(a) $(P \vee Q) \wedge (\neg P \vee \neg Q)$.
(b) $(P \vee Q) \wedge (\neg P \wedge \neg Q)$.
(c) $(P \vee Q) \vee (\neg P \vee \neg Q)$.
(d) $[P \wedge (Q \vee \neg R)] \vee (\neg P \vee R)$.

10. Use truth tables to check these laws:
(a) The second DeMorgan's law. (The first was checked in the text.)
(b) The distributive laws.

*11. Use the laws stated in the text to find simpler formulas equivalent to these formulas. (See Examples 1.2.5 and 1.2.7.)
(a) $\neg(\neg P \wedge \neg Q)$.
(b) $(P \wedge Q) \vee (P \wedge \neg Q)$.
(c) $\neg(P \wedge \neg Q) \vee (\neg P \wedge Q)$.

12. Use the laws stated in the text to find simpler formulas equivalent to these formulas. (See Examples 1.2.5 and 1.2.7.)
(a) $\neg(\neg P \vee Q) \vee (P \wedge \neg R)$.
(b) $\neg(\neg P \wedge Q) \vee (P \wedge \neg R)$.
(c) $(P \wedge R) \vee [\neg R \wedge (P \vee Q)]$.

13. Use the first DeMorgan's law and the double negation law to derive the second DeMorgan's law.

*14. Note that the associative laws say only that parentheses are unnecessary when combining *three* statements with $\wedge$ or $\vee$. In fact, these laws can be used to justify leaving parentheses out when more than three statements are combined. Use associative laws to show that $[P \wedge (Q \wedge R)] \wedge S$ is equivalent to $(P \wedge Q) \wedge (R \wedge S)$.

15. How many lines will there be in the truth table for a statement containing n letters?

*16. Find a formula involving the connectives $\wedge$, $\vee$, and $\neg$ that has the following truth table:

P	Q	???
F	F	T
F	T	F
T	F	T
T	T	T

17. Find a formula involving the connectives $\wedge$, $\vee$, and $\neg$ that has the following truth table:

P	Q	???
F	F	F
F	T	T
T	F	T
T	T	F

18. Suppose the conclusion of an argument is a tautology. What can you conclude about the validity of the argument? What if the conclusion is a contradiction? What if one of the premises is either a tautology or a contradiction?

1.3. Variables and Sets

In mathematical reasoning it is often necessary to make statements about objects that are represented by letters called *variables*. For example, if the variable x is used to stand for a number in some problem, we might be interested in the statement "x is a prime number." Although we may sometimes use a single letter, say P, to stand for this statement, at other times we will revise this notation slightly and write $P(x)$, to stress that this is a statement *about* x. The latter notation makes it easy to talk about substituting some number for x in the statement. For example, $P(7)$ would represent the statement "7 is a prime number," and $P(a+b)$ would mean "$a+b$ is a prime number." If a statement contains more than one variable, our abbreviation for the statement will include a list of all the variables involved. For example, we might represent the statement "p is divisible by q" by $D(p, q)$. In this case, $D(12, 4)$ would mean "12 is divisible by 4."

Although you have probably seen variables used most often to stand for numbers, they can stand for anything at all. For example, we could let $M(x)$ stand for the statement "x is a man," and $W(x)$ for "x is a woman." In this case, we are using the variable x to stand for a person. A statement might even contain several variables that stand for different kinds of objects. For example, in the statement "x has y children," the variable x stands for a person, and y stands for a number.

Statements involving variables can be combined using connectives, just like statements without variables.

Example 1.3.1. Analyze the logical forms of the following statements:

1. x is a prime number, and either y or z is divisible by x.
2. x is a man and y is a woman and x likes y, but y doesn't like x.

Solutions

1. We could let P stand for the statement "x is a prime number," D for "y is divisible by x," and E for "z is divisible by x." The entire statement would then be represented by the formula $P \land (D \lor E)$. But this analysis, though not incorrect, fails to capture the relationship between the statements

Since the empty set has no elements, the statement $x \in \varnothing$ is an example of a statement that is always false, no matter what x is.

Another common notation for the empty set is based on the fact that any set can be named by listing its elements between braces. Since the empty set has no elements, we write nothing between the braces, like this: $\varnothing = \{\ \}$. Note that $\{\varnothing\}$ is not correct notation for the empty set. Just as we saw earlier that 2 and $\{2\}$ are not the same thing, $\varnothing$ is not the same as $\{\varnothing\}$. The first is a set with no elements, whereas the second is a set with one element, that one element being $\varnothing$, the empty set.

Exercises

*1. Analyze the logical forms of the following statements:
   (a) 3 is a common divisor of 6, 9, and 15. (Note: You did this in exercise 2 of Section 1.1, but you should be able to give a better answer now.)
   (b) x is divisible by both 2 and 3 but not 4.
   (c) x and y are natural numbers, and exactly one of them is prime.

2. Analyze the logical forms of the following statements:
   (a) x and y are men, and either x is taller than y or y is taller than x.
   (b) Either x or y has brown eyes, and either x or y has red hair.
   (c) Either x or y has both brown eyes and red hair.

*3. Write definitions using elementhood tests for the following sets:
   (a) {Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune, Pluto}.
   (b) {Brown, Columbia, Cornell, Dartmouth, Harvard, Princeton, University of Pennsylvania, Yale}.
   (c) {Alabama, Alaska, Arizona, ..., Wisconsin, Wyoming}.
   (d) {Alberta, British Columbia, Manitoba, New Brunswick, Newfoundland and Labrador, Northwest Territories, Nova Scotia, Nunavut, Ontario, Prince Edward Island, Quebec, Saskatchewan, Yukon}.

4. Write definitions using elementhood tests for the following sets:
   (a) $\{1, 4, 9, 16, 25, 36, 49, \ldots\}$.
   (b) $\{1, 2, 4, 8, 16, 32, 64, \ldots\}$.
   (c) $\{10, 11, 12, 13, 14, 15, 16, 17, 18, 19\}$.

*5. Simplify the following statements. Which variables are free and which are bound? If the statement has no free variables, say whether it is true or false.
   (a) $-3 \in \{x \in \mathbb{R} \mid 13 - 2x > 1\}$.
   (b) $4 \in \{x \in \mathbb{R}^- \mid 13 - 2x > 1\}$.
   (c) $5 \notin \{x \in \mathbb{R} \mid 13 - 2x > c\}$.

below disjoint from any of the others? Are any of the sets below subsets of any others?

(a) $A \cap B$.
(b) $(A \cup B) \setminus C$.
(c) $A \cup (B \setminus C)$.

2. Let $A = \{$United States, Germany, China, Australia$\}$, $B = \{$Germany, France, India, Brazil$\}$, and $C = \{x \mid x$ is a country in Europe$\}$. List the elements of the following sets. Are any of the sets below disjoint from any of the others? Are any of the sets below subsets of any others?
   (a) $A \cup B$.
   (b) $(A \cap B) \setminus C$.
   (c) $(B \cap C) \setminus A$.
3. Verify that the Venn diagrams for $(A \cup B) \setminus (A \cap B)$ and $(A \setminus B) \cup (B \setminus A)$ both look like Figure 5, as stated in this section.

*4. Use Venn diagrams to verify the following identities:
   (a) $A \setminus (A \cap B) = A \setminus B$.
   (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

5. Verify the identities in exercise 4 by writing out (using logical symbols) what it means for an object x to be an element of each set and then using logical equivalences.
6. Use Venn diagrams to verify the following identities:
   (a) $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.
   (b) $A \cup (B \setminus C) = (A \cup B) \setminus (C \setminus A)$.
7. Verify the identities in exercise 6 by writing out (using logical symbols) what it means for an object x to be an element of each set and then using logical equivalences.

*8. For each of the following sets, write out (using logical symbols) what it means for an object x to be an element of the set. Then determine which of these sets must be equal to each other by determining which statements are equivalent.
   (a) $(A \setminus B) \setminus C$.
   (b) $A \setminus (B \setminus C)$.
   (c) $(A \setminus B) \cup (A \cap C)$.
   (d) $(A \setminus B) \cap (A \setminus C)$.
   (e) $A \setminus (B \cup C)$.

9. It was shown in this section that for any sets A and B, $(A \cup B) \setminus B \subseteq A$. Give an example of two sets A and B for which $(A \cup B) \setminus B \neq A$.

*10. It is claimed in this section that you cannot make a Venn diagram for four sets using overlapping circles.

One of the reasons it's so easy to confuse a conditional statement with its converse is that in everyday speech we sometimes use a conditional statement when what we mean to convey is actually a biconditional. For example, you probably wouldn't say "The lecture will be given if at least ten people are there" unless it was also the case that if there were fewer than ten people, the lecture wouldn't be given. After all, why mention the number ten at all if it's not the minimum number of people required? Thus, the statement actually suggests that the lecture will be given *iff* there are at least ten people there. For another example, suppose a child is told by his parents, "If you don't eat your dinner, you won't get any dessert." The child certainly expects that if he *does* eat his dinner, he *will* get dessert, although that's not literally what his parents said. In other words, the child interprets the statement as meaning "Eating your dinner is a necessary *and sufficient* condition for getting dessert."

Such a blurring of the distinction between *if* and *iff* is never acceptable in mathematics. Mathematicians always use a phrase such as *iff* or *necessary and sufficient condition* when they want to express a biconditional statement. You should never interpret an if-then statement in mathematics as a biconditional statement, the way you might in everyday speech.

Exercises

*1. Analyze the logical forms of the following statements:
   (a) If this gas either has an unpleasant smell or is not explosive, then it isn't hydrogen.
   (b) Having both a fever and a headache is a sufficient condition for George to go to the doctor.
   (c) Both having a fever and having a headache are sufficient conditions for George to go to the doctor.
   (d) If $x \neq 2$, then a necessary condition for x to be prime is that x be odd.

2. Analyze the logical forms of the following statements:
   (a) Mary will sell her house only if she can get a good price and find a nice apartment.
   (b) Having both a good credit history and an adequate down payment is a necessary condition for getting a mortgage.
   (c) John will kill himself, unless someone stops him. (Hint: First try to rephrase this using the words *if* and *then* instead of *unless*.)
   (d) If x is divisible by either 4 or 6, then it isn't prime.

3. Analyze the logical form of the following statement:
   (a) If it is raining, then it is windy and the sun is not shining.

Now analyze the following statements. Also, for each statement determine whether the statement is equivalent to either statement (a) or its converse.

(b) It is windy and not sunny only if it is raining.
(c) Rain is a sufficient condition for wind with no sunshine.
(d) Rain is a necessary condition for wind with no sunshine.
(e) It's not raining, if either the sun is shining or it's not windy.
(f) Wind is a necessary condition for it to be rainy, and so is a lack of sunshine.
(g) Either it is windy only if it is raining, or it is not sunny only if it is raining.

*4. Use truth tables to determine whether or not the following arguments are valid:
  (a) Either sales or expenses will go up. If sales go up, then the boss will be happy. If expenses go up, then the boss will be unhappy. Therefore, sales and expenses will not both go up.
  (b) If the tax rate and the unemployment rate both go up, then there will be a recession. If the GNP goes up, then there will not be a recession. The GNP and taxes are both going up. Therefore, the unemployment rate is not going up.
  (c) The warning light will come on if and only if the pressure is too high and the relief valve is clogged. The relief valve is not clogged. Therefore, the warning light will come on if and only if the pressure is too high.

5. (a) Show that $P \leftrightarrow Q$ is equivalent to $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.
  (b) Show that $(P \rightarrow Q) \vee (P \rightarrow R)$ is equivalent to $P \rightarrow (Q \vee R)$.

*6. (a) Show that $(P \rightarrow R) \wedge (Q \rightarrow R)$ is equivalent to $(P \vee Q) \rightarrow R$.
  (b) Formulate and verify a similar equivalence involving $(P \rightarrow R) \vee (Q \rightarrow R)$.

7. (a) Show that $(P \rightarrow Q) \wedge (Q \rightarrow R)$ is equivalent to $(P \rightarrow R) \wedge [(P \leftrightarrow Q) \vee (R \leftrightarrow Q)]$.
  (b) Show that $(P \rightarrow Q) \vee (Q \rightarrow R)$ is a tautology.

*8. Find a formula involving only the connectives $\neg$ and $\rightarrow$ that is equivalent to $P \wedge Q$.

9. Find a formula involving only the connectives $\neg$ and $\rightarrow$ that is equivalent to $P \leftrightarrow Q$.

10. Which of the following formulas are equivalent?
  (a) $P \rightarrow (Q \rightarrow R)$.
  (b) $Q \rightarrow (P \rightarrow R)$.
  (c) $(P \rightarrow Q) \wedge (P \rightarrow R)$.
  (d) $(P \wedge Q) \rightarrow R$.
  (e) $P \rightarrow (Q \wedge R)$.

Now analyze the following statements. Also, for each statement determine whether the statement is equivalent to either statement (a) or its converse.

(b) It is windy and not sunny only if it is raining.
(c) Rain is a sufficient condition for wind with no sunshine.
(d) Rain is a necessary condition for wind with no sunshine.
(e) It's not raining, if either the sun is shining or it's not windy.
(f) Wind is a necessary condition for it to be rainy, and so is a lack of sunshine.
(g) Either it is windy only if it is raining, or it is not sunny only if it is raining.

*4. Use truth tables to determine whether or not the following arguments are valid:

(a) Either sales or expenses will go up. If sales go up, then the boss will be happy. If expenses go up, then the boss will be unhappy. Therefore, sales and expenses will not both go up.
(b) If the tax rate and the unemployment rate both go up, then there will be a recession. If the GNP goes up, then there will not be a recession. The GNP and taxes are both going up. Therefore, the unemployment rate is not going up.
(c) The warning light will come on if and only if the pressure is too high and the relief valve is clogged. The relief valve is not clogged. Therefore, the warning light will come on if and only if the pressure is too high.

5. (a) Show that $P \leftrightarrow Q$ is equivalent to $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.
(b) Show that $(P \rightarrow Q) \vee (P \rightarrow R)$ is equivalent to $P \rightarrow (Q \vee R)$.

*6. (a) Show that $(P \rightarrow R) \wedge (Q \rightarrow R)$ is equivalent to $(P \vee Q) \rightarrow R$.
(b) Formulate and verify a similar equivalence involving $(P \rightarrow R) \vee (Q \rightarrow R)$.

7. (a) Show that $(P \rightarrow Q) \wedge (Q \rightarrow R)$ is equivalent to $(P \rightarrow R) \wedge [(P \leftrightarrow Q) \vee (R \leftrightarrow Q)]$.
(b) Show that $(P \rightarrow Q) \vee (Q \rightarrow R)$ is a tautology.

*8. Find a formula involving only the connectives $\neg$ and $\rightarrow$ that is equivalent to $P \wedge Q$.

9. Find a formula involving only the connectives $\neg$ and $\rightarrow$ that is equivalent to $P \leftrightarrow Q$.

10. Which of the following formulas are equivalent?

(a) $P \rightarrow (Q \rightarrow R)$.
(b) $Q \rightarrow (P \rightarrow R)$.
(c) $(P \rightarrow Q) \wedge (P \rightarrow R)$.
(d) $(P \wedge Q) \rightarrow R$.
(e) $P \rightarrow (Q \wedge R)$.

6. This means that for every natural number x, the statement $\forall y(x < y)$ is true. But as we saw in the third statement, there isn't even *one* value of x for which this statement is true. Thus, $\forall x \forall y(x < y)$ is false.

Exercises

*1. Analyze the logical forms of the following statements.
   (a) Anyone who has forgiven at least one person is a saint.
   (b) Nobody in the calculus class is smarter than everybody in the discrete math class.
   (c) Everyone likes Mary, except Mary herself.
   (d) Jane saw a police officer, and Roger saw one too.
   (e) Jane saw a police officer, and Roger saw him too.
2. Analyze the logical forms of the following statements.
   (a) Anyone who has bought a Rolls Royce with cash must have a rich uncle.
   (b) If anyone in the dorm has the measles, then everyone who has a friend in the dorm will have to be quarantined.
   (c) If nobody failed the test, then everybody who got an A will tutor someone who got a D.
   (d) If anyone can do it, Jones can.
   (e) If Jones can do it, anyone can.
3. Analyze the logical forms of the following statements. The universe of discourse is $\mathbb{R}$. What are the free variables in each statement?
   (a) Every number that is larger than x is larger than y.
   (b) For every number a, the equation $ax^2 + 4x - 2 = 0$ has at least one solution iff $a \geq -2$.
   (c) All solutions of the inequality $x^3 - 3x < 3$ are smaller than 10.
   (d) If there is a number x such that $x^2 + 5x = w$ and there is a number y such that $4 - y^2 = w$, then w is between -10 and 10.
*4. Translate the following statements into idiomatic English.
   (a) $\forall x[(H(x) \wedge \neg\exists y M(x, y)) \rightarrow U(x)]$, where $H(x)$ means "x is a man," $M(x, y)$ means "x is married to y," and $U(x)$ means "x is unhappy."
   (b) $\exists z(P(z, x) \wedge S(z, y) \wedge W(y))$, where $P(z, x)$ means "z is a parent of x," $S(z, y)$ means "z and y are siblings," and $W(y)$ means "y is a woman."
5. Translate the following statements into idiomatic mathematical English.
   (a) $\forall x[(P(x) \wedge \neg(x = 2)) \rightarrow O(x)]$, where $P(x)$ means "x is a prime number" and $O(x)$ means "x is odd."

(b) $\exists x[P(x) \land \forall y(P(y) \to y \leq x)]$, where $P(x)$ means "x is a perfect number."

6. Are these statements true or false? The universe of discourse is the set of all people, and $P(x, y)$ means "x is a parent of y."
   (a) $\exists x \forall y P(x, y)$.
   (b) $\forall x \exists y P(x, y)$.
   (c) $\neg\exists x \exists y P(x, y)$.
   (d) $\exists x \neg\exists y P(x, y)$.
   (e) $\exists x \exists y \neg P(x, y)$.

*7. Are these statements true or false? The universe of discourse is $\mathbb{N}$.
   (a) $\forall x \exists y(2x - y = 0)$.
   (b) $\exists y \forall x(2x - y = 0)$.
   (c) $\forall x \exists y(x - 2y = 0)$.
   (d) $\forall x(x < 10 \to \forall y(y < x \to y < 9))$.
   (e) $\exists y \exists z(y + z = 100)$.
   (f) $\forall x \exists y(y > x \land \exists z(y + z = 100))$.

8. Same as exercise 7 but with $\mathbb{R}$ as the universe of discourse.
9. Same as exercise 7 but with $\mathbb{Z}$ as the universe of discourse.

2.2. Equivalences Involving Quantifiers

In our study of logical connectives in Chapter 1 we found it useful to examine equivalences between different formulas. In this section, we will see that there are also a number of important equivalences involving quantifiers.

For example, in Example 2.1.2 we represented the statement "Nobody's perfect" by the formula $\neg\exists x P(x)$, where $P(x)$ meant "x is perfect." But another way to express the same idea would be to say that everyone fails to be perfect, or in other words $\forall x \neg P(x)$. This suggests that these two formulas are equivalent, and a little thought should show that they are. No matter what $P(x)$ stands for, the formula $\neg\exists x P(x)$ means that there's no value of x in the universe of discourse for which $P(x)$ is true. But that's the same as saying that for every value of x in the universe, $P(x)$ is false, or in other words $\forall x \neg P(x)$. Thus, $\neg\exists x P(x)$ is equivalent to $\forall x \neg P(x)$.

Similar reasoning shows that $\neg\forall x P(x)$ is equivalent to $\exists x \neg P(x)$. To say that $\neg\forall x P(x)$ means that it is not the case that for all values of x, $P(x)$ is true. That's equivalent to saying there's at least one value of x for which $P(x)$ is false, which is what it means to say $\exists x \neg P(x)$. For example, in Example 2.1.2 we translated "Someone didn't do the homework" as $\exists x \neg H(x)$, where $H(x)$ stands for "x did the homework." An equivalent statement would be "Not everyone did the homework," which would be represented by the formula $\neg\forall x H(x)$.