely minimizing notion decouples the considerations from the boundary condition; it defines a class of functions without regard to behavior at the boundary of U. We might then consider the problem: find $u\colon \overline{U} \to \mathbb{R}$ such that

$$u \text{ is absolutely minimizing for } \mathcal{F} \text{ on } U \text{ and } u = b \text{ on } \partial U. \tag{0.4}$$

It is not quite clear that a solution of this problem minimizes $\mathcal{F}$ among functions which agree with b on ∂U. Indeed, it is not true for $\mathcal{F} = \mathcal{F}_\infty, \mathrm{Lip}, \mathcal{F}_2$ if U is unbounded (Exercise 2). The notion also does not require the existence of a function u satisfying the boundary condition for which $\mathcal{F}(u, U) < \infty$.

The theory of absolutely minimizing functions and the problem (0.4) for the functional Lip is quicker and slicker than that for $\mathcal{F}_\infty$, and we present this first, ignoring $\mathcal{F}_\infty$ for a while. However, it is shown in Section 6 that a function which is absolutely minimizing for Lip is also absolutely minimizing for $\mathcal{F}_\infty$ and conversely. It turns out that the absolutely minimizing functions for Lip and $\mathcal{F}_\infty$ are precisely the viscosity solutions of the famous partial differential equation

$$\Delta_\infty u = \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0. \tag{0.5}$$

The operator Δ_∞ is called the "∞-Laplacian" and solutions of (0.5) are said to be ∞-harmonic. The reason for this nomenclature is given in Section 8. The notion of a "viscosity solution" is given in Definition 2.6 and again in Section 8, together with more information about them than is in the main text.

An all-important class if ∞-harmonic functions (equivalently, absolutely minimizing functions for $\mathcal{F} = \mathcal{F}_\infty$ or $\mathcal{F} = \mathrm{Lip}$) is the class of *cone functions*:

$$C(x) = a|x - z|, \tag{0.6}$$

where $a \in \mathbb{R}$ and $z \in \mathbb{R}^n$; C is absolutely minimizing on $\mathbb{R}^n \setminus \{0\}$. It turns out that the ∞-harmonic functions are precisely those that have a comparison property with respect to cone functions. All the basic theory of ∞-harmonic functions can be derived by *comparison with cones*. This is explained and exploited in Sections 2 - 7.

The table of contents gives an impression of how these notes are organized, and we will not belabor that here, except for some comments. Likewise, the reader will have noticed a lack of references in this introduction. We will give only two, including the research/expository article [8]. These notes are closely related to [8]. However, [8] treats the situation in which $|\cdot|$ is a general norm on $\mathbb{R}^n$ rather than the Euclidean norm. There is perhaps a cost in elegance in this generality, and we welcome the opportunity to write the story of the Euclidean case by itself. In particular, at the time of this writing, it is not quite settled whether or not the absolutely minimizing property is equivalent to a partial differential equation in the case of a general norm. It has, however, been shown that this is true for, for example, the l_∞ and l_1 norms on $\mathbb{R}^n$, which was unknown at the writing of [8]. There is a rather complete set of references in [8], along with comments, up to the time it was written. We rely partly on [8] in this regard.

In Section 8 we give an informal outline of the nearly 40 year long saga of the theory of the ∞-Laplace equation. This section is intended to be readable immediately after this introduction; it corresponds to a talk the author gave at a conference in honor of G. Aronsson, the initiator of the theory, in 2004. Selected references are given in Section 8. In addition, in Section 9, we attempt to give a feeling for the many generalizations and the scope of recent activity in this active area, going well beyond the basic case we study here in some detail, including sufficient references and pointers to provide the interested reader entree into whatever part of the evolving landscape suits their interests.

What is new in the current article, relative to [8], besides many details of the organization and presentation and Sections 8, 9? Quite a number of things, of which we point out the direct derivation of the ∞-Laplace equation in the viscosity sense from comparison with cones in Section 2.2 (a refinement of an original argument in [8]), the gradient flow curves in Section 6, the outline of a new uniqueness proof in Section 5, a new result in Section 7, a variety of items near the end of Section 4 and a little proof we'd like to share in Section 7.2. In addition, the current text is supplemented by exercises. Most of them fall in the range from straightforward to very easy and of some of them are solved in the main text of [8]. Whether or not the reader attempts them, they should be read. While we have largely restricted this exposition to the case of the Euclidean norm, the knowledgeable reader will see that many of the new elements herein immediately generalize to the case of a general norm, and beyond, and the presentation has this in mind.

Regarding our exposition, we build in some redundancy to ease the flow of reading. In particular, Section 8 is largely accessible now, after this introduction.

In the lectures to which these notes correspond, the author spent considerable time on a recent result of O. Savin [50]. Savin proved that ∞-harmonic functions are C^1 if $n = 2$. This was a big event. Whether or not this is true if $n > 2$, and the author would bet that it is, remains the most prominent open problem in the area. The author had hoped to include an exposition of Savin's results here, but did not in the end do so, due to lack of time. In any case, any such exposition would have been very close to the original article. If it is proved that ∞-harmonic functions are C^1 in general, this article will need substantial revision; however, most of the theory is about ∞-subharmonic functions, which are not C^1, so it is unlikely the material herein will be rendered obsolete. In any case, we'll get on with the current version, right after expressing our thanks to Gunnar Aronsson for comments which improved this manuscript.

Contents

1. Notation

In these notes, U, V, W are always open subsets of $\mathbb{R}^n$. The closure of U is $\overline{U}$ and its boundary is ∂U. The statement $V \ll U$ means that $\overline{V}$ is a compact subset of U.

If $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, then

$$|x| := \left(\sum_{j=1}^n x_j^2\right)^{1/2} \qquad \text{and} \qquad \langle x, y\rangle := x_1y_1 + \cdots + x_ny_n.$$

The notation $A := B$ means that A is defined to be B. If $K \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, then

$$\operatorname{dist}(x, K) := \inf_{y \in K} |x - y|.$$

If $x^j \in \mathbb{R}^n$, $j = 1, 2, \dots,$ then

$$x^j \to y \quad \text{means} \quad y \in \mathbb{R}^n \text{ and } \lim_{j\to\infty} x^j = y.$$

The space of continuous real valued functions on a topological space K is denoted by $C(K)$. The notation "$u \in C^k$" indicates that u is a real-valued function on a subset of $\mathbb{R}^n$

and it is k-times continuously differentiable. $L^\infty(U)$ is the standard space of essentially bounded Lebesgue measurable function with the usual norm,

$$\|v\|_{L^\infty(U)} = \inf\{M \in \mathbb{R} : |v(x)| \le M \text{ a.e in } U\}.$$

We also use pointwise differentiability and twice differentiability. For example, if $w : U \to \mathbb{R}$ and $y \in U$, then w is twice differentiable at y if there exist $p \in \mathbb{R}^n$ and a real symmetric $n \times n$ matrix such that

$$w(x) = w(y) + \langle p, x-y\rangle + \frac{1}{2}\langle X(x-y), x-y\rangle + \mathrm{o}(|x-y|^2). \tag{1.1}$$

In this event, we write $Dw(y) = p$ and $D^2w(y) = X$.

Exercise 1. Show that (1.1) can hold for at most one pair p, X. If (1.1) holds and y is a local maximum point for w, then $p = 0$ and $X \le 0$ (in the usual ordering of symmetric matrices: $X \le 0$ iff $\langle X\zeta, \zeta\rangle \le 0$ for $\zeta \in \mathbb{R}^n$).

Balls are denoted as follows:

$$B_r(x) := \{y \in \mathbb{R}^n : |y-x| < r\}, \ \overline{B}_r(x) := \{y \in \mathbb{R}^n : |y-x| \le r\},$$

If $w, z \subset \mathbb{R}^n$, then

$$[w, z] := \{w + t(z-w) : 0 \le t \le 1\}$$

is the line segment from w to z. Similarly, $(w, z) := \{w + t(z-w) : 0 < t < 1\}$ and so on.

2. The Lipschitz Extension/Variational Problem

Let $b \in C(\partial U)$. We begin by considering the problem: find u such that

$$\begin{cases} u \in C(\overline{U}), u = b \text{ on } \partial U \text{ and} \\ \mathrm{Lip}(u, \overline{U}) = \min\left\{\mathrm{Lip}(v, \overline{U}) : v \in C(\overline{U}), v = b \text{ on } \partial U\right\} \end{cases} \tag{2.1}$$

The notation "b" for the boundary data above is intended as a mnemonic. It is clear that if $u \in C(\overline{U})$, then $\mathrm{Lip}(u, \partial U) \le \mathrm{Lip}(u, \overline{U}) = \mathrm{Lip}(u, U)$. Thus, if $\mathrm{Lip}(b, \partial U) = \infty$, any continuous extension of b into U is a solution of (2.1). Moreover, if $\mathrm{Lip}(b, \partial U) < \infty$ and $u \in C(\overline{U})$ agrees with b on ∂U, then $\mathrm{Lip}(u, U) = \mathrm{Lip}(b, \partial U)$ guarantees that u solves (2.1).

Assuming that $\mathrm{Lip}(b, \partial U) < \infty$, it is easy to see that (2.1) has a maximal and a minimal solution which in fact satisfy $\mathrm{Lip}(u, U) = \mathrm{Lip}(b, \partial U)$. Indeed, if $\mathrm{Lip}(u, U) = \mathrm{Lip}(b, \partial U)$, $z, y \in \partial U$, $x \in U$ and $L = \mathrm{Lip}(b, \partial U)$, then u must satisfy

$$\begin{aligned} &b(z) - L|x-z| = u(z) - L|x-z| \le u(x) \text{ and} \\ &u(x) \le u(y) + L|x-y| = b(y) + L|x-y|. \end{aligned}$$

This implies

$$\sup_{z \in \partial U}(b(z) - L|x-z|) \le u(x) \le \inf_{y \in \partial U}(b(y) + L|x-y|). \tag{2.2}$$

Denote the left-hand side of (2.2) by $\mathcal{M}W_*(b)(x)$ and the right hand side of (2.2) by $\mathcal{M}W^*(b)(x)$. The notation is in honor of McShane and Whitney, see Section 8. Since infs and sups over functions with a given Lipschitz constant possess the same Lipschitz constant, $\mathrm{Lip}(\mathcal{M}W_*(b), \mathbb{R}^n), \mathrm{Lip}(\mathcal{M}W^*(b), \mathbb{R}^n) \le L = \mathrm{Lip}(b, \partial U)$. It is also obvious that $\mathcal{M}W^*(b) = \mathcal{M}W_*(b) = b$ on ∂U. Thus $\mathcal{M}W^*(b)$ ($\mathcal{M}W_*(b)$) provides a *maximal* (respectively, *minimal*) solution of (2.1). Since $\mathcal{M}W^*(b), \mathcal{M}W_*(b)$ have the same Lipschitz constant as b, they are regarded as solving the "Lipschitz extension problem," in that they extend b to $\mathbb{R}^n$ while preserving the Lipschitz constant. "Extension" has different overtones than does "variational."

There is no reason for these extremal solutions to coincide, and it is rare that they do. The example below shows this, no matter how nice U might be.

Example 2.1. Let $U = \{x \in \mathbb{R}^2 : |x| < 1\}$ be the unit disc in $\mathbb{R}^2$ and $b \in C(\partial U)$. Arrange that $-1 \le b \le 1$; this can be done with arbitrarily large $L = \mathrm{Lip}(b, \partial U)$. Then, via (2.2), $\mathcal{M}W_*(b)(0) < \mathcal{M}W^*(b)(0)$ provided that there is a $\delta > 0$ for which

$$\begin{aligned} b(z) - L|z| + \delta &= b(z) - L + \delta \\ &< b(y) + L|y| = b(y) + L \text{ for } |y| = |z| = 1. \end{aligned}$$

Since $-1 \le b \le 1$, this is satisfied if $L > 1$, for then

$$\delta - 2L \le -1 - 1 = -2 \le b(y) - b(z)$$

with $\delta = (L - 1)$.

A primary difference between the functional Lip and more standard integral functionals is that Lip is not "local," as explained in the introduction. The next exercise shows this.

Exercise 2. Let $n = 1, U = (-1, 0) \cup (0, 1)$ and $b(-1) = b(0) = 0, b(1) = 1$. Find $\mathcal{M}W^*(b)$ and $\mathcal{M}W_*(b)$. Conclude that the functional Lip is not local. Modify this example to show that even if U is bounded, then it is not necessarily true that $\mathcal{M}W^*(b) \le \max_{\partial U} b$ in U, nor does $b \le \tilde{b}$ necessarily imply that $\mathcal{M}W^*(b) \le \mathcal{M}W^*(\tilde{b})$ in U (for example).

We recall the localized version of the idea that u "minimizes the functional Lip."

Definition 2.2. Let $u \colon U \to \mathbb{R}$. Then is an *absolute minimizer* for the functional Lip on U provided that $u \in C(U)$ and

$$\mathrm{Lip}(u, V) \le \mathrm{Lip}(v, V) \text{ whenever } V \ll U, v \in C(\overline{V}) \text{ and } u = v \text{ on } \partial V. \tag{2.3}$$

We will use various ways to refer to this notion, saying, equivalently, that "u is an absolute minimizer for Lip on U," or "u is absolutely minimizing for Lip on U," or "u is absolutely minimizing Lipschitz" or merely writing

$$u \in \mathrm{AML}(U).$$

The notion is evidently local in the sense that if $u \in \mathrm{AML}(U)$ and $V \subset U$, then $u \in \mathrm{AML}(V)$.

Exercise 3. With the notation of Exercise 2, determine all the continuous functions on $\mathbb{R}$ and all the continuous functions on $[-1,1]$ which agree with b on ∂U and which are absolute minimizers for Lip on $\mathbb{R} \setminus \partial U$ and U, respectively. More generally, show that if $\mathcal{I}$ is an interval and $u \in C(\mathcal{I})$ is an absolute minimizer for Lip on $\mathcal{I}$, then u is linear.

Note that the notion of an absolute minimizer does not involve boundary conditions; it is a property of functions defined on open sets. We recast the problem (2.1) and in terms of this notion. This results in: find u with the properties

$$u \in C(\overline{U}) \cap \mathrm{AML}(U) \text{ such that } u = b \text{ on } \partial U. \tag{2.4}$$

It is not clear that (2.4) has any solutions, whether or not $\mathrm{Lip}(b, \partial U) = \infty$. Nor is it clear that a solution is unique if it exists. Your solution of Exercise 3 shows that solutions are not unique in general. We will see in Section 5 that solutions exist very generally and are unique if U is bounded.

Theorem 2.3. *The following are equivalent conditions on a function $u \in C(U)$.*

(a) $u \in \mathrm{AML}(U)$.
(b) *If $w = u$ or $w = -u$, then for every $a \in \mathbb{R}$, $V \ll U$ and $z \notin V$*

$$w(x) - a|x - z| \le \max_{y \in \partial V}(w(y) - a|y - z|) \text{ for } x \in V. \tag{2.5}$$

(c) *If $w = u$ or $w = -u$ and $\varphi \in C^2(U)$ and $w - \varphi$ has a local maximum at $\hat{x} \in U$, then*

$$\Delta_\infty \varphi(\hat{x}) := \sum_{i,j=1}^{n} \varphi_{x_i}(\hat{x})\varphi_{x_j}(\hat{x})\varphi_{x_i x_j}(\hat{x}) \ge 0. \tag{2.6}$$

Definition 2.4. If $a \in \mathbb{R}$ and $z \in \mathbb{R}^n$, we call the function $C(x) = a|x - z|$ a *cone function*. The *slope of* C is a and its *vertex* is z. The half-line $\{z + t(x - z), t \ge 0\}$ is the *ray* of C through x.

Definition 2.5. A function $w \in C(U)$ with the property (2.5) is said to *enjoy comparison with cones from above in U*. If $-w$ enjoys comparison with cones from above, equivalently

$$w(x) - C(x) \ge \min_{y \in \partial V}(w(y) - C(y)) \text{ for } x \in V$$

for $V \ll U$ and cone functions C whose vertices are not in V, w is said to enjoy comparison with cones from below. If w enjoys comparison with cones from above and from below, then it enjoys comparison with cones. Thus, condition (b) of Theorem 2.3 is that "u enjoys comparion with cones."

Definition 2.6. The differential operator given by

$$\Delta_\infty \varphi := \sum_{i,j=1}^{n} \varphi_{x_i}\varphi_{x_j}\varphi_{x_i x_j} = \left\langle D^2\varphi D\varphi, D\varphi \right\rangle \tag{2.7}$$

on smooth functions φ is called the "∞-Laplacian. Here $D\varphi = (\varphi_{x_1}, \ldots, \varphi_{x_n})$ is the gradient of φ and $D^2\varphi = \left(\varphi_{x_i x_j}\right)$ is the Hessian matrix of second derivatives of φ. A function $w \in C(U)$ such that (2.6) holds for every φ in $C^2(U)$ and local maximum $\hat{x}$ of $w - \varphi$ is said to be *a viscosity subsolution of* $\Delta_\infty w = 0$; equivalently, it is a viscosity solution of $\Delta_\infty w \geq 0$ or w is ∞-*subharmonic.* If $-w$ is a ∞-subharmonic, equivalently, at any local minimum $\hat{x} \in U$ of $w - \varphi$ where $\varphi \in C^2$, one has

$$\Delta_\infty \varphi(\hat{x}) \leq 0,$$

then w is ∞-*superharmonic.* If w is both ∞-subharmonic and ∞-superharmonic, then it is ∞-harmonic and we write $\Delta_\infty w = 0$.

Important Notice: Hereafter the modifier "viscosity" will be often be dropped, as was already done in defining, for example, "∞-subharmonic." The viscosity notions are the right ones here, and are taken as primary. One does not, in general, compute the expression "$\Delta_\infty w$" and evaluate it, as in $\Delta_\infty w(x)$, to determine whether or not u is ∞-harmonic. Instead, one checks the conditions of the definition above, or some equivalent, as in Theorem 2.3. However, the expression $\Delta_\infty w(x)$ does have a pointwise meaning if w is twice differentiable at x, that is,

$$w(z) = w(x) + \langle p, z - x\rangle + \frac{1}{2}\langle X(z-x), z-x\rangle + \mathrm{o}(|z-x|^2) \tag{2.8}$$

for some $p \in \mathbb{R}^n$ and real symmetric $n \times n$ matrix X. Then

$$\Delta_\infty w(x) = \left\langle D^2 w(x) Dw(x), Dw(x)\right\rangle = \langle Xp, p\rangle .$$

Noting, for example, that if (2.8) holds, $\varphi \in C^2$, and $w - \varphi$ has a maximum at x, then $p = D\varphi(x)$ and $X \leq D^2\varphi(x)$ (see Exercise 1), we find that

$$\Delta_\infty w(x) = \langle Xp, p\rangle = \langle X D\varphi(x), D\varphi(x)\rangle \leq \left\langle D^2\varphi(x) D\varphi(x), D\varphi(x)\right\rangle .$$

It follows that if $w \in C^2$, then $\Delta_\infty w \geq 0$ in the pointwise sense implies $\Delta_\infty w \geq 0$ in the viscosity sense. Similarly, if (2.8) holds, then for $\varepsilon > 0$

$$z \mapsto w(z) - \left(w(x) + \langle p, z-x\rangle + \frac{1}{2}\langle (X + \varepsilon I)(z-x), z-x\rangle \right)$$

has a maximum at $z = x$, so if w is a viscosity solution of $\Delta_\infty w \geq 0$, we must have $\langle (X+\varepsilon I)p, p\rangle \geq 0$. Letting $\varepsilon \downarrow 0$, we find $\Delta_\infty w(x) \geq 0$. Thus the viscosity notions are entirely consistent with the pointwise notion at points of twice differentiability.

Remark 2.7. In Exercises 7, 8 below you will show that the function defined on $\mathbb{R}^2$ by $u(x,y) = x^{4/3} - y^{4/3}$ is ∞-harmonic on $\mathbb{R}^2$. As it is not twice differentiable on the coordinate axes, this cannot be checked via pointwise computation of $\Delta_\infty u$. The viscosity notions, which we are taking as primary here, give a precise meaning to the claim that $\Delta_\infty u = 0$.

We break out the proof of Theorem 2.3 in several simple parts.

2.1. **Absolutely Minimizing Lipschitz iff Comparison With Cones.** We begin with a useful triviality. If $g\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$, then

$$\begin{aligned}\text{Lip}(g,[a,b])| = \frac{|g(b)-g(a)|}{|b-a|} &\implies \\ g(a+t(b-a)) &= g(a)+t(g(b)-g(a)) \text{ for } 0 \le t \le 1.\end{aligned} \tag{2.9}$$

This has the following obvious consequence: If $u\colon [w,z] \subset \mathbb{R}^n \to \mathbb{R}$, then

$$\begin{aligned}\frac{|u(z)-u(w)|}{|z-w|} = \text{Lip}(u,[w,z]) &\implies \\ u(w+t(z-w)) &= u(w)+t(u(z)-u(w)) \text{ for } 0 \le t \le 1.\end{aligned} \tag{2.10}$$

Assume now that $u \in C(U)$ enjoys comparison with cones (Definition (2.5)). Note that the comparison with cones from above is equivalent to the condition (2.11) (a) below. Likewise, comparison with cones from below may be restated as (2.11) (b). If $a, c \in \mathbb{R}$ and $z \notin V$, then

$$\begin{aligned}&\text{(a) } u(x) \le c + a|x-z| \text{ for all } x \in V \text{ if it holds for } x \in \partial V,\\ &\text{(b) } c + a|x-z| \le u(x) \text{ for all } x \in V \text{ if it holds for } x \in \partial V.\end{aligned} \tag{2.11}$$

We show that, for any $x \in V$,

$$\text{Lip}(u, \partial(V \setminus \{x\})) = \text{Lip}(u, \partial V \cup \{x\}) = \text{Lip}(u, \partial V). \tag{2.12}$$

To see this we need only check that if $y \in \partial V$, then

$$u(y) - \text{Lip}(u,\partial V)|x-y| \le u(x) \le u(y) + \text{Lip}(u,\partial V)|x-y|. \tag{2.13}$$

As each of the above inequalities holds for $x \in \partial V$ and u enjoys comparison with cones, the inequalities indeed hold if $x \in V$. Let $x, y \in V$. Using (2.12) twice,

$$\text{Lip}(u,\partial V) = \text{Lip}(u, \partial(V \setminus \{x\})) = \text{Lip}(u, \partial(V \setminus \{x,y\})).$$

Since $x, y \in \partial(V \setminus \{x,y\})$, we have $|u(x)-u(y)| \le \text{Lip}(u,\partial V)|x-y|$, and hence $u \in \text{AML}(U)$.

Let $C(x) = a|x-z|$ be a cone function. Notice that $\text{Lip}(C,[w,y]) = |a|$ whenever w, y are distinct points on the same ray of C. Thus $\text{Lip}(C,V) = |a|$ for any nonempty open set V and $\text{Lip}(C,\partial V) = |a|$ if V is bounded, nonempty and does not contain the vertex z of C.

Suppose now that $u \in \text{AML}(U)$. Assume that $V \ll U$, $z \notin V$ and set

$$W = \{x \in V : u(x) - a|x-z| > \max_{w \in \partial V}(u(w) - a|w-z|)\}. \tag{2.14}$$

We want to show that W is empty. If it is not empty, then it is open and

$$u(x) = a|x-z| + \max_{w\in\partial V}(u(w)-a|w-z|) =: C(x) \text{ for } x \in \partial W. \tag{2.15}$$

Therefore $u = C$ on ∂W and $\text{Lip}(u,W) = \text{Lip}(C,\partial W) = |a|$ since u is absolutely minimizing. Now if $x_0 \in W$, the ray of C through x^0, $t \mapsto z + t(x_0 - z)$, $t \ge 0$, contains a segment in

W containing x_0 which meets ∂W at its endpoints. Since $t \mapsto C(z+t(x_0-z)) = at|x_0-z|$ is linear on this segment, with slope $a|x_0-z|$, while $t \mapsto u(z+t(x_0-z))$ also has $|a||x_0-z|$ as a Lipschitz constant and the same values at the endpoints of the segment; therefore it is the same function ((2.10)). Thus

$$u(z+t(x_0-z)) = C(z+t(x_0-z))$$

on the segment, which contains x^0, whence $u(x_0) = C(x_0)$, a contradiction to $x_0 \in W$. Thus W is empty.

Remark 2.8. In showing that comparison with cones implies AML, we only used comparison with cones with nonnegative slopes from above, and comparison with cones with nonpositive slopes from below.

Exercise 4. Let $u \in C(U)$ be absolutely minimizing for $\mathcal{F}_\infty$, that is, whenever $V \ll U$, $v \in C(\overline{V})$ and $u = v$ on ∂V, then $\mathcal{F}_\infty(u,V) \le \mathcal{F}_\infty(v,V)$. Show that u enjoys comparison with cones. Hint: the proof above needs only minor tweaking.

2.2. **Comparison With Cones Implies ∞-Harmonic.** You may prefer other proofs, but the one below is direct; it does not use contradiction. First we use comparison with cones from above, which implies that, using the form (2.11) (a) of this condition,

$$u(x) \le u(y) + \max_{\{w:|w-y|=r\}} \left(\frac{u(w)-u(y)}{r}\right)|x-y| \tag{2.16}$$

for $x \in B_r(y) \ll U$. The inequality (2.16) holds as asserted because it trivially holds for $x \in \partial(B_r(y)\setminus\{y\})$.

Rewrite (2.16) as

$$u(x)-u(y) \le \max_{\{w:|w-y|=r\}} (u(w)-u(x))\frac{|x-y|}{r-|x-y|} \tag{2.17}$$

for $x \in B_r(y) \ll U$. If u is twice differentiable at x, namely, if there is a $p \in \mathbb{R}^n$ and a symmetric $n \times n$ matrix X such that

$$\begin{aligned} u(z) =& u(x) + \langle p, z-x\rangle + \frac{1}{2}\langle X(z-x), z-x\rangle + \mathrm{o}\left(|z-x|^2\right), \\ &\text{so that } p := Du(x),\ X := D^2u(x), \end{aligned} \tag{2.18}$$

we will show that

$$\Delta_\infty u(x) = \left\langle D^2u(x)Du(x), Du(x)\right\rangle = \langle Xp, p\rangle \ge 0. \tag{2.19}$$

That is, comparison with cones from above implies $\Delta_\infty u \ge 0$ at points of twice differentiability.

We are going to plug (2.18) into (2.17) with two choices of z. First, on the left of (2.17), we choose $z = y = x - \lambda p$ where p is from (2.18), and expand $u(x) - u(y)$ according to (2.18). Next, let $w_{r,\lambda}$ be a value of w for which the maximum on the right of (2.17) is attained and expand $u(w_{r,\lambda}) - u(x)$ according to (2.18). This yields, after dividing by $\lambda > 0$,

$$
\begin{aligned}
&|p|^2 + \lambda \frac{1}{2} \langle Xp, p \rangle + \mathrm{o}(\lambda) \\
&\quad \le \left(\langle p, w_{r,\lambda} - x \rangle + \frac{1}{2} \langle X(w_{r,\lambda} - x), w_{r,\lambda} - x \rangle + \mathrm{o}((r+\lambda)^2) \right) \frac{|p|}{r - \lambda |p|}
\end{aligned} \tag{2.20}
$$

Sending $\lambda \downarrow 0$ yields

$$
\begin{aligned}
|p|^2 &\le \left(\left\langle p, \frac{w_r - x}{r} \right\rangle + \frac{1}{2} \left\langle X \left(\frac{w_r - x}{r} \right), w_r - x \right\rangle \right) |p| + |p| \mathrm{o}(r) \\
&\le |p|^2 + \frac{1}{2} \left\langle X \left(\frac{w_r - x}{r} \right), w_r - x \right\rangle |p| + |p| \mathrm{o}(r),
\end{aligned} \tag{2.21}
$$

where w_r is a any limit point of the $w_{r,\lambda}$ as $\lambda \downarrow 0$ and therefore $w_r \in \partial B_r(x)$ - so $(w_r - x)/r$ is a unit vector. Since the second term inside the parentheses on the right of the first inequality above has size r and $(w_r - x)/r$ is a unit vector, it follows from the first inequality that $(w_r - x)/r \to p/|p|$ as $r \downarrow 0$. (We are assuming that $p \neq 0$, as we may.) Then the inequality of the extremes in (2.21), after dividing by r and letting $r \downarrow 0$, yields $0 \le \langle Xp, p \rangle$, as desired.

The above proof contains a bit more information than $0 \le \langle Xp, p \rangle$ if $p = Du(x) = 0$. In this case, choosing y so that $|x - y| = r/2$, we have

$$u(x) - u(y) = \mathrm{O}(r^2)$$

and then (2.17) yields

$$\mathrm{O}(r^2) \le \frac{1}{2} \left\langle X \left(\frac{w_{r,y} - x}{r} \right), w_{r,y} - x \right\rangle + \mathrm{o}(r)$$

where $(w_{r,y} - y)/r$ is a unit vector. Dividing by r, sending $r \downarrow 0$ and using compactness, any limit point of $(w_{r,y} - x)/r$ as $r \downarrow 0$ is a unit vector q for which $0 \le \langle Xq, q \rangle$. In particular, if $Du(x) = 0$, then

$$D^2 u(x) \quad \text{has a nonnegative eigenvalue.} \tag{2.22}$$

This set up is a bit more awkward than is necessary for the results obtained so far. It is set up this way to make the next remark easy. If x is a local maximum point of $u - \varphi$ for some smooth φ, then

$$\varphi(x) - \varphi(y) \le u(x) - u(y) \text{ and } u(w) - u(x) \le \varphi(w) - \varphi(x)$$

for y, w near x. That is, we may replace u by φ in (2.17). By what was just shown, it follows that $\Delta_\infty \varphi(x) \ge 0$. That is, by definition, u is a viscosity solution of $\Delta_\infty u \ge 0$ if it satisfies comparison with cones from above. We record this again: if u enjoys comparison with cones from above, then

$$\varphi \in C^2, u - \varphi \text{ has a local max at } x \implies \Delta_\infty \varphi(x) \ge 0. \tag{2.23}$$

In addition, if $D\varphi(x) = 0$, then

$$D^2\varphi(x) \quad \text{has a nonnegative eigenvalue.} \tag{2.24}$$

Similarly, if u enjoys comparison with cones from below, then

$$\varphi \in C^2, u - \varphi \text{ has a local min at } x \implies \Delta_\infty \varphi(x) \le 0. \tag{2.25}$$

In addition, if $D\varphi(x) = 0$, then

$$D^2\varphi(x) \quad \text{has a nonpositive eigenvalue.} \tag{2.26}$$

These results follow directly from what was already shown because $-u$ enjoys comparison with cones from above. See Section 3.

2.3. **∞-Harmonic Implies Comparison With Cones.** Suppose that $\Delta_\infty u \ge 0$ on the bounded set U. Computing the ∞-Laplacian on a radial function $x \mapsto G(|x|)$ yields

$$\Delta_\infty G(|x|) = G''(|x|)G'(|x|)^2$$

if $x \neq 0$ and from this we find that

$$\Delta_\infty(a|x-z| - \gamma|x-z|^2) = -2\gamma(a - 2\gamma|x-z|)^2 < 0$$

for all $x \in U$, $x \neq z$, if $\gamma > 0$ is small enough. But then if $\Delta_\infty u \ge 0$, $u(x) - (a|x-z| - \gamma|x-z|^2)$ cannot have a local maximum in $V \ll U$ different from z, by the very definition of a viscosity solution of $\Delta_\infty u \ge 0$. Thus if $z \notin V \ll U$ and $x \in V$, we have

$$u(x) - (a|x-z| - \gamma|x-z|^2) \le \max_{w \in \partial V}(u(w) - (a|w-z| - \gamma|w-z|^2)).$$

Now let $\gamma \downarrow 0$. The full assertions now follow from Section 3.

2.4. **Exercises and Examples.** Below we explore with curves. A C^1 unit speed curve $\gamma : \mathcal{I} \to U$ on some open interval $\mathcal{I} = (t^-, t^+)$ will be called "maximal" in U if the following holds: (i) if $t^+ < \infty$, then $\lim_{t \uparrow t^+} \gamma(t) =: \gamma(t^+) \in \partial U$ (the limit exists, since γ is unit speed) and (ii) If $-\infty < t^-$, then $\gamma(t^-) \in \partial U$. We also consider the variant where $\mathcal{I} = [0, t^+)$, with the obvious modification.

Exercise 5. Let $u \in C^2(U)$ and $Du(x^0) \neq 0$. Let $\gamma : \mathcal{I} \to U$, where $\mathcal{I}$ is an open interval in $\mathbb{R}$, be such that $Du(\gamma(t)) \neq 0$. Assume that

$$\dot\gamma(t) = \frac{Du(\gamma(t))}{|Du(\gamma(t))|} \text{ for } t \in \mathcal{I}. \tag{2.27}$$

Show that

$$\frac{d}{dt}u(\gamma(t)) = |Du(\gamma(t))| \text{ and } \frac{d}{dt}|Du(\gamma(t))|^2 = \frac{2}{|Du(\gamma(t))|}\Delta_\infty u(\gamma(t)). \tag{2.28}$$

Conclude that u is ∞-harmonic iff for every $x^0 \in U$ with $Du(x^0) \neq 0$, there is a maximal unit speed curve $\gamma : \mathcal{I} \to U$ with the following properties: $0 \in \mathcal{I}$, $\gamma(0) = x^0$, $|Du(\gamma(t))|$ is constant, and $u(\gamma(t)) = u(x^0) + t|Du(x^0)|$.

Exercise 6. Show that cone functions are ∞-harmonic on the complement of their vertices and that linear functions are ∞-harmonic. Show that $x \mapsto |x|$ is ∞-subharmonic on $\mathbb{R}^n$.

Exercise 7. Show that in $n = 2$ and $u(x, y) = x^{4/3} - y^{4/3}$, then for each $(x_0, y_0) \neq (0, 0)$, there is a unit speed C^1 curve $\gamma\colon (-\infty, \infty) \to \mathbb{R}^2$ such that (2.27) holds, $\gamma(0) = (x_0, y_0)$, and $|Du(\gamma(t))| \geq |Du(x_0, y_0)|$.

Exercise 8. Let $u \in C^1(U)$ and suppose that for each $x^0 \in U$ with $Du(x^0) \neq 0$ there is a maximal unit speed C^1 curve $\gamma : [0, t^+) \to U$ with $\gamma(0) = x^0$ such that $\langle Du(\gamma(t)), \dot\gamma(t)\rangle \geq |Du(x^0)|$. Show that u enjoys comparison with cones from above (and so it is ∞-subharmonic). Formulate a similar condition which guarantees that u enjoys comparison with cones from below in U and conclude that the u of Exercise 7 is ∞-harmonic. Hints: Suppose $V \ll U$ and $C(x) = a|x - z|$ is a cone function with $z \notin V$. Assuming that $u - C \leq c$ on ∂V, the issue is to show that $u - C \leq c$ in V. If not, there exists $x^0 \in V$ such that

$$u(x^0) - C(x^0) = \max_V (u - C) > c. \tag{2.29}$$

Start the curve γ at x^0 and note $Du(x^0) = DC(x^0)$, so $\langle Du(\gamma(t)), \dot\gamma(t)\rangle \geq |a|$. Show that $u(\gamma(t)) - C(\gamma(t))$ is nondecreasing, and conclude that (2.29) cannot hold.

Exercise 9. Show that if $u \in C^1(U)$ and satisfies the eikonal equation $|Du| = 1$, then u is ∞-harmonic. Note that this class of functions is not C^2 in general, an example being the distance to an interval on the complement of the interval. Hint: There are a number of ways to do this. For example, one involves showing that $\max_{\overline{B_r(y)}} u = u(y) + r$ and looking ahead to Lemma 4.1, and another (closely related) involves showing that any curve $\gamma(t)$ satisfying (2.27) is a line on which Du is constant.

Exercise 10. Let $n = 2, u(x) = x_1, v(x) = |x|$.

(a) Construct an example of a bounded set $U \subset \mathbb{R}^2 \setminus \{0\}$ such that $u < v$ on ∂U except at two points, and $u = v$ holds on the line segment joining these two points (there is no "strong comparison theorem" for ∞-harmonic functions).

(b) Show that the function

$$w(x) = \begin{cases} u(x) = x_1 \text{ for } x_1 > 0, x_2 > 0, \\ v(x) = |x| \text{ for } x_1 > 0, x_2 \leq 0, \end{cases}$$

is ∞-harmonic in $\{x_1 > 0\}$ ("unique continuation" does not hold for ∞-harmonic functions). There are a number of ways to do this, including using Exercise 9.

Exercise 11. The point of (a) of this exercise is to give a proof that if $C(x) = a|x - z|$ is a cone function, then $C \in \mathrm{AML}(\mathbb{R}^n \setminus \{z\})$, which generalizes to the case in which $|\cdot|$ is *any* norm on $\mathbb{R}^n$, and, moreover, if b in (2.4) is $C|_{\partial U}$, then the only solution of (2.1) is C, provided that U is bounded. However, so we don't have to talk too much, assume that $|\cdot|$ is the Euclidean norm here.

(a) Show that $C \in \text{AML}(\mathbb{R}^n \setminus \{z\})$ without using calculus. Hint: Review Section 2.1. Similarly, show that linear functions are in $\text{AML}(\mathbb{R}^n)$.
(b) Let u be a solution of (2.4) where U is bounded, $b = C|_{\partial U}$ and $z \notin U$. Show that for $\varepsilon > 0$ the set $V := \{x \in U : u(x) > C(x) + \varepsilon\}$ is empty. Hint: If it is not, then $u - C = \varepsilon$ on ∂V. Continue to conclude that $u = C$ on U.
(c) Show that if $b(x) = \langle p, x\rangle$ on ∂U in (2.4) and U is bounded, then $u(x) = \langle x, p\rangle$ is the unique solution of (2.4) and it is the maximal ∞-subharmonic function with the property $u = b$ on ∂U.

Remark 2.9. The only comparisons used by Savin in [50] are those of Exercise 11.

3. From ∞-Subharmonic to ∞ - Superharmonic

We put these short remarks in a section of their own, so that they stand out. The theory we are discussing splits naturally into two halves. Owing to our biases, we present results about ∞-subharmonic functions directly; this is the first half. Then, if u is ∞-superharmonic, the second half is obtained by applying the result for ∞-subharmonic functions to $-u$. In contrast to, say, the Laplace equation $\Delta u = 0$, where one might prove the mean value property without splitting it into halves, we *do not* have this kind of option. The reason lies in the very notion of a viscosity solution (even if this theory applies very well to the Laplace equation). The u, $-u$ game we can play here is a reflection of the definitions or, if you prefer, the fact that $\Delta_\infty(-u) = -\Delta_\infty u$ if u is smooth.

This is true of the other properties we use. For example, u enjoys comparison with cones from above iff $-u$ enjoys comparison with cones from below, and so on.

In this spirit, it is notable that we have not split the "absolutely minimizing" notions into halves, and we will not. However, this can be done. See Section 4 of [8], where further equivalences are given beyond what is discussed in these notes.

The main message is that we often present results or proofs *only* in the ∞-subharmonic case and assume that then the reader knows the corresponding result or proof in the ∞-superharmonic case, and therefore the ∞-harmonic case.

4. More Calculus of ∞ - Subharmonic Functions

While we have been assuming that our functions are continuous so far, we will show below that upper-semicontinous functions which enjoy comparison with cones from above are necessarily locally Lipschitz continuous. Thus there is no generality lost in working with continuous functions at the outset (we could have used upper-semicontinuous functions earlier).

The basic calculus type results that we derive about ∞-subharmonic functions below are all consequences of the particular case (2.16) of comparison with cones from above. We have already seen an example of this in Section 2.2. This inequality is equivalent to comparison with cones from above or ∞-subharmonicity by what has already been shown; we also review this in the following proposition. We regard (2.16) as the ∞-subharmonic analogue of the mean value property of ordinary subharmonic functions which estimates the value of such a function at the center of a ball by its average over the ball or the bounding sphere.

The first assertion of the next proposition is that the sphere in (2.16) can be replaced by the ball. The other assertions comprise, together with their consequences in Lemma 4.6, the rest of the most basic facts about ∞-subharmonic functions.

Lemma 4.1. *Let $u\colon U \to I\!R$ be upper-semicontinuous.*

(a) *Assume that*

$$u(x) \le u(y) + \max_{\{w:|w-y|\le r\}} \left(\frac{u(w)-u(y)}{r}\right)|x-y|. \tag{4.1}$$

for $y \in U$, $r > 0$, and $x \in \overline{B}_r(y) \subset U$. Then

$$\max_{\{w:|w-y|\le r\}} u(w) = \max_{\{w:|w-y|=r\}} u(w); \tag{4.2}$$

in particular, (2.16) holds.

(b) *If u satisfies the conditions of (a), then u is locally Lipschitz continuous in U.*

(c) *If u satisfies the conditions of (a), then u is ∞-subharmonic and enjoys comparison with cones from above.*

(d) *If u satisfies the conditions of (a), then the quantity*

$$S^+(y,r) := \max_{\{w:|w-y|\le r\}} \left(\frac{u(w)-u(y)}{r}\right) = \max_{\{w:|w-y|=r\}} \left(\frac{u(w)-u(y)}{r}\right) \tag{4.3}$$

is nonnegative and nondecreasing in r, $0 < r < \operatorname{dist}(y, \partial U)$. Moreover,

$$\begin{gathered} \text{if } |w-y| = r \text{ and } S^+(y,r) = \frac{u(w)-u(y)}{r}, \text{ then} \\ S^+(y,r) \le S^+(w,s) \text{ for } 0 < s < \operatorname{dist}(y,\partial U) - r. \end{gathered} \tag{4.4}$$

(e) *u satisfies the conditions of (a) if and only if for every $y \in U$*

$$r \mapsto \max_{\overline{B}_r(y)} u \tag{4.5}$$

is convex on $0 \le r < \operatorname{dist}(y, \partial U)$.

Proof. Assume that $y \in U$, (4.1) holds, $\overline{B}_r(y) \subset U$, $|x-y| < r$ and $u(x) = \max_{\overline{B}_r(y)} u$. Then we may replace $u(w)$ by $u(x)$ in (4.1) to conclude that

$$u(x)(1 - |x-y|/r) \le u(y)(1 - |x-y|/r),$$

which implies that $u(x) \le u(y)$. Since also $u(x) \ge u(y)$, we conclude that $u(x) = u(y)$. Since this is true for all y such that $x \in B_r(y) \ll U$ and $u(x) = \max_{\overline{B}_r(y)} u$, it is true if $\overline{B}_R(x) \subset U$, $u(x) = \max_{\overline{B}_R(x)} u$ and $|y-x| < R/2$. Thus if u has a local maximum point, it is constant in a ball around that point. We record this: if $\overline{B}_R(x) \ll U$,

$$u \text{ satisfies (4.1) and } u(x) = \max_{\overline{B}_R(x)} u, \text{ then } u \text{ is constant on } B_{R/2}(x)\,. \tag{4.6}$$

This guarantees that if u assumes its maximum value at any point of a connected open set, then it is constant in that set, and hence that the maximum of u over any closed ball is attained in the boundary. There is no difference between (4.1) and (2.16) and (a) is proved.

We turn to (b). Assume, to begin, that $u \le 0$. Then, as $u(w) \le 0$ in (4.1), the $u(w)$ on the right can be dropped. Thus we have, written three equivalent ways,

$$\begin{aligned} &\text{(a) } u(x) \le \left(1 - \frac{|x-y|}{r}\right) u(y), \\ &\text{(b) } -u(y) \le -u(x)\left(\frac{r}{r-|x-y|}\right), \\ &\text{(c) } u(x) - u(y) \le -\frac{|x-y|}{r} u(y). \end{aligned} \tag{4.7}$$

Either of (4.7) (a) or (b) is a Harnack inequality; (b) is displayed just because you might prefer the nonnegative function $-u$, which is lower-semicontinuous and enjoys comparison with cones from below, to u. If $u(x) \neq 0$, either estimates the ratio $u(y)/u(x)$ by quantities not depending on u. Taking the limit inferior as $y \to x$ on the right of (4.7)(a), we find that u is lower-semicontinuous as well as upper-semicontinuous, so it is continuous. If also $B_r(x) \ll U$, we may interchange x and y in (4.7) (c) and conclude, from the two relations, that

$$|u(x) - u(y)| \le -\min(u(x), u(y)) \frac{|x-y|}{r - |x-y|}.$$

As u is locally bounded, being continuous, we conclude that it is also locally Lipschitz continuous. If $u \le 0$ does not hold and $x, y \in B_r(z)$, where $\overline{B}_{2r}(z) \subset U$, replace u by $u - \max_{\overline{B}_{2r}(z)} u$. We thus learn that u is Lipschitz continuous in $B_r(z)$ if $\overline{B}_{2r}(z) \subset U$.

We turn to (c). The assumptions of (a) imply that if $|x-y| = s \le r$, then

$$\frac{u(x) - u(y)}{s} \le \max_{\{w : |w-y| = r\}} \left(\frac{u(w) - u(y)}{r}\right).$$

The monotonicity of $S^+(r, y)$ in r follows upon maximizing the left-hand side with respect to x, $|x-y| = s$. The quantity $S^+(y, r)$ is nonnegative by what was already shown - u attains its maximum over a ball on the boundary.

To prove (d), let the assumptions of (4.4) hold: $r < \operatorname{dist}(y, \partial U)$ and

$$|w - y| = r, \ u(w) = \max_{\overline{B}_r(y)} u. \tag{4.8}$$

Let $0 < s < \operatorname{dist}(y, \partial U)$ and for $0 \le t \le 1$ put $y_t := y + t(w-y)$. By our assumptions,

$$u(y_t) - u(y) \le \left(\frac{u(w) - u(y)}{r}\right) |y_t - y| = t(u(w) - u(y));$$

equivalently,

$$\frac{u(w)-u(y)}{|w-y|} \le \frac{u(w)-u(y_t)}{|w-y_t|}$$

which implies, using the choice of w and monotonicity of S^+, that

$$S^+(y,r) = \frac{u(w)-u(y)}{|w-y|} \le \frac{u(w)-u(y_t)}{|w-y_t|} \le S^+(y_t,s)$$

for $s \ge |w-y_t| = (1-t)|w-y|$. Letting $t \uparrow 1$ and using the continuity of $S^+(x,s)$ in x, this yields

$$S^+(y,r) \le S^+(w,s).$$

We turn to (e). If

$$M(r) := \max_{|w-y|\le r} u(w) \text{ is convex on } 0 \le r < \operatorname{dist}(y,\partial U) \tag{4.9}$$

we note that then $M(0) = u(y)$ and so

$$\frac{M(s)-M(0)}{s} \le \frac{M(r)-M(0)}{r} \text{ for } 0 < s \le r \tag{4.10}$$

says that

$$u(x) \le u(y) + \max_{\{w:|w-y|\le r\}} \left(\frac{u(w)-u(y)}{r}\right) s$$

for $|x-y| \le s$, and (4.1) holds.

To prove the converse, recall that if (4.1) holds, then so does (2.16). By the the proof of Section 2.2, u is ∞-subharmonic. By Section 2.3, u enjoys comparison with cones from above. Therefore we have the following variant of (2.16):

$$u(x) \le \left(\max_{|w-y|\le r} u(w)\right)\frac{|x-y|-s}{r-s} + \left(\max_{|w-y|\le s} u(w)\right)\frac{r-|x-y|}{r-s} \tag{4.11}$$

for $0 \le s \le |x-y| \le r$. The inequality is obvious for $|x-y| = s, r$, so it holds as asserted. Taking the maximum of $u(x)$ over $\overline{B}_\tau(y)$, where $s \le \tau \le r$ yields

$$\max_{|w-y|\le \tau} u(w) \le \left(\max_{|w-y|\le r} u(w)\right)\frac{\tau-s}{r-s} + \left(\max_{|w-y|\le s} u(w)\right)\frac{r-\tau}{r-s}, \tag{4.12}$$

which says that

$$r \mapsto \max_{|w-y|\le r} u(w) \text{ is convex.} \tag{4.13}$$

□

We now know that (2.16) guarantees that u is locally Lipschitz continuous. To use this fact and the estimate implicit in (2.16) efficiently, we introduce the local Lipschitz constant $L(v,x)$ of a function v at a point x.

Definition 4.2. Let $v\colon U \to \mathbb{R}$ and $x \in U$. Then

$$L(v,x) := \lim_{r\downarrow 0} \operatorname{Lip}(v, B_r(x)) = \inf_{0<r<\operatorname{dist}(x,\partial U)} \operatorname{Lip}(v, B_r(x)). \tag{4.14}$$

Of course, $L(u,x) = \infty$ is quite possible.

Lemma 4.3. *Let $v\colon U \to \mathbb{R}$.*

(a) *$L(v,x)$ is upper-semicontinuous in $x \in U$.*
(b) *If v is differentiable at $y \in U$, then then $L(v,y) \geq |Dv(y)|$.*
(c) *If $y \in U$ and $L(v,y) = 0$, then v is differentiable at y and $Dv(y) = 0$.*
(d) *Let the line segment $[y,z] \subset U$. Then*

$$|v(y) - v(z)| \leq \left(\max_{w\in[y,z]} L(v,w) \right) |y - z|.$$

In consequence, $\operatorname{Lip}(v, B_r(y)) \leq \max_{x\in \overline{B}_r(y)} L(v,x)$.

(e) *$Dv \in L^\infty(U)$ holds in the sense of distributions if and only if $L(v,x)$ is bounded on U and then*

$$\sup_{x\in U} L(v,x) = \||Dv|\|_{L^\infty(U)} \ \textit{and}\ L(v,x) = \lim_{r\downarrow 0} \||Dv|\|_{L^\infty(B_r(x))}. \tag{4.15}$$

Proof. To establish (a), note that if $x^j \to x$, then $B_{r-|x-x^j|}(x^j) \subset B_r(x)$ and therefore, for large j,

$$L(u,x^j) \leq \operatorname{Lip}(u, B_{r-|x-x^j|}(x^j)) \leq \operatorname{Lip}(u, B_r(x))$$

Let $j \to \infty$ and then $r \downarrow 0$ to conclude that $\limsup_{j\to\infty} L(u,x^j) \leq L(u,x)$.

We prove (b) assuming, as we may, that $p := Du(y) \neq 0$, choose $x = y + \lambda p$ with small $\lambda > 0$ to find

$$\operatorname{Lip}(v, B_{\lambda|p|}(y)) \geq \frac{|v(y+\lambda p) - v(y)|}{|\lambda p|} \geq \frac{\lambda |p|^2}{\lambda|p|} + \mathrm{o}(1) \text{ as } \lambda \downarrow 0.$$

so, recalling $Dv(y) = p$, $L(v,y) \geq |p|$.

The example $n = 1$, $v(x) = x^2 \sin(1/x)$ and $y = 0$ shows that, in general, $L(v,y) > |Dv(y)|$.

Now assume that $L(v,y) = 0$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |v(x) - v(y)| \leq \operatorname{Lip}(v, B_\delta(y))|x - y| \leq \varepsilon|x - y|,$$

which proves (c).

We turn to (d). For $t \in [0,1)$, $0 \leq \delta < 1 - t$ and $r > \delta|z - y|$, and

$$g(t) = |v(y + t(z-y)) - v(y)|,$$

we have

$$\begin{aligned} \frac{g(t+\delta) - g(t)}{\delta} &\leq \frac{|v(y + (t+\delta)(z-y)) - v(y + t(z-y))|}{\delta} \\ &\leq \operatorname{Lip}(v, B_r(y + t(z-y)))|z - y| \text{ for } r \geq \delta|z-y|. \end{aligned}$$

Letting $\delta \downarrow 0$ yields

$$\limsup_{\delta\downarrow 0} \frac{g(t+\delta)-g(t)}{\delta} \leq \operatorname{Lip}(v, B_r(y+t(z-y)))|z-y|$$

and then, letting $r \downarrow 0$,

$$\limsup_{\delta\downarrow 0} \frac{g(t+\delta)-g(t)}{\delta} \leq L(v, y+t(z-y))|z-y| \leq \max_{w\in[y,z]} L(v,w)|z-y|.$$

This implies, using elementary facts about Dini derivatives (for example), that

$$g(1)-g(0) = |v(z)-v(y)| \leq \max_{w\in[y,z]} L(v,w)|z-y|,$$

as claimed.

The claim (e) follows easily from (d) if we take as known that $Dv \in L^\infty(U)$ and $\||Dv|\|_{L^\infty(U)} = L$ if and only if $[y,z] \subset U$ implies

$$|v(z)-v(y)| \leq L|z-y|$$

and L is the least such constant. □

Definition 4.4. If u is ∞-subharmonic in U and $x \in U$, then

$$S^+(x) := \lim_{r\downarrow 0} S^+(x,r) = \lim_{r\downarrow 0} \max_{\{w:|w-x|=r\}} \left(\frac{u(w)-u(x)}{r}\right). \tag{4.16}$$

Similarly, if u is ∞-superharmonic, then

$$S^-(x) := \lim_{r\downarrow 0} S^-(x,r) := \lim_{r\downarrow 0} \min_{\{w:|w-x|=r\}} \left(\frac{u(w)-u(x)}{r}\right). \tag{4.17}$$

Remark 4.5. Note the notational peculiarities. S^+ is used both with two arguments and with a single argument; no confusion should arise from this once noted, but the meaning changes with the usage. L has two arguments, and, in contrast to S^+, one of them is the function itself. (We did not want to write $S^+(u,x,r)$ or some variant.) We will want to display the function argument later, and the identities (4.18), (4.19) below will allow us to use $L(u,x)$ in place of $S^+(x), S^-(x)$ when we need to indicate the function involved. Also, the notation has been changed from that in [8], wherein our $L(u,x)$ was written $T_u(x)$.

Lemma 4.6. *Let $u \in C(U)$ be ∞-subharmonic. Then for $x \in U$ and $B_r(x) \ll U$,*

$$L(u,x) = S^+(x). \tag{4.18}$$

In consequence, if u is ∞-harmonic in U and $x \in U$, then

$$S^-(x) = -S^+(x). \tag{4.19}$$

Moreover, $u \in C(U)$ satisfies

$$L(u,x) \le \max_{\{w:|w-x|\le r\}} \left(\frac{u(w)-u(x)}{r} \right) \tag{4.20}$$

for $\overline{B}_r(x) \subset U$ if and only if u is ∞-subharmonic in U. Finally, at any point y of differentiability of u,

$$|Du(y)| = S^+(y) = L(u,y). \tag{4.21}$$

Proof. First, via Lemma 4.6, we have that $\max_{\overline{B}_r(x)} L(u,x) = \mathrm{Lip}(u, \overline{B}_r(x))$. Thus

$$S^+(x,r) = \max_{\{w:|w-x|\le r\}} \left(\frac{u(w)-u(x)}{r} \right) \le \max_{\{w:|w-x|\le r\}} L(u,w).$$

The upper-semicontinuity of $L(u,x)$ in x (Lemma 4.3 (a)) then implies, upon taking the limit $r \downarrow 0$, that $S^+(x) \le L(u,x)$. To obtain the other inequality, let $[w,z] \subset U$. By the local Lipschitz continuity of u, $g(t) := u(w+t(z-w))$ is Lipschitz continuous in $t \in [0,1]$. Fix $t \in (0,1)$ and observe that the definition of S^+ implies, for small $h > 0$,

$$\begin{aligned} \frac{g(t+h)-g(t)}{h} &= \frac{u(w+(t+h)(z-w)) - u(w+t(z-w))}{h|z-w|}|z-w|. \\ &\le S^+(w+t(z-w), h|z-w|)|z-w|. \end{aligned}$$

The inequality follows from the definition of $S^+(w+t(z-w),r)$. Sending $h \downarrow 0$ we find

$$\begin{aligned} \limsup_{h\downarrow 0} \frac{g(t+h)-g(t)}{h} &\le S^+(w+t(z-w))|z-w| \\ &\le \left(\sup_{y\in[w,z]} S^+(y) \right) |z-w|. \end{aligned}$$

Thus

$$u(z)-u(w) = g(1)-g(0) \le \left(\sup_{y\in[w,z]} S^+(y) \right) |z-w|.$$

Interchanging z and w we arrive at

$$|u(z)-u(w)| \le \left(\sup_{y\in[w,z]} S^+(y) \right) |z-w|. \tag{4.22}$$

Using also the monotonicity of $S^+(x,r)$ in r (Lemma 4.1), if $\delta > 0$ is small, we have, via (4.22),

$$\mathrm{Lip}(u, B_r(x^0)) \le \sup_{x\in\overline{B}_r(x^0)} S^+(x) \le \sup_{x\in B_r(x^0)} S^+(x,\delta).$$

Now send $r \downarrow 0$ and then $\delta \downarrow 0$ to find, via the continuity of $S^+(x,\delta)$ for $\delta > 0$,

$$L(u,x^0) \le S^+(x^0,\delta) \downarrow S^+(x^0) \quad \text{as } \delta \downarrow 0;$$

this completes the proof of (4.18).

To establish (4.19), note that $L(u,x) = L(-u,x)$ and that S^- for u is just $-S^+$ for $-u$ and invoke (4.18).

If u is differentiable at y and $|w-y| = r$, then

$$u(w) = u(y) + \langle Du(y), w-y\rangle + \mathrm{o}(r) \implies \max_{\{w:|w-y|=r\}} \left(\frac{u(w)-u(y)}{r}\right) = |Du(y)| + \frac{\mathrm{o}(r)}{r}.$$

Now use (4.18) to conclude that (4.21) holds.

We next show that (4.20) implies (4.1), and thus that u is ∞-subharmonic. Suppose that $t \to \gamma(t)$ is a C^1 curve in U. Using (4.20) with $\gamma(t)$ in place of x, one easily checks, using the local Lipschitz continuity of $t \mapsto u(\gamma(t))$, that, almost everywhere,

$$\begin{aligned} \left|\frac{d}{dt}u(\gamma(t))\right| &\le L(u,\gamma(t))|\dot\gamma(t)| \\ &\le \max_{w\in \overline{B}_r(\gamma(t))}\left(\frac{u(w)-u(\gamma(t))}{r}\right)|\dot\gamma(t)| \text{ for } \ r < \operatorname{dist}(\gamma(t),\partial U). \end{aligned} \tag{4.23}$$

It is convenient to rewrite (4.23) as

$$\pm\frac{d}{dt}u(\gamma(t)) + \frac{|\dot\gamma(t)|}{r}u(\gamma(t)) \le \left(\max_{w\in \overline{B}_r(\gamma(t))} u(w)\right)\frac{|\dot\gamma(t)|}{r} \text{ for } \ r < \operatorname{dist}(\gamma(t),\partial U). \tag{4.24}$$

If $x \in B_r(y) \ll U$ and $\gamma(t) = y + t(x-y)$, then $\operatorname{dist}(\gamma(t),\partial U) > r - t|x-y|$. Moreover, $B_r(y) \supset B_{r-t|x-y|}(\gamma(t))$. We use this information in (4.24) to deduce that

$$\pm\frac{d}{dt}u(\gamma(t)) + \frac{|x-y|}{r-t|x-y|}u(\gamma(t)) \le \left(\max_{|w-y|\le r} u(w)\right)\frac{|x-y|}{r-t|x-y|}. \tag{4.25}$$

This simple differential inequality taken with the "+" sign and integrated over $0 \le t \le 1$ yields (4.1). □

Note that the generality of the "±" above is superfluous; it corresponds to reversing the direction of γ.

Exercise 12. Perform the integration of (4.25) referred to just above.

Exercise 13. Show that the map $C(U) \ni u \mapsto L(u,x)$ ($x \in U$ is fixed) is lower-semicontinuous but it is not continuous. Show, however, that the restriction of this mapping to the set of ∞-subharmonic functions u is continuous and $(u,x) \mapsto L(u,x)$ is upper-semicontinuous.

Exercise 14. If u is ∞-subharmonic in U and $u \le 0$, use (4.20) to conclude that

$$|Du(x)| \le -\frac{u(x)}{\operatorname{dist}(x,\partial U)} \text{ if } u \text{ is differentiable at } x \in U.$$

Exercise 15. Show that (4.20) always holds with equality for cone functions $C(x) = a|x - z|$ with nonnegative slopes, so C is ∞-subharmonic on $\mathbb{R}^n$ and (4.20) is sharp. Observe that (4.20) fails to hold for cones with negative slopes.

Exercise 16. Let u be ∞-subharmonic in U and $u \le 0$. It then follows from (4.24) that

$$\frac{d}{dt}u(\gamma(t)) + \frac{|\dot\gamma(t)|}{\text{dist}\,(\gamma(t), \partial U)}u(\gamma(t)) \le 0. \tag{4.26}$$

(i) If $\gamma\colon [0,1] \to U$, conclude that

$$u(\gamma(1)) \le \exp\left(-\int_0^1 \frac{|\dot\gamma(t)|}{\text{dist}\,(\gamma(t), \partial U)}\, dt\right) u(\gamma(0)). \tag{4.27}$$

(ii) Use (i) to show that if $x, y \in B_r(z) \subset B_R(z) \subset U$, then

$$e^{\frac{|x-y|}{R-r}} u(x) \le u(y). \tag{4.28}$$

This is another Harnack inequality.

(iii) Let $\mathcal{H} = \{(x_1, \dots, x_n) : x_n > 0\}$ be a half-space, $\Delta_\infty u \ge 0$ and $u \le 0$ in $\mathcal{H}$. Using (4.26), show that $x_n \mapsto x_n u(x_1, \dots, x_n)$ is nonincreasing and $x_n \mapsto u(x_1, \dots, x_n)/x_n$ is nondecreasing on $\mathcal{H}$. (Here one only needs to check the sign of a derivative.)

(iv) Let $u \le 0$ be ∞-subharmonic in $B_R(0)$. Use (4.26) to show that if $x \in \mathbb{R}^n$, then

$$t \mapsto \frac{u(tx)}{R - t|x|} \text{ is nonincreasing on } 0 \le t|x| < R.$$

In particular, if $x \in B_R(0)$, then $u(x)/(R - |x|) \le u(0)/R$.

Remark 4.7. See Sections 8 and 9 concerning citations for Exercises 14 and 16. Clearly, in Exercise 16 we are showing how to organize various things in the literature in an new efficient way.

5. Existence and Uniqueness

We have nothing new to offer regarding existence, and as far as we know Theorem 3.1 of [8] remains the state of the art in this regard. The result allows U to be unbounded and a boundary function b which grows at most linearly.

Theorem 5.1. *Let U be an open subset of $\mathbb{R}^n$, $0 \in \partial U$, and $b \in C(\partial U)$. Let $A^\pm, B^\pm \in \mathbb{R}$, $A^+ \ge A^-$, and*

$$A^-|x| + B^- \le b(x) \le A^+|x| + B^+ \text{ for } x \in \partial U. \tag{5.1}$$

Then there exists $u \in C(\overline{U})$ which is ∞-harmonic in U, satisfies $u = b$ on ∂U and which further satisfies

$$A^-|x| + B^- \le u(x) \le A^+|x| + B^+ \quad \text{for } x \in \overline{U}. \tag{5.2}$$

The proof consists of a rather straight-foward application of the Perron method, using the equivalence between ∞-harmonic and comparison with cones. In the statement, the assumption $0 \in \partial U$ is made to simplify notation and interacts with the assumption (5.1). A translation handles the general case, but this requires naming a point of ∂U. The Perron method runs by defining $\underline{h}, \overline{h} : \mathbb{R}^n \to \mathbb{R}$ by

$$\underline{h}(x) = \sup\{\underline{C}(x) : \underline{C}(x) = a|x-z| + c,\ a < A^-, c \in \mathbb{R}, z \in \partial U,\ \underline{C} \le b \text{ on } \partial U\},$$
$$\overline{h}(x) = \inf\{\overline{C}(x) : \overline{C}(x) = a|x-z| + c,\ a > A^+, c \in \mathbb{R}, z \in \partial U,\ \overline{C} \ge b \text{ on } \partial U\}.$$

and then showing that u below has the desired properties.

$$u(x) := \sup\left\{v(x) : \underline{h} \le v \le \overline{h} \text{ and } v \text{ enjoys comparison with cones from above}\right\}. \tag{5.3}$$

See [8] for details, or give the proof as an exercise.

Uniqueness has always been a sore spot for the theory, in the sense that it took a long time for Jensen [36] to give the first, quite tricky, proof and then another proof, still tricky, but more in line with standard viscosity solution theory, was given by Barles and Busca [9]. A self-contained presentation of the proof of [9], which does not require familiarity with viscosity solution theory, is given in [8].

Here we give the skeleton of a third proof, from [28], in which unbounded domains are treated for the first time. This time, however, we fully reduce the result to standard arguments from viscosity solution theory, and we do not render our discussion self-contained in this regard.

The result, proved first by Jensen and with a second proof by Barles and Busca, is

Theorem 5.2. *Let U be bounded, $u, v \in C(\overline{U})$, $\Delta_\infty u \ge 0$ and $\Delta_\infty v \le 0$ in U. Then if $u \le v$ on ∂U, we have $u \le v$ in U.*

We just sketch the new proof, as it applies to the case of a bounded domain. The main point is an approximation result.

Proposition 5.3. *Let U be a (possibly unbounded) subset of $\mathbb{R}^n$ and $u \in C(\overline{U})$ be ∞-subharmonic. Let $\varepsilon > 0$,*

$$U_\varepsilon := \{x \in U : L(u,x) \ge \varepsilon\} \text{ and } V_\varepsilon := \{x \in U : L(u,x) < \varepsilon\}. \tag{5.4}$$

Then there is function u_ε with the properties (a)-(f) below:

(a) *$u_\varepsilon \in C(\overline{U})$ and $u_\varepsilon = u$ on ∂U.*
(b) *$u_\varepsilon = u$ on U_ε.*
(c) *$\varepsilon - |Du_\varepsilon| = 0$ (in the viscosity sense) on V_ε.*
(d) *$L(u_\varepsilon, x) \ge \varepsilon$ for $x \in U$.*
(e) *u_ε is ∞-subharmonic in U.*
(f) *$u_\varepsilon \le u$ and $\lim_{\varepsilon\downarrow 0} u_\varepsilon = u$.*

Remark 5.4. In (c), "viscosity sense" means that at a maximum (minimum) $\hat{x}$ of $u_\varepsilon - \varphi$ we have $\varepsilon - |D\varphi(\hat{x})| \le 0$ ($\varepsilon - |D\varphi(\hat{x})| \ge 0$). For example, $-\varepsilon|x|$ has this property on $\mathbb{R}^n$, while $\varepsilon|x|$ does not. This is consistent with the conventions of [31].

The assertions (a)-(c) are really a prescription of how to construct u_ε, which is then given by a standard formula. See [28] for the details. The trickiest point is (e), which we discuss below. Full details are available in [28]; see also Barron and Jensen [13], Proposition 5.1, where related observations were first made, although with a different set of details; one does not find Proposition 5.3 in [13], but there is an embedded proof of (e), different from the one we will sketch. In fact, these authors show that if u is ∞-harmonic, then the u_ε above solves $\max(-\Delta_\infty u_\varepsilon, \varepsilon - |Du_\varepsilon|) = 0$; this variational inequality played a fundamental role in [36] (see Section 8). However, they do not consider subsolutions nor note the approximation property (f) in this case.

We first explain how Proposition 5.3 reduces Theorem 5.2 to a routine citation of results in [31]. After that, we present a proof of the most subtle point, (e), of Proposition 5.3; that discussion will also verify (d).

If we can show that

$$u_\varepsilon(x) - v(x) \le \max_{\partial U}(u_\varepsilon - v) \text{ for } x \in U \tag{5.5}$$

then (f) allows us to replace u_ε by u and the right and Theorem 5.2 follows in the limit $\varepsilon \downarrow 0$. Thus, using (d), we may assume, without further ado, that

$$L(u, x) \ge \varepsilon \text{ for } x \in U. \tag{5.6}$$

Some version of the following, which is the step that allows us to take advantage of (5.6), is used in all comparison proofs for Δ_∞ below, beginning with the proof of Jensen [36]. This sort of nonlinear change of variables is a standard tool in comparison theory of viscosity solutions. Let $\lambda > 0$ and $\sup_U u < \frac{1}{2\lambda}$ and then define w by

$$u(x) = G(w(x)) := w(x) - \frac{\lambda}{2} w(x)^2 \text{ and } w(x) < \frac{1}{\lambda}. \tag{5.7}$$

That is, $w(x) = (1 - \sqrt{1 - 2\lambda u})/\lambda$. Clearly $w \to u$ uniformly as $\lambda \downarrow 0$. It therefore suffices to show that

$$w(x) - v(x) \le \max_{\partial U}(w - v) \text{ for } x \in U \tag{5.8}$$

To this end, one formally computes that

$$\begin{aligned} 0 \le \Delta_\infty u &= \left\langle D^2 u Du, Du \right\rangle \\ &= \left\langle \left(G''(w)(Dw \otimes Dw) + G'(w) D^2 w\right) G'(w) Dw, G'(w) Dw \right\rangle \\ &= G''(w) G'(w)^2 |Dw|^4 + G'(w)^3 \left\langle D^2 w Dw, Dw \right\rangle, \end{aligned} \tag{5.9}$$

which implies, using (5.6), that

$$\begin{aligned} \Delta_\infty w = \left\langle D^2 w Dw, Dw \right\rangle &\ge -\frac{G''(w)}{G'(w)} |Dw|^4 \\ &= -\frac{G''(w)}{G'(w)^5} |Du|^4 \ge \frac{\lambda}{(1 - \lambda w)^3} \varepsilon^5 \ge \kappa > 0 \end{aligned} \tag{5.10}$$

for some κ. It is straightforward to check that these computations are valid in the viscosity sense (see [28]). It follows immediately from Theorem 3.2 in [31], used in the standard way, that $\Delta_\infty w \geq \kappa > 0$ and $\Delta_\infty v \leq 0$ together imply (5.8).

We now discuss key elements of the proof of Proposition 5.3. Note that V_ε is open by the upper-semicontinuity of $L(u, \cdot)$ while $U_\varepsilon = U \setminus V_\varepsilon$ is closed relative to U.

It is known that any function solving $\varepsilon - |Du_\varepsilon| = 0$ in the viscosity sense in an open set is ∞-subharmonic in that set. In fact, from the general theory of viscosity solutions of first order equations, one has the formula

$$u_\varepsilon(x) = \max_{\{y:|y-z|=r\}} (u_\varepsilon(y) - \varepsilon|x-y|) \text{ for } x \in \overline{B}_r(z), \tag{5.11}$$

whenever $\overline{B}_r(z) \subset V_\varepsilon$, which makes the claim evident (in our context, we could use that the class of functions enjoying comparison with cones from above is closed under taking supremums in place of the analogous statement about viscosity subsolutions). The function u_ε is produced by a similar standard formula, with ∂V_ε in place of $\partial B_r(z)$ and the distance interior to V_ε in place of $|x-y|$ (which is the distance from x to y interior to $\overline{B}_r(z)$).

In the end, given the tools in hand, the main point one needs to establish in order to prove (e) is this: assuming that we have constructed u_ε, then

$$x \in (\partial V_\varepsilon) \cap U \implies L(u_\varepsilon, x) \leq L(u, x). \tag{5.12}$$

Since $x \in U_\varepsilon \supset (\partial V_\varepsilon) \cap U$, we have $L(u, x) \geq \varepsilon$, which renders the ε's appearing in estimates appearing below harmless as regards establishing (5.12).

To establish (5.12), it suffices to show that if $[y, z] \subset B_r(x) \ll U$, then

$$|u_\varepsilon(z) - u_\varepsilon(y)| \leq \text{Lip}(u, B_r(x))|y - z|, \tag{5.13}$$

for then $\text{Lip}(u_\varepsilon, B_r(x)) \leq \text{Lip}(u, B_r(x))$.

First note that if *any* open interval $(y, z) \subset V_\varepsilon$, then $|Du_\varepsilon| \leq \varepsilon$ a.e. on the open set V_ε implies

$$|u_\varepsilon(z) - u_\varepsilon(y)| \leq \varepsilon|y - z|. \tag{5.14}$$

In particular, if $[y, z] \subset V_\varepsilon$, then (5.13) holds. If $[y, z] \cap U_\varepsilon$ is not empty, we proceed as follows. The set U_ε is closed in U, so $[y, z] \cap U_\varepsilon$ is closed and there exists a least $t_0 \in [0, 1]$ and a greatest $t_1 \in [0, 1]$ such that

$$x^0 := y + t_0(w - y) \in U_\varepsilon, \; x^1 := y + t_1(w - y) \in U_\varepsilon. \tag{5.15}$$

Then

$$\begin{aligned} |u_\varepsilon(z) - u_\varepsilon(y)| &= |u_\varepsilon(z) - u_\varepsilon(x^1) + u_\varepsilon(x^1) - u_\varepsilon(x^0) + u_\varepsilon(x^0) - u_\varepsilon(y)| \\ &= |u_\varepsilon(z) - u_\varepsilon(x^1) + u(x^1) - u(x^0) + u_\varepsilon(x^0) - u_\varepsilon(y)| \\ &\leq |u_\varepsilon(z) - u_\varepsilon(x^1)| + |u(x^1) - u(x^0)| + |u_\varepsilon(x^0) - u_\varepsilon(y)| \\ &\leq |u_\varepsilon(z) - u_\varepsilon(x^1)| + \text{Lip}(u, B_r(x))|x^1 - x^0| + |u_\varepsilon(x^0) - u_\varepsilon(y)| \end{aligned} \tag{5.16}$$

Now each interval $[y, x^0)$ and $(x^1, z]$ is either empty (as is the case, for example, for $[y, x^0)$ if $y = x^0$) or lies entirely in V_ε. Using (5.14) twice, with $z = x^0$ and then with $y = x^1$,

$$|u_\varepsilon(z) - u_\varepsilon(x^1)| \le \varepsilon|z - x^1|, \ |u_\varepsilon(x^0) - u_\varepsilon(y)| \le \varepsilon|x^0 - y|.$$

Combining this with (5.16), (5.13) follows from

$$|z - y| = |z - x^1| + |x^1 - x^0| + |x^0 - y|.$$

We continue. Let u_ε be as in the proposition and $x \in U$. According to Lemma 4.6, if we can show that

$$L(u_\varepsilon, x) \le \max_{\{w:|w-z|\le r\}} \left(\frac{u_\varepsilon(w) - u(x)}{r}\right) \tag{5.17}$$

for $r < \operatorname{dist}(x, \partial U)$, we are done. If $x \in U_\varepsilon$, then (5.17) holds. Indeed, then, using (5.12) if necessary (i.e, if $x \in \partial V_\varepsilon$), we have, as explained below,

$$L(u_\varepsilon, x) \le L(u, x) \le \frac{u(w_r) - u(x)}{r} = \frac{u_\varepsilon(w_r) - u_\varepsilon(x)}{r} \tag{5.18}$$

for some w_r, $|w_r - x| = r$. The final equality holds owing to (4.4), which implies that $L(u, w_r) \ge L(u, x) \ge \varepsilon$, so $w_r \notin V_\varepsilon$, and so $u(w_r) = u_\varepsilon(w_r)$. Hence (5.17) holds if $x \in U_\varepsilon$.

To handle the case $x \in V_\varepsilon$, one first observes that (5.11) implies that if $B_r(z) \subset V_\varepsilon$ then

$$\max_{y \in \overline{B}_r(z)} u_\varepsilon(y) = u_\varepsilon(z) + \varepsilon r, \tag{5.19}$$

and $L(u_\varepsilon, z) = \varepsilon$. Recalling $L(u, x) \ge \varepsilon$ for $x \in U_\varepsilon$, between (5.18) and (5.19), we learn that for every $z \in U$ there is an $r_z > 0$ such that

$$\max_{w \in \overline{B}_r(z)} u_\varepsilon(w) \ge u_\varepsilon(z) + \varepsilon r \text{ for } 0 \le r \le r_z.$$

This implies, with a little continuation argument, that

$$\varepsilon \le \max_{\{w:|w-z|\le r\}} \left(\frac{u_\varepsilon(w) - u_\varepsilon(x)}{r}\right) \text{ for } r < \operatorname{dist}(x, \partial U),$$

and we are done.

Exercise 17. Provide the little continuation argument.

6. The Gradient Flow and the Variational Problem for $\||Du|\|_{L^\infty}$

Let us note right away that if u is absolutely minimizing for $\mathcal{F}_\infty$, then it is absolutely minimizing for Lip.

Proposition 6.1. *Let $u \in C(U)$ be absolutely minimizing for $\mathcal{F}_\infty$, that is, whenever $V \ll U$, $v \in C(\overline{V})$ and $u = v$ on ∂V, then $\mathcal{F}_\infty(u, V) \le \mathcal{F}_\infty(v, V)$. Then u is absolutely minimizing for* Lip *(and hence ∞-harmonic).*

Proof. One proof was already indicated in Exercise 4. Here is another. Let $v \in C(\overline{V})$ and $u = v$ on ∂V. Assume $\mathrm{Lip}(v, \partial V) < \infty$ and replace v by $\mathcal{M}W_*(v|_{\partial V})$ so that we may assume that $\mathrm{Lip}(v, V) = \mathrm{Lip}(v, \partial V)$. Then, by assumption, $\mathcal{F}_\infty(u, V) \le \mathcal{F}_\infty(v, V)$, which is at most $\mathrm{Lip}(v, V)$. Now use Exercise 18 below. □

Exercise 18. Show that if $u \in C(\overline{V})$ and $\mathcal{F}_\infty(u, V) \le \mathrm{Lip}(u, \partial V)$, then $\mathrm{Lip}(u, V) = \mathrm{Lip}(u, \partial V)$.

The main result of this section is the tool we will use to prove the converse to the proposition above.

Proposition 6.2. *Let u be ∞-subharmonic in U, $x \in U$. Then there is a $T > 0$ and Lipschitz continuous curve $\gamma \colon [0, T) \to U$ with the following properties:*

(6.1)
$$
\begin{aligned}
&\text{(i) } \gamma(0) = x\\
&\text{(ii) } |\dot\gamma(t)| \le 1 \textit{ a.e. on } [0, T).\\
&\text{(iii) } L(u, \gamma(t)) \ge L(u, x) \textit{ on } [0, T).\\
&\text{(iv) } u(\gamma(t)) \ge u(x) + tL(u, x) \textit{ on } [0, T).\\
&\text{(v) } t \mapsto u(\gamma(t)) \textit{ is convex on } [0, T).\\
&\text{(vi) } \textit{Either } T = \infty \textit{ or } \lim_{t \uparrow T} \gamma(t) \in \partial U.
\end{aligned}
$$

The motivation for this result is contained in Exercises 5-8. We are able, in the general case, to obtain curves with similar properties to those discussed in the exercises.

Before proving this result, let us give an application.

Theorem 6.3. *Let $u \in \mathrm{AML}(U)$. Let $V \ll U$, $v \in C(\overline{V})$ and $u = v$ on ∂V. Then $\sup_V L(u, x) \le \sup_V L(v, x)$. In other words, if u is ∞-harmonic, then it is absolutely minimizing for $\mathcal{F}_\infty$.*

Proof. Under the assumptions of the theorem, if the conclusion does not hold, then there exists $z \in V$ and $\delta > 0$ such that $L(u, z) \ge L(v, x) + \delta$ for $x \in V$. Let $\gamma : [0, T) \to V$ be the curve provided by Proposition 6.2 which starts at z (regarding V as the U of the proposition). Since u is bounded on $\overline{V}$, it follows from (iv), (vi) of the proposition that $T < \infty$ and $\lim_{t \uparrow T} \gamma(t) =: \gamma(T) \in \partial V$. By (iii) of the proposition, $L(u, \gamma(t)) \ge L(u, z) > \sup_V L(v, x)$ and so, using (ii), almost everywhere,

$$\frac{d}{dt} v(\gamma(t)) \le L(v, \gamma(t)) < L(u, z).$$

Integrating over $[0, T]$ and using (ii), (iv) of the proposition,

$$v(\gamma(T)) - v(z) < TL(u, z) \le u(\gamma(T)) - u(z).$$

Now $-u$ is also ∞-subharmonic and it is related to $-v$ as u was to v, so there is another curve $\tilde\gamma(t) \colon [0, \tilde T) \to V$ with $\tilde\gamma(\tilde T) \in \partial V$ such that

$$-v(\tilde\gamma(\tilde T)) + v(z) < \tilde T L(u, z) \le -u(\tilde\gamma(\tilde T)) + u(z).$$

Adding the two inequalities yields

$$v(\gamma(T)) - v(\tilde\gamma(\tilde T)) < u(\gamma(T)) - u(\tilde\gamma(\tilde T)),$$

which contradicts $u = v$ on ∂U. □

Proof of Proposition 6.2

The main idea is to build a discrete version of the desired γ by using the "increasing slope estimate," Lemma 4.1 (d). We may assume that $L(u,x) > 0$, for otherwise we may take $\gamma(t) \equiv x$. Fix δ, $0 < \delta < \operatorname{dist}(x^0, \partial U)$. Form a sequence $\left\{x_\delta^j\right\}_{j=0}^J \subset U$ is according to $x_\delta^0 := x$ and

$$|x_\delta^{j+1} - x_\delta^j| = \delta,\ u(x_\delta^{j+1}) = \max_{\overline{B}_\delta(x_\delta^j)} u \text{ for } j = 1, \dots, J-1. \tag{6.2}$$

We allow J to be finite or infinite. The value of J is determined by checking, after the successful determination of some x^j, if $\overline{B}_\delta(x_\delta^j) \subset U$ or not. If so, then x_δ^{j+1} is determined and $j+1 \le J$. If not, then $J = j$, and we stop. Clearly, J is at least the greatest integer less than $\operatorname{dist}(x^0, \partial U)/\delta$. According to the increasing slope estimate (4.4), we then have

$$S^+(x_\delta^{j+1}) \ge \frac{u(x_\delta^{j+1}) - u(x_\delta^j)}{\delta} = S^+(x_\delta^j, \delta) \ge S^+(x_\delta^j); \tag{6.3}$$

Thus $S^+(x_\delta^j) \ge S^+(x_\delta^0) = S^+(x)$ for $j = 0, 1, \dots J-1$ and (recall $x_\delta^0 = x$)

$$u(x_\delta^{j+1}) - u(x_\delta^j) \ge \delta S^+(x_\delta^0) \implies u(x_\delta^j) - u(x) \ge j\delta S^+(x). \tag{6.4}$$

Form the piecewise linear curve defined by $\gamma_\delta(0) = x$ and

$$\gamma_\delta(t) = x_\delta^j + (t - j\delta)\left(\frac{x_\delta^{j+1} - x_\delta^j}{\delta}\right) \text{ on } j\delta \le t \le (j+1)\delta, j = 0, \dots, J-1.$$

For $j = 0, 1, \dots, J$ we have, by the construction,

$$\begin{aligned} &\text{(i) } \gamma_\delta(0) = x \\ &\text{(ii) } |\dot\gamma_\delta(t)| = 1 \text{ a.e. on } [0, J\delta). \\ &\text{(iii) } L(u, \gamma_\delta(j\delta)) \ge L(u,x). \\ &\text{(iv) } u(\gamma_\delta(j\delta)) \ge u(x) + j\delta L(u,x). \end{aligned} \tag{6.5}$$

By construction, $\delta J \ge \operatorname{dist}(x, \partial U) - \delta$. By compactness, there is then a sequence $\delta_k \downarrow 0$ and a $\gamma\colon [0, \operatorname{dist}(x, \partial U)) \to U$ such that $\gamma_{\delta_k}(t) \to \gamma(t)$ uniformly on compact subsets of $[0, \operatorname{dist}(x, \partial U))$. Clearly $\operatorname{dist}(\gamma(t), \partial U) \ge \operatorname{dist}(x, \partial U) - t$. Moreover, if $0 \le t < \operatorname{dist}(u, \partial U)$, there exist j_k such that $j_k\delta_k \to t$. Passing to the limit in the relations (6.5) with $\delta = \delta_k$ and $j = j_k$, using the upper-semicontinuity of $L(u, \cdot)$, yields all of the relations of (6.1) except (v), (vi). To see that (v) holds, note that the piecewise linear function $g_k(t)$ whose value at $j\delta_k$ is $u(\gamma_{\delta_k}(j\delta_k))$ is convex by (6.3). Moreover, by the continuity of u and the uniform convergence of γ_{δ_k} to γ, g_k converges to $u(\gamma(t))$, which is therefore convex.

The property (vi) can now be obtained by the standard continuation argument of ordinary differential equations. There is a curve γ with the properties of (6.1) defined on a maximal interval of existence of the form $[0,T)$. Assume now that $T<\infty$ and $\lim_{t\uparrow T}\gamma(t)=:\gamma(T)$ and $\gamma(T)\notin\partial U$. The proof concludes by arguing that then γ was not maximal. Indeed, clearly

$$\lim_{t\uparrow T}\frac{u(\gamma(T))-u(\gamma(t))}{T-t}\le L(u,\gamma(T)), \tag{6.6}$$

so if we use the construction above, starting at $\gamma(T)$, to extend γ to a curve $\tilde\gamma:[0,T+\operatorname{dist}(\gamma(T),\partial U))\to U$, we obtain a strict extension of γ with all the right properties, producing a contradiction. The property (6.6), together with $u(\tilde\gamma(t))\ge(t-T)L(u,\gamma(T))+u(\gamma(T))$ for $t\ge T$ and the convexity of $u(\tilde\gamma(t))$ on $T\le t<T+\operatorname{dist}(\gamma(T),\partial U)$ is what guarantees that $u(\tilde\gamma(t))$ is convex; it all glues together right, just as in ode. □

Remark 6.4. All of the conclusions which can be obtained using the curves γ of Proposition 6.2 can also be obtained using the discrete ingredients from which they are built, extended "maximally." See [29], [8]. However, it is considerably more elegant to use curves instead of the discrete versions. Barron and Jensen first constructed analogous curves in [13]; however, the technical surroundings in [13] are a bit discouraging from the point of view of extracting this information. In addition, they do not discuss the maximality and there is an small oversight in their proof of (v) (we learned to include (v) in the list of properties from [13]).

Exercise 19. Let $u\in \mathrm{AML}(U)\cap C(\overline{U})$ and U be bounded. Show that $\mathrm{Lip}(u,U)=\mathrm{Lip}(u,\partial U)$. Hint: Exercise 18.

Exercise 20. Let $u,v\in C(\overline{U})$ and U be bounded. If u is ∞-harmonic in U and $u=v$ on ∂U, show that $\mathcal{F}_\infty(u,U)\le\mathcal{F}_\infty(v,U)$. That is, u solves the minimum problem for $\mathcal{F}_\infty$ with $b=u|_{\partial U}$.

7. Linear On All Scales

7.1. **Blow Ups and Blow Downs are Tight on a Line.** The information in this section remains the main evidence we have regarding the primary open problem in the subject: are ∞-harmonic functions C^1? Savin has proved that they are in the case $n=2$, using the information herein, in the form of Exercise 23, which is a version of Proposition 7.1 (a). His paper also requires Proposition 7.1 (b), which is original here, in one argument in which (a) is erroneously cited in support of the argument. This minor point is set right in the paper [52] by Wang and Yu.

Let $\Delta_\infty u\le 0$ in U, $x^0\in U$ and $r>0$. Then

$$v_r(x):=\frac{u(rx+x^0)-u(x^0)}{r} \tag{7.1}$$

satisfies

$$\mathrm{Lip}(v_r,B_R(0))=\mathrm{Lip}(u,B_{rR}(x^0)) \tag{7.2}$$

as is seen by a simple calculation. We are interested in what sort of functions are subsequential limits as $r \downarrow 0$ and $r \uparrow \infty$ (in which case we need $U = \mathbb{R}^n$).

Proposition 7.1.

(a) *Let u be ∞-harmonic on $B_1(x^0)$. Then the set of functions $\{v_r : 0 < r < 1/(2R)\}$ is precompact in $C(B_R(0))$ for each $R > 0$ and if $r_j \downarrow 0$ and $v_{r_j} \to v$ locally uniformly on $\mathbb{R}^n$, then $v(x) = \langle p, x\rangle$ for some p satisfying $|p| = S^+(x^0)$.*

(b) *Let u be ∞-harmonic on $\mathbb{R}^n$ and $\mathrm{Lip}(u, \mathbb{R}^n) < \infty$. Then the set of functions $\{v_r : r > 0\}$ is precompact in $C(\mathbb{R}^n)$ and if $R_j \uparrow \infty$ and $v_{R_j} \to v$ locally uniformly on $\mathbb{R}^n$, then $v(x) = \langle p, x\rangle$ for some p satisfying $|p| = \mathrm{Lip}(u, \mathbb{R}^n)$.*

Proof. We prove (b), which is new, and relegate (a), which is known, to Exercise 21 below. Let the assumptions of (b) hold. First we notice that

$$\mathrm{Lip}(u, \mathbb{R}^n) = \lim_{R\to\infty} \frac{\max_{\overline{B}_R(0)} u}{R}. \tag{7.3}$$

Indeed, for every $x \in \mathbb{R}^n$,

(7.4)

$$L(u, x) = S^+(x) \le \max_{w\in\overline{B}_R(x)} \frac{u(w) - u(x)}{R} \le \max_{w\in\overline{B}_{R+|x|}(0)} \frac{u(w)}{R} - \frac{u(x)}{R} \to \lim_{R\to\infty} \frac{\max_{\overline{B}_R(0)} u}{R}$$

as $R \to \infty$. This proves (7.3) with "$\le$" in place of the equal sign. Letting x_R be a maximum point of u relative to $\partial B_R(0)$, we also have

$$\frac{\max_{\overline{B}_R(0)} u}{R} = \frac{u(x_R) - u(0)}{R} + \frac{u(0)}{R} \le \mathrm{Lip}(u, \mathbb{R}^n) + \frac{u(0)}{R} \to \mathrm{Lip}(u, \mathbb{R}^n),$$

which provides the other side of (7.3).

Now let $r = R_j$ where $R_j \uparrow \infty$. We set $x^0 = 0$ and now denote $v_j := v_{R_j}$. Clearly $\mathrm{Lip}(v_j, \mathbb{R}^n) \le \mathrm{Lip}(u, \mathbb{R}^n)$. Passing to a subsequence for which $v_j \to v$ locally uniformly, we claim that v is linear. Clearly $v(0) = 0$. As u is ∞-subharmonic, for $R > 0$ there is a unit vector x^j_R such that

$$v_j(Rx^j_R) = \max_{B_R(0)} v_j = \frac{\max_{\overline{B}_{R_jR}(0)} u - u(0)}{R_j}.$$

Passing to a subsequence along which $x^j_R \to x^+_R \in \partial B_1(0)$ and taking the limit $j \to \infty$ above, we have, via (7.3),

$$v(Rx^+_R) = R\mathrm{Lip}(u, \mathbb{R}^n) \ge R\mathrm{Lip}(v, \mathbb{R}^n).$$

As u is also ∞-superharmonic, there is also a unit vector x^-_R, such that

$$v(Rx^-_R) = -R\mathrm{Lip}(u, \mathbb{R}^n) \le -R\mathrm{Lip}(v, \mathbb{R}^n).$$

It follows that

$$2R\mathrm{Lip}(v, \mathbb{R}^n) \le 2R\mathrm{Lip}(u, \mathbb{R}^n) = v(x^+_R) - v(x^-_R) \le R\mathrm{Lip}(v, \mathbb{R}^n)|x^+_R - x^-_R|.$$

If $\mathrm{Lip}(u, \mathbb{R}^n) = 0$, then $v \equiv 0$ and we are done. If $\mathrm{Lip}(u, \mathbb{R}^n) > 0$, then we deduce that $|x_R^+ - x_R^-| = 2$, which implies that $x_R^+ = -x_R^-$, as well as $\mathrm{Lip}(v, \mathbb{R}^n) = \mathrm{Lip}(u, \mathbb{R}^n)$. In particular, the points x_R^+, x_R^- are unique. By (2.10), v is linear on each segment $[Rx_R^-, Rx_R^+]$, and this implies, together with the uniqueness, that $x_R^-, x_R^+ = \omega$ is a unit vector independent of R. Altogether we have

$$v(t\omega) = t\mathrm{Lip}(v, \mathbb{R}^n) = t\mathrm{Lip}(u, \mathbb{R}^n).$$

The assertion (b) now follows from Section 7.2 below. □

Exercise 21. Prove, by similar arguments, (a) of the proposition.

Definition 7.2. In the case (a) of Proposition 7.1, if $r_j \downarrow 0$ and $v_{r_j} \to \langle p, x\rangle$ on $\mathbb{R}^n$, we call p a derivate of u at x^0. Similarly, in case (b), one defines "derivates of u" at ∞.

Exercise 22. Show, in case (a) of Proposition 7.1, that if the set of derivates of u at x^0 consists of a single point, then u is differentiable at x^0. Conclude that if $S^+(x^0) > 0$, then u is differentiable at x^0 if and only if

$$1 = |\omega_r| \text{ and } u(x^0 + r\omega_r) = \max_{\overline{B}_r(x^0)} u,$$

then $\lim_{r\downarrow 0} \omega_r$ exists.

Exercise 23. Use (a) of Proposition 7.1 to show that

$$\lim_{r\downarrow 0} \min_{|p|=S^+(x^0)} \max_{|x|\le r} \left| \frac{u(x_0 + x) - u(x_0) - \langle p, x\rangle}{r} \right| = 0.$$

7.2. **Implications of Tight on a Line Segment.** Please forgive us the added generality we are about to explain. The costs aren't large, and we wanted to share this little proof. It is a straightforward generalization and extension of a proof learned from [3] in the Euclidean case. However, the reader may just assume that $|\cdot|^* = |\cdot|$ and $F(x) = x/|x|$ if $x \neq 0$ below to restrict to the Euclidean case.

In this section we use the notation $|x|$, as usual, to denote the norm of $x \in \mathbb{R}^n$. However, the norm *need not be the Euclidean norm.* However, we do assume that $x \to |x|$ is differentiable at any $x \neq 0$. This is equivalent to the dual norm, denoted by $|\cdot|^*$, being strictly convex, where

$$|x|^* := \max\{\langle x, y\rangle : |y| \le 1\} \tag{7.5}$$

and $\langle x, y\rangle$ denotes the usual inner-product of $x, y \in \mathbb{R}^n$. The duality map F (from $(\mathbb{R}^n, |\cdot|)$ to $(\mathbb{R}^n, |\cdot|^*)$) is defined by

$$\begin{cases} |F(x)|^* = 1 \quad \text{and} \quad |x| = \langle x, F(x)\rangle \quad \text{if} \quad x \neq 0 \\ F(0) = \{y : |y|^* \le 1\}. \end{cases} \tag{7.6}$$

When $x \neq 0$ there is only one vector with the properties assigned to $F(x)$ because of the strict convexity of $|\cdot|^*$; moreover, if $x \neq 0$,

$$|x + y| = |x| + \langle y, F(x)\rangle + \mathrm{o}(y) \quad \text{as} \quad y \to 0. \tag{7.7}$$

This follows from the relation

$$\frac{d}{dt}|x+ty| = \langle y, F(x+ty)\rangle \tag{7.8}$$

when $x+ty \neq 0$ and continuity of F away from 0.

Lemma 7.3. *Let $U \subset \mathbb{R}^n$ be open and $u\colon U \to \mathbb{R}$ satisfy*

$$|u(x)-u(y)| \le |x-y| \quad \textit{for} \quad x, y \in U. \tag{7.9}$$

Let $\mathcal{I}$ be an open interval containing 0, $x^0 \in U$, $p \in \mathbb{R}^n$, $|p| = 1$ and $x^0 + tp \in U$ for $t \in \mathcal{I}$. Let

$$u(x^0+tp) = u(x^0) + t \quad \textit{for} \quad t \in \mathcal{I}. \tag{7.10}$$

Then:

(a) *u is differentiable at x^0 and $Du(x^0) = F(p)$.*
(b) *If also $\mathcal{I} = \mathbb{R}$, then u is the restriction to U of the affine function*

$$v(x^0+x) = u(x^0) + \langle x, F(p)\rangle$$

on $\mathbb{R}^n$.

Proof. We may assume that $x^0 = 0$ and $u(x^0) = 0$ without loss of generality. Let P be "projection" along p;

$$Px = \langle x, p^*\rangle\, p \quad \text{where} \quad p^* = F(p).$$

For small $x \in U$ and small $r \in \mathbb{R}$, (7.9) and (7.10) yield

$$\begin{aligned} &\text{(i) } \langle x, p^*\rangle + r - |(x-Px) - rp| = u(Px+rp) - |(x-Px)-rp| \le u(x),\\ &\text{(ii) } u(x) \le u(Px+rp) + |(x-Px)-rp| = \langle x, p^*\rangle + r + |(x-Px)-rp|. \end{aligned} \tag{7.11}$$

These relations are valid whenever $\langle x, p^*\rangle \pm r \in \mathcal{I}$ and $x \in U$. Rearranging (7.11) yields

$$r - |(x-Px)+rp| \le u(x) - \langle x, p^*\rangle \le r + |(x-Px)+rp|. \tag{7.12}$$

We seek to estimate the extreme expressions above. To make the left-most extreme small, we take $r > 0$. Then, by differentiability of $|\cdot|$,

$$\begin{aligned} |(x-Px)-rp| - r &= r\left(|p - \frac{x-Px}{r}| - 1\right)\\ &= r\left(|p| - \frac{1}{r}\langle x-Px, p^*\rangle - 1\right) + r\mathrm{o}\left(\frac{|x-Px|}{r}\right)\\ &= r\mathrm{o}\left(\frac{|x-Px|}{r}\right). \end{aligned} \tag{7.13}$$

Here we used $|p| = 1$ and $\langle x-Px, p^*\rangle = \langle x, p^*\rangle - \langle x, p^*\rangle\langle p, p^*\rangle = 0$. We use (7.13) two ways. First, if r is bounded and bounded away from 0, we have

$$r\mathrm{o}\left(\frac{|x-Px|}{r}\right) = \mathrm{o}(|x-Px|) = \mathrm{o}(|x|) \quad \text{as} \quad x \to 0.$$

To make the right-most extreme of (7.12) small, we take $r < 0$ and proceed similarly. Returning to (7.12), this verifies the differentiability of u at 0 and $Du(0) = p$. On the other hand, if we may send $|r| \to \infty$, the extremes of (7.13) tend to 0. Returning to (7.12), this verifies $u(x) = \langle x, p^* \rangle$ in U. □

Exercise 24. Let $|\cdot|$ be the Euclidean norm. Let K be a closed subset of $\mathbb{R}^n$. Suppose $z \in \mathbb{R}^n \setminus K$ and

$$y \in K \text{ and } |w - y| \leq |w - z| \text{ for } z \in K.$$

Show that $x \mapsto \operatorname{dist}(x, K)$ is differentiable at each point of $[y, w)$ and compute its derivative. Give an example where differentiability fails at w.

Exercise 25. Let $U = \mathbb{R}^n \setminus \partial U$. Let $b \in C(\partial U)$ and $L := \mathrm{Lip}(b, \partial U) < \infty$. Show that if $x \in U$ and $\mathcal{MW}_*(b)(x) = \mathcal{MW}^*(b)(x)$, then both $\mathcal{MW}_*(b)$ and $\mathcal{MW}_*(b)$ are differentiable at x. Show that $\mathcal{MW}_*(b) = \mathcal{MW}^*(b)$ if and only if they are both in $C^1(U)$ and satisfy the eikonal equation $|Du| = L$ in U. Hint: For the first part, show that x lies in the line segment $[y, z]$ when $y, z \in \partial U$ and $b(y) - L|y - x| = b(z) + L|z - x|$.

8. An Impressionistic History Lesson

The style of this section is quite informal; we seek to convey the flow of things, hopefully with enough clarity, but without distracting precision. It is assumed that the reader has read the introduction, but not the main text of this article. We do include some pointers, often parenthetical, to appropriate parts of the main text.

8.1. **The Beginning and Gunnar Aronsson.** It all began with Gunnar Aronsson's 1967 paper [3]. The functional Lip is primary in this paper, but two others are mentioned, including $\mathcal{F}_\infty$. Aronsson observed that $\mathrm{Lip} = \mathcal{F}_\infty$ if U is convex, while this is not generally the case if U is not convex.

The problem of minimizing $\mathcal{F} = \mathrm{Lip}$ subject to a Dirichlet conditions was known to have a largest and a smallest solution, given by explicit formulas ($\mathcal{MW}_*$, $\mathcal{MW}^*$ of (2.2)), via the works of McShane and Whitney [44], [53]. Aronsson derived, among other things, interesting information about the set on which these two functions coincide (Exercise 25)) and the derivatives of any solution on this "contact set". In particular, he established that minimizers for Lip are unique iff there is a function $u \in C^1(U) \cap C(\overline{U})$ which satisfies

$$|Du| \equiv \mathrm{Lip}(b, \partial U) \text{ in } U,\ u = b \text{ on } \partial U,$$

which is then the one and only solution. This is a very special circumstance. Moreover, in general, the McShane-Whitney extensions have a variety of unpleasant properties (Exercise 2). The following question naturally arose: is it possible to find a canonical Lipschitz constant extension of b into U that would enjoy comparison and stability properties? Furthermore, could this special extension be unique once the boundary data is fixed? The point of view was that the problem was an "extension" problem - the problem of extending the boundary data b into U without increasing the Lipschitz constant, hopefully in a manner which had these other good properties. Aronsson's - eventually successful -

proposal in this regard was to introduce the class of absolutely minimizing functions for Lip, which generalized notions already appearing in works of his in one dimension ([1], [2]). Aronsson further gave the outlines of an existence proof not so different from the one sketched in Section 5, but using the McShane-Whitney extensions rather than cones. This required the boundary data to be Lipschitz continuous (in contrast with Theorem 5.1). He could not, however, prove the uniqueness or stability (Theorem 5.2).

Thinking in terms of $\mathcal{F}_\infty = \text{Lip}$ on convex sets, Aronsson was led to the now famous pde:

$$\Delta_\infty u = \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j} = 0.$$

He discovered this by heuristic reasoning: first, by standard reasoning, if $1 < p$, then

$$u \text{ minimizes } \mathcal{F}_p(u,U) := \||Du|\|_{L^p(U)} \text{ among functions}$$
$$\text{satisfying } u = b \text{ on } \partial U \text{ iff } u = b \text{ on } \partial U \text{ and} \frac{1}{p-2}|Du|^2 \Delta u + \Delta_\infty u = 0,$$

provided that $\mathcal{F}_p(u,U) < \infty$.

Letting $p \to \infty$ yields $\Delta_\infty u = 0$. Moreover, $\mathcal{F}_p(u,U) \to \mathcal{F}_\infty(u,U)$ if $|Du| \in L^\infty(U)$.

Aronsson further observed that for $u \in C^2$, $\Delta_\infty u = 0$ amounts to the constancy of $|Du|$ on the lines of the gradient flow (Section 2.4). He went on to prove that if $u \in C^2(U)$, then u is absolutely minimizing for Lip in U iff $\Delta_\infty u = 0$ in U (Section 2.4).

With the technology of the times, this is about all anyone could have proved. The gaps between $\Delta_\infty u = 0$ being the "Euler equation" if $u \in C^2$ and his existence proof, which produced a function only known to be Lipschitz continuous, could not be closed at that time. In particular, Aronsson already knew that classical solutions of the eikonal equation $|Du| = \text{constant}$, which might not be C^2, are absolutely minimizing. However, he offered no satisfactory way to interpret them as solutions of $\Delta_\infty u = 0$. Moreover, the question of uniqueness of the function whose existence Aronsson proved would be unsettled for 26 years!

Aronsson himself made the gap more evident in the paper [4] in which he produced examples of U, b for which he could show that the problem had no C^2 solution. This work also contained a penetrating analysis of classical solutions of the pde. However, all of these results are false in the generality of *viscosity solutions* of the equation (see below), which appear as the perfecting instrument of the theory.

The best known explicit irregular absolutely minimizing function - outside of the relatively regular solutions of eikonal equations - was exhibited again by Aronsson, who showed in the 1984 paper [6] that $u(x,y) = x^{4/3} - y^{4/3}$ is absolutely minimizing in R^2 for Lip and $\mathcal{F}_\infty$ (Exercises 7, 8).

Most of the interesting results for classical solutions in 2 dimensions proved by Aronsson are falsified by this example. These include: $|Du|$ is constant on trajectories of the gradient flow, global absolutely minimizing functions are linear, and Du cannot vanish unless u is locally constant. A rich supply of other solutions was provided as well.

8.2. **Enter Viscosity Solutions and R. Jensen.** Let $u \in \mathrm{USC}(U)$ (the upper semi-continuous functions on U). Then u is a viscosity subsolution of $\Delta_\infty u = 0$ (equivalently, a viscosity solution of $\Delta_\infty u \geq 0$) in U if: whenever $\varphi \in C^2(U)$ and $u - \varphi$ has a local maximum at $\hat{x} \in U$, then $\Delta_\infty \varphi(\hat{x}) \geq 0$.

Let $u \in \mathrm{LSC}(U)$. Then u is a viscosity supersolution of $\Delta_\infty u = 0$ (equivalently, a viscosity solution of $\Delta_\infty u \leq 0$) in U if whenever $\varphi \in C^2(U)$ and $u - \varphi$ has a local minimum at $\hat{x} \in U$, then $\Delta_\infty \varphi(\hat{x}) \leq 0$.

The impetus for this definition arises from the standard maximum principle argument at a point $\hat{x}$ where $u - \varphi$ has a local maximum. Let $D^2 u = \left(u_{x_i, x_j}\right)$ be the Hessian matrix of the second order partial derivatives of u. Then

$$\begin{aligned} \Delta_\infty u = \left\langle D^2 u Du, Du \right\rangle \geq 0,\ Du(\hat{x}) = D\varphi(\hat{x}) \text{ and } D^2 u(\hat{x}) \leq D^2 \varphi(\hat{x}) \\ \implies \left\langle D^2 \varphi(\hat{x}) D\varphi(\hat{x}), D\varphi(\hat{x}) \right\rangle \geq 0. \end{aligned}$$

This puts the derivatives on the "test function" φ via the maximum principle, a device used by L. C. Evans in 1980 in [32].

The theory of viscosity solutions of much more general equations, born in first order case the 1980's, and to which Jensen made major contributions, contained strong results of the form:

Comparison Theorem: *Let $u, -v \in USC(\overline{U})$,*

$$G(x, u, Du, D^2 u) \geq 0 \text{ and } G(x, v, Dv, D^2 v) \leq 0 \text{ in } U$$

in the viscosity sense, and $u \leq v$ on ∂U. Then $u \leq v$ in U.

The developers of these results in the second order case were Jensen, Ishii, Lions, Souganidis, $\cdots$. They are summarized in [31], which contains a detailed history. In using [31], note that we are thinking here of G being an increasing function of the Hessian, $D^2 u$, not a decreasing function. Hence solutions of $G \geq 0$ are subsolutions.

Of course, there are structure conditions needed on G in order that the Comparison Theorem hold in addition to standard maximum principle enabling assumptions, and these typically imply that the solutions of $G \geq 0$ perturb to a solution of $G > 0$ or that some change of variables such as $u = g(w)$ produces a new problem with this property. This had not been not accomplished in any simple way for Δ_∞, except at points where "$Du \neq 0$" until the method explained in Section 5, which gives an approximation procedure for subsolutions which produces subsolutions with nonvanishing gradients, was developed. This is coupled with a change of variable such as $u = w - \frac{\lambda}{2} w^2$, under which $\Delta_\infty u \geq 0$ implies $\Delta_\infty w \geq \frac{\lambda}{1 - \lambda w} |Dw|^4$.

Moreover, Hitoshi Ishii introduced the Perron method in the theory of viscosity solutions, which provides "existence via uniqueness," that is, roughly speaking,

Existence Theorem: *When the Comparison Theorem is true and there exist u, v satisfying its assumptions, continuous on $\overline{U}$ and satisfying*

$$u = b = v \text{ on } \partial U$$

then there exists a (unique) viscosity solution $u \in C(\overline{U})$ of
$$0 \le G(x,u,Du,D^2u) \le 0 \text{ in } U \text{ and } u = b \text{ on } \partial U$$

See Section 4 of [31].

We wrote $0 \le G \le 0$ above to highlight that a viscosity solution of $G = 0$ is exactly a function which is a viscosity solution of both $G \le 0$ AND $G \ge 0$; there is no other notion of $G = 0$ in the viscosity sense.

NOTE: hereafter all references to "solutions", "subsolutions", $\Delta_\infty u \le 0$, etc, are meant in the viscosity sense!

As mentioned, in 1993 R. Jensen proved, in [36], that

(J1) Absolute minimizers u for $\mathcal{F}_\infty$ are characterized by $\Delta_\infty u = 0$.

(J2) The comparison theorem holds for $G = \Delta_\infty$.

Jensen's proof of the comparison theorem was remarkable. In order to deal with the difficulties associated with points where $Du = 0$, he used approximations via the "obstacle problems"

$$\max\left\{\varepsilon - |Du^+|, \Delta_\infty u^+\right\} = 0, \ \min\left\{|Du^-| - \varepsilon, \Delta_\infty u^-\right\} = 0!$$

These he "solved" by approximation with modifications of $\mathcal{F}_p$ and then letting $p \to \infty$, although they are amenable to the general theory discussed above. It is easy enough to show that

$$u^- \le u \le u^+$$

when $\Delta_\infty u = 0$ and $u = u^+ = u^- = b$ on ∂U. Comparison then followed from an estimate, involving Sobolev inequalities, which established $u^+ - u^- \le \kappa(\varepsilon)$ where $\kappa(0+) = 0$.

The first assertion of (J1) was proved directly, via a modification of Aronsson's original proof, while the "conversely" was a consequence of existence and uniqueness. The relation between "absolutely minimizing" relative to $\mathcal{F}_\infty$ and relative to Lip had become even more murky. Jensen referred as well to another "$\mathcal{F}$", as had Aronsson earlier, namely the Lipschitz constant relative to the "interior distance" between points:

$$\operatorname{dist}_U(x,y) = \text{infimum of the lengths of paths in } U \text{ joining } x \text{ and } y$$

and the ordinary Lipschitz constant did not play a role in his work.

Thus, after *26* years, the existence of absolutely minimizing functions assuming given boundary values was known (Aronsson and Jensen), and, at last, the uniqueness (Jensen).

Jensen's work generated considerable interest in the theory. Among other contributions was a lovely new uniqueness proof by G. Barles and J. Busca in [9]. Roughly speaking, this proof couples some penetrating observations to the standard machinery of viscosity solutions to reach the same conclusions as Jensen, but without obstacle problems or integral estimates. This was the state of the art until the method of Section 5 was deployed.

After existence and uniqueness, one wants to know about regularity.

8.3. **Regularity.** Aronsson's example, $u(x, y) = x^{4/3} - y^{4/3}$, sets limits on what might be true. The first derivatives of u are Hölder continuous with exponent 1/3; second derivatives do not exist on the lines $x = 0$ and $y = 0$. It is still not known whether or not every ∞-harmonic function has this regularity.

8.3.1. *Modulus of Continuity.* The first issue is the question of a modulus of continuity for absolutely minimizing functions. In Aronsson's framework, he dealt only with locally Lipschitz continuous functions. In our Lemma 4.1 we establish the local Lipschitz continuity of USC ∞-harmonic functions via comparison with cones. Jensen also gave similar arguments to establish related results. See also [37], [19]. As Lipschitz continuous functions are differentiable almost everywhere, so are ∞-harmonic functions.

8.3.2. *Harnack and Liouville.* Aronsson's original "derivation" of $\Delta_\infty u = 0$ as the "Euler equation" corresponding to the property of being absolutely minimizing and Jensen's existence and uniqueness proofs closely linked letting $p \to \infty$ in the problem

$$\Delta_p u_p := |Du|^{p-4}\left(|Du|^2\Delta u + (p-2)\Delta_\infty u\right) = 0 \quad \text{in } U$$

and $u_p = b$ on ∂U with our problem

$$\Delta_\infty u = 0 \text{ in } U \text{ and } u = b \text{ on } \partial U.$$

With this connection in mind and using estimates learned from the theory of Δ_p, P. Lindqvist and J. Manfredi [42] proved that if $u \geq 0$ is a *variational* solution of $\Delta_\infty u \leq 0$ (i.e, a limit of solutions of $\Delta_p u_p \leq 0$), then one has the *Harnack Inequality*

$$u(x) \leq e^{\frac{|x-y|}{R-r}} u(y) \text{ for } x, y \in B_r(x_0) \subset B_R(x_0) \subset U.$$

They derived this from the elegant *Gradient Estimate*

$$|Du(x)| \leq \frac{1}{\operatorname{dist}(x, \partial U)}(u(x) - \inf_U u),$$

valid at points where $Du(x)$ exists. (Our equation (4.20) is more precise and the proof is considerably more elementary.) The appropriate Harnack inequality is closely related to regularity issues for classes of elliptic and parabolic equations, which is one of the reasons to be interested in it. However, so far, it has not played a similar role in the theory of Δ_∞. In particular, note that the gradient estimate implied the Harnack inequality in the reasoning of [42]; estimating the gradient came first.

The same authors extended this result, a generalization of an earlier result of Evans for smooth functions, to all ∞-superharmonic functions, ie, solutions of $\Delta_\infty u \leq 0$, in [43], by showing that ALL ∞-superharmonic functions are variational. This perfected the relationship between $\Delta_p, \mathcal{F}_p$ and $\Delta_\infty, \mathcal{F}_\infty$. By the way, the original observation that if solutions of $\Delta_p u_p = 0$ have a limit $u_p \to u$ as $p \to \infty$, then $\Delta_\infty u = 0$ is due to Bhattacharya, Di Benedetto and Manfredi, [24] (1989). This sort of observation is a routine matter in the viscosity solution theory; there is much else in that paper. One point of concern is the relationship between the notions of viscosity solutions and solutions in the sense of distributions for the p-Laplace equation. See Juutinen, Lindqvist and Manfredi, [39] and Ishii, [35].

Lindqvist and Manfredi also showed that

(LM1) If $u(x) - v(x) \geq \min_{\partial V}(u - v)$ when $\Delta_\infty v = 0$, and $x \in V \subset\subset U$, then $\Delta_\infty u \leq 0$.

(LM2) If $\Delta_\infty u = 0$ in $\mathbb{R}^n$ and u is bounded below, then u is constant.

8.3.3. *Comparison with Cones, Full Born.* Subsequently, Crandall, Evans and Gariepy [29] showed that it suffices to take functions of the form

$$v(x) = a|x - z|, \quad \text{also known as a “cone function”},$$

where $z \notin V$, in (LM1), and introduced the terminology “comparison with cones” (Definition 2.4 and Theorem 2.3).

The assumptions of (LM1) with v a cone function as above is called comparison with cones from below; the corresponding relation for $\Delta_\infty u \geq 0$ is called comparison with cones from above. When u enjoys both of these, it enjoys comparison with cones. All the information contained in $\Delta_\infty u \leq 0, \Delta_\infty u \leq 0, \Delta_\infty u = 0$ is contained in the corresponding comparison with cones property.

Also proved in [29], with a 2-cone argument, was the generalization

$$\Delta_\infty u \leq 0 \text{ and } \ u(x) \geq a + \langle p, x\rangle \ \text{ in } \mathbb{R}^n \implies u(x) = u(0) + \langle p, x\rangle \,.$$

of (LM2). With this generality, it follows that if $\Delta_\infty u \leq 0$, $\varphi \in C^1$ and $\hat{x}$ is a local minimum of $u - \varphi$, then u is differentiable at $\hat{x}$. At last: a result asserting the existence of a derivative at a particular point.

More importantly, it had become clear that approximation by Δ_p is probably not the most efficient path to deriving properties of ∞-sub and super harmonic and ∞-harmonic functions. Of course, comparison with cones had already been used by Jensen to derive Lipschitz continuity, and was used contemporaneously with [29] by Bhattacharya [19], etc., but it was now understood that this approach made use of all the available information.

Recall that in Jensen's organization, he showed that if $\Delta_\infty u = 0$ fails, then u is not absolutely minimizing for $\mathcal{F}_\infty$; equivalently, if u is absolutely minimizing, then $\Delta_\infty u = 0$. Then he used existence/uniqueness in order to establish the converse: if $\Delta_\infty u = 0$, then u is absolutely minimizing. In [29] it was proved directly - without reference to existence or uniqueness - that comparison with cones, and hence $\Delta_\infty u = 0$, implies that u is absolutely minimizing for $\mathcal{F}_\infty$ (Proposition 6.1).

8.3.4. *Blowups are Linear.* The next piece of evidence in the regularity mystery was provided by Crandall and Evans, [30]. Using tools from [29] and some new arguments, all rather simple.

They proved that if $\Delta_\infty u = 0$ near x_0 and

$$r_j \downarrow 0,\ v_j(x) = \frac{u(x_0 + r_j x) - u(x_0)}{r_j},\ v(x) = \lim_{j\to\infty} v_j(x)$$

then $v(x) = \langle p, x\rangle$ for some $p \in \mathbb{R}^n$ (Section 7).

Note that since u is Lipschitz in each ball $B_R(x_0)$ in its domain, each v_j is Lipschitz with the same constant in $B_{R/r_j}(0)$. Thus any sequence $r_j \downarrow 0$ has a subsequence along which the v_j converge locally uniformly in $\mathbb{R}^n$. In consequence, if x_0 is a point for which

$$|z_r - x_0| = r, u(z_r) = \max_{\overline{B}_r(x_0)} u \implies \lim_{r\downarrow 0} \frac{z_r - x_0}{|z_r - x_0|} \text{ exists,}$$

then u is differentiable at x_0. However, no one has been able to show that the maximum points have a limiting direction, except as a consequence of Ovidiu Savin's results in [50], in the case $n = 2$.

8.3.5. *Savin's Theorem.* Savin [50] showed that ∞-harmonic functions are C^1 if $n = 2$. Savin does not work on the "directions" of $z_r - x_0$ mentioned above directly. He does start from a reformulation of the "$v(x) = \langle p, x\rangle$" result above (Exercise (23)). It does not appear that Savin's arguments contain any clear clues about the case $n > 2$, as he uses the topology of $\mathbb{R}^2$ very strongly, and the question of whether ∞-harmonic functions are necessarily C^1 in general remains the most prominent open problem in the area. Moreover, while Savin provides a modulus of continuity for Du when u is ∞-harmonic, this modulus is not explicit. It would be quite interesting to have more information about it. Is it Hōlder 1/3, as for $x^{4/3} - y^{4/3}$? Probably not. Savin shows, in consequence of his other results, that if u is ∞-harmonic on $\mathbb{R}^2$ and globally Lipschitz continuous, then u is affine. The corresponding question in the case $n > 2$ is also open, and it is certainly related to the C^1 question.

Savin's results have been generalized to solutions of the Aronsson equation for absolutely minimizing functions with respect to suitable convex functions $H(p)$ (see below) in [52].

9. Generalizations, Variations, Recent Developments and Games

First of all, there is by now a substantial literature concerning optimization problems with supremum type functionals. Much of this theory was developed by N. Barron and R. Jensen in collaboration with various coauthors. We refer the reader to the review article by Barron [10] for an overview up to the time of its writing, and to Jensen, Barron and Wang [11], [12] for more recent advances. In particular, [12] is concerned with vector-valued functions u in the set up we explain below for scalar functions u. However, in the vector case, existence of minimizers and not absolute minimizers is the focus (unless $n = 1$).

We take the following point of view in giving selected references here. If one goes to MathSciNet, for example, and brings up the review of Jensen's paper [36], there will be over 30 reference citations (which is a lot). This will reveal papers with titles involving homogenization, Γ-limits, eigenvalue problems, free boundary problems, and so on. None of these topics are mentioned in this work of limited aims. Likewise, one can bring up a list of the papers coauthored by Barron and/or Jensen, etc., or any of the authors which popped up as referencing Jensen [36], and then search the web to find the web sites of authors of articles that interest you, it is a new world. Perhaps it is worth mentioning that the Institute for Scientific Information's *Web of Science* generally provides a more complete cited reference search (Jensen's article gets over 50 citations on the Web of Science).

So we are selective, sticking to variants of the main thrust of this article.

9.1. **What is Δ_∞ for $H(x, u, Du)$?** A very natural generalization of the theory of the preceding sections arises by replacing the functional $\mathcal{F}_\infty$ by a more general functional

$$\mathcal{F}_\infty^H(u, U) := \|H(x, u, Du(x))\|_{L^\infty(U)} \tag{9.1}$$

for suitable functions H. We write the generic arguments of H as $H(x, r, p)$. H should be reasonable, and for us $r \in \mathrm{I\!R}$ is real, corresponding to $u : U \to \mathrm{I\!R}$. The case discussed in these notes is $H(x, r, p) = |p|$, but we could as well put $H(x, r, p) = |p|^2$, which has the virtue that H is now smooth, along with being convex and quite coercive. It turns out that for part of the theory, it is not convexity of H which is primary, but instead "quasi-convexity," which means that each sublevel set of H is convex:

$$\{p : H(x, r, p) \le \lambda\} \text{ is convex for each } x \in \overline{U}, r, \lambda \in \mathrm{I\!R}. \tag{9.2}$$

We are going to suppress more technical assumptions on H, such as the necessary regularity, coercivity, and so on, needed to make statements precise in most of this discussion. The reader should go to the references given for this, if it is omitted.

The operator corresponding to Δ_∞ in this generality is

$$\mathcal{A}(x, r, p, X) = \langle H_x(x, r, p) + H_r(x, r, p)p + XH_p(x, r, p), H_p(x, r, p)\rangle\,. \tag{9.3}$$

By name, we call this the "Aronsson operator" associated with H. It is defined on arguments (x, r, p, X), where X is a symmetric $n \times n$ real matrix. The notations H_x, H_p stand for the gradients of H in the x and p variables, while H_r is $\partial H/\partial r$. The Aronsson equation is $\mathcal{A}[u] := \mathcal{A}(x, u, Du, D^2u) = 0$. In this form, it is more easily remembered as

$$\mathcal{A}[u] = \langle H_p(x, u(x), Du(x)), D_x(H(x, u(x), Du(x))\rangle = 0.$$

Observe that if $H = (1/2)|p|^2$, then $\mathcal{A}[u] = \Delta_\infty u$, while if $H = |p|$ we would have instead

$$\mathcal{A}(x, r, p, X) = \langle X\hat{p}, \hat{p}\rangle \text{ where } \hat{p} = \frac{p}{|p|}. \tag{9.4}$$

There is a viscosity interpretation of equations with singularities such as (9.4), and at $p = 0$ this interpretation just leads to our relations (2.24), (2.26). It was shown by Barron, Jensen and Wang [11] that if u is absolutely minimizing for $\mathcal{F}_\infty^H$, then $\mathcal{A}[u] = 0$ in the viscosity sense. The technical conditions under which these authors established this are more severe than those given in [27], corresponding to the more transparent proof given in this paper. It remains an interesting question if one assumption common to [11], [27] can be removed, namely, is it sufficient to have $H \in C^1$ (rather than C^2)?

It remained a question as to whether or not $\mathcal{A}[u] = 0$ implied that u is absolutely minimizing for $\mathcal{F}_\infty^H$. Y. Yu [55] proved several things in this direction. First, if $H = H(x, p)$

is convex in p and sufficiently coercive, the answer is yes. Secondly, he provided an example to show that the answer is no in general if H is merely quasi-convex, but otherwise nice enough. He takes $n = 1$, $H(x,p) = (p^2 - 2p)^3 + V(x)$ and designs V to create the counterexample. Likewise, Yu showed that in the case $H = H(r,p)$, the Aronsson equation does not guarantee absolutely minimizing. Subsequently, Gariepy, Wang and Yu [34] showed that if $H = H(p)$, that is, H does not depend on x, r, and is merely quasi-convex, then, indeed, $\mathcal{A}[u] = 0$ implies absolutely minimizing.

Moreover, Yu also showed that there is no uniqueness theorem in the generality of $\mathcal{F}_\infty^H$. This is not an issue of smoothness of H. Yu gives the simple example of $n = 1$, $U = (0, 2\pi)$, $H(x,p) = |p|^2 + \sin^2(x)$ and notes that both $u \equiv 0$ and $u = \sin(x)$ solve

$$\mathcal{A}[u] = 2u_x(2\cos(x)\sin(x) + 2u_x u_{xx}) = 0, \quad u(0) = u(2\pi) = 0.$$

9.2. **Generalizing Comparison with Cones.** Are there cones in greater generality than in the archetypal case $H = |p|$ and $|\cdot|$ is any norm on $\mathbb{R}^n$? Note that in this case, if $a \geq 0$, we have the cone functions

$$a|x - z|^* = \sup_{H(p)=a} \langle x - z, p \rangle ,$$

where $|\cdot|^*$ is the dual norm. That is, with this H, absolutely minimizing for $\mathcal{F}_\infty^H$ couples with comparison with cones defined via $|\cdot|^*$ and absolutely minimizing for Lip where Lipschitz constants are computed with respect to this dual norm. This is the generality of [8]. In this case, u is absolutely minimizing for $\mathcal{F}_\infty^H$ if and only if u and $-u$ enjoy comparison with these cone functions from above, per [8], and these properties are equivalent to u being a solution of the Aronsson equation if $|\cdot|^*$ and $|\cdot|$ are C^1 off of the origin, and in other cases as well.

In the general quasi-convex case in which $H(p)$ is C^2 and coercive, Gariepy, Wang and Yu show that the same formula yields a class of functions $C_H^a(x - z)$ which has the same properties. The generality of the situation requires a more delicate analysis to show that the Aronsson equation implies comparison with these generalized cones.

The case in which $H(x,p)$ is convex in p, along with other technical assumptions, Yu [55] uses comparison type arguments to show similar results.

The general case $H(x,p)$ is treated as well, in full quasi-convex generality and with minimum regularity on H, in Champion and De Pascale, [25]. These authors supply a "comparison with distance functions" equivalence for the property of being absolutely minimizing with respect to $\mathcal{F}_\infty^H$. This is a bit too complex to explain here (while quite readable in [25]), so we content ourselves by noting that this property is defined by Mc Shane - Whitney type operators relative to distance within a given set, coupled with appropriate structures related to the definition of generalized cones just above. There are other interesting things in this paper.

One key difference between works in this direction and the simpler case treated in this paper, is that they do not use any evident analogue of the interplay between the functionals Lip and $\mathcal{F}_\infty$, which we used to help our organization.

9.3. **The Metric Case.** Let us take this case to include abstract metric spaces as well as metrics arising from differential geometric considerations. In the abstract case, we mention primarily Champion and De Pascale [25], Section 5, as this seems to be the state of the art in this direction. Here the Lipschitz functional is primary, and the authors show that the associated absolutely minimizing property is equivalent to a straightforward comparison with distance functions property. In particular, it makes clear that the fact that the distance functions do not themselves satisfy a full comparison with distance function property noted in [8] is not an impediment to the full characterization, properly put. Existence had previously been treated in Juutinen [38] and Mil'man [46], [47].

For papers which treat various geometrical structures, we mention the the work Bieske [15], which was the first, as well as [16], [17], [18] and Wang [51]. The paper of Wang, currently available on his website, contains a very nice introduction to which we refer for a further overview.

9.4. **Playing Games.** It was recently discovered by Peres, Schramm, Sheffield and Wilson [49] that the value function for a random turn "tug of war" game, in which the players take turns according to the outcome of a coin toss, and a certain limit is taken, is ∞-harmonic. This striking emergence of the ∞-Laplacian in a completely new arena supports the idea that this operator is liable to arise in many situations. This framework leads to many other operators as its ingredients are varied, and one can currently read about this in a preprint of Barron, Evans and Jensen, [14] (which is available on Evans' website as of this writing). Much more is contained in this interesting article, and many variations are derived in several different ways: many different operators, inhomogeneous equations, time dependent versions, — .

In this context, the results explained herein are quite special, corresponding, say, to merely deriving the basic properties of harmonic functions via their mean value property, and all sorts of generalizations are treated later by various theories, not using the mean value property (Poisson equations, more general elliptic operators, time dependent versions and so on). However, one should understand the Laplace equation as a starting point, and in this case, we do not know enough yet about ∞-harmonic functions.

The interesting functional equation

$$u^\varepsilon(x) = \frac{1}{2}\left(\max_{|y|\le\varepsilon} u^\varepsilon(x+\varepsilon y) + \max_{|z|\le\varepsilon} u^\varepsilon(x+\varepsilon z)\right)$$

arises in the game theoretic considerations (see [14]). This same elegant relation plays a role in approximation arguments given in Le Gruyer [41], which associates the ∞-Laplace equation with constructions originating in Le Guyer and Archer, [40]. This latter article contains the first analysis of functional relations like the above of which we are aware. Oberman [48] uses closely related ideas to generate numerical approximations. The articles [41] and [40] could also have been mentioned under the "metric space" heading above. A preprint of [41] is available on arXiv.

9.5. **Miscellany.** We want to mention a few further papers and topics. There is the somewhat speculative offering of Evans and Yu [33], which ponders, among other things, the relation of the question of whether or not ∞-harmonic functions are C^1 to standard pde approaches to this question. The paper Mikalelyan [45] shows that ∞-harmonic functions may fail to be twice differentiable on a dense set.

Then there are the results of Bhattacharya [20]-[23] on some more refined properties of ∞-harmonic functions in special situations. For example, in [21], the author shows that a nonnegative ∞-harmonic function on a half-space which is continuous on the closure of the half-space is neccesarily a scalar multiple of the distance to the boundary. This result does not seem to follow in any simple way from the theory we have presented so far; it makes a more sophisticated use of consequences of the Harnack inequality, taking it up to the boundary and comparing two different functions. The results of Exercise 16 are very present, however, in these works.

We could go on, but it is time to stop.

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