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Sentential Logic

1.1. Deductive Reasoning and Logical Connectives

As we saw in the introduction, proofs play a central role in mathematics, and deductive reasoning is the foundation on which proofs are based. Therefore, we begin our study of mathematical reasoning and proofs by examining how deductive reasoning works.

Example 1.1.1. Here are three examples of deductive reasoning:

1. It will either rain or snow tomorrow.
   It's too warm for snow.
   Therefore, it will rain.
2. If today is Sunday, then I don't have to go to work today.
   Today is Sunday.
   Therefore, I don't have to go to work today.
3. I will go to work either tomorrow or today.
   I'm going to stay home today.
   Therefore, I will go to work tomorrow.

In each case, we have arrived at a *conclusion* from the assumption that some other statements, called *premises*, are true. For example, the premises in argument 3 are the statements "I will go to work either tomorrow or today" and "I'm going to stay home today." The conclusion is "I will go to work tomorrow," and it seems to be forced on us somehow by the premises.

But is this conclusion really correct? After all, isn't it possible that I'll stay home today, and then wake up sick tomorrow and end up staying home again? If that happened, the conclusion would turn out to be false. But notice that in that case the first premise, which said that I would go to work either tomorrow or today, would be false as well! Although we have no guarantee that the conclusion is true, it can only be false if at least one of the premises is also false. *If* both premises are true, we can be sure that the conclusion is also true. This is the sense in which the conclusion is forced on us by the premises, and this is the standard we will use to judge the correctness of deductive reasoning. We will say that an argument is *valid* if the premises cannot all be true without the conclusion being true as well. All three of the arguments in our example are valid arguments.

Here's an example of an invalid deductive argument:

Either the butler is guilty or the maid is guilty.
Either the maid is guilty or the cook is guilty.
Therefore, either the butler is guilty or the cook is guilty.

The argument is invalid because the conclusion could be false even if both premises are true. For example, if the maid were guilty, but the butler and the cook were both innocent, then both premises would be true and the conclusion would be false.

We can learn something about what makes an argument valid by comparing the three arguments in Example 1.1.1. On the surface it might seem that arguments 2 and 3 have the most in common, because they're both about the same subject: attendance at work. But in terms of the reasoning used, arguments 1 and 3 are the most similar. They both introduce two possibilities in the first premise, rule out the second one with the second premise, and then conclude that the first possibility must be the case. In other words, both arguments have the form:

P or Q.
not Q.
Therefore, P.

It is this form, and not the subject matter, that makes these arguments valid. You can see that argument 1 has this form by thinking of the letter P as standing for the statement "It will rain tomorrow," and Q as standing for "It will snow tomorrow." For argument 3, P would be "I will go to work tomorrow," and Q would be "I will go to work today."

Replacing certain statements in each argument with letters, as we have in stating the form of arguments 1 and 3, has two advantages. First, it keeps us from being distracted by aspects of the arguments that don't affect their validity. You don't need to know anything about weather forecasting or work habits to recognize that arguments 1 and 3 are valid. That's because both arguments have the form shown earlier, and you can tell that this argument form is valid without even knowing what P and Q stand for. If you don't believe this, consider the following argument:

> Either the framger widget is misfiring, or the wrompal mechanism is out of alignment.
> I've checked the alignment of the wrompal mechanism, and it's fine.
> Therefore, the framger widget is misfiring.

If a mechanic gave this explanation after examining your car, you might still be mystified about why the car won't start, but you'd have no trouble following his logic!

Perhaps more important, our analysis of the forms of arguments 1 and 3 makes clear what *is* important in determining their validity: the words *or* and *not*. In most deductive reasoning, and in particular in mathematical reasoning, the meanings of just a few words give us the key to understanding what makes a piece of reasoning valid or invalid. (Which are the important words in argument 2 in Example 1.1.1?) The first few chapters of this book are devoted to studying those words and how they are used in mathematical writing and reasoning.

In this chapter, we'll concentrate on words used to combine statements to form more complex statements. We'll continue to use letters to stand for statements, but only for unambiguous statements that are either true or false. Questions, exclamations, and vague statements will not be allowed. It will also be useful to use symbols, sometimes called *connective symbols*, to stand for some of the words used to combine statements. Here are our first three connective symbols and the words they stand for:

Symbol	Meaning
$\vee$	or
$\wedge$	and
$\neg$	not

Thus, if P and Q stand for two statements, then we'll write $P \vee Q$ to stand for the statement "P or Q," $P \wedge Q$ for "P and Q," and $\neg P$ for "not P" or "P is false." The statement $P \vee Q$ is sometimes called the *disjunction* of P and Q, $P \wedge Q$ is called the *conjunction* of P and Q, and $\neg P$ is called the *negation* of P.

Example 1.1.2. Analyze the logical forms of the following statements:

1. Either John went to the store, or we're out of eggs.
2. Joe is going to leave home and not come back.
3. Either Bill is at work and Jane isn't, or Jane is at work and Bill isn't.

Solutions

1. If we let P stand for the statement "John went to the store" and Q stand for "We're out of eggs," then this statement could be represented symbolically as $P \lor Q$.
2. If we let P stand for the statement "Joe is going to leave home" and Q stand for "Joe is not going to come back," then we could represent this statement symbolically as $P \land Q$. But this analysis misses an important feature of the statement, because it doesn't indicate that Q is a negative statement. We could get a better analysis by letting R stand for the statement "Joe is going to come back" and then writing the statement Q as $\neg R$. Plugging this into our first analysis of the original statement, we get the improved analysis $P \land \neg R$.
3. Let B stand for the statement "Bill is at work" and J for the statement "Jane is at work." Then the first half of the statement, "Bill is at work and Jane isn't," can be represented as $B \land \neg J$. Similarly, the second half is $J \land \neg B$. To represent the entire statement, we must combine these two with *or*, forming their disjunction, so the solution is $(B \land \neg J) \lor (J \land \neg B)$.

Notice that in analyzing the third statement in the preceding example, we added parentheses when we formed the disjunction of $B \land \neg J$ and $J \land \neg B$ to indicate unambiguously which statements were being combined. This is like the use of parentheses in algebra, in which, for example, the product of $a + b$ and $a - b$ would be written $(a + b) \cdot (a - b)$, with the parentheses serving to indicate unambiguously which quantities are to be multiplied. As in algebra, it is convenient in logic to omit some parentheses to make our expressions shorter and easier to read. However, we must agree on some conventions about how to read such expressions so that they are still unambiguous. One convention is that the symbol $\neg$ always applies only to the statement that comes immediately after it. For example, $\neg P \land Q$ means $(\neg P) \land Q$ rather than $\neg(P \land Q)$. We'll see some other conventions about parentheses later.

Example 1.1.3. What English sentences are represented by the following expressions?

1. $(\neg S \land L) \lor S$, where S stands for "John is stupid" and L stands for "John is lazy."
2. $\neg S \land (L \lor S)$, where S and L have the same meanings as before.
3. $\neg(S \land L) \lor S$, with S and L still as before.

Solutions

1. Either John isn't stupid and he is lazy, or he's stupid.
2. John isn't stupid, and either he's lazy or he's stupid. Notice how the placement of the word *either* in English changes according to where the parentheses are.
3. Either John isn't both stupid and lazy, or John is stupid. The word *both* in English also helps distinguish the different possible positions of parentheses.

It is important to keep in mind that the symbols $\wedge$, $\vee$, and $\neg$ don't really correspond to all uses of the words *and*, *or*, and *not* in English. For example, the symbol $\wedge$ could not be used to represent the use of the word *and* in the sentence "John and Bill are friends," because in this sentence the word *and* is not being used to combine two statements. The symbols $\wedge$ and $\vee$ can only be used *between two statements*, to form their conjunction or disjunction, and the symbol $\neg$ can only be used *before a statement*, to negate it. This means that certain strings of letters and symbols are simply meaningless. For example, $P\neg \wedge Q$, $P \wedge/\vee Q$, and $P\neg Q$ are all "ungrammatical" expressions in the language of logic. "Grammatical" expressions, such as those in Examples 1.1.2 and 1.1.3, are sometimes called *well-formed formulas* or just *formulas*. Once again, it may be helpful to think of an analogy with algebra, in which the symbols $+$, $-$, $\cdot$, and $\div$ can be used *between two numbers*, as operators, and the symbol $-$ can also be used *before a number*, to negate it. These are the only ways that these symbols can be used in algebra, so expressions such as $x - \div y$ are meaningless.

Sometimes, words other than *and*, *or*, and *not* are used to express the meanings represented by $\wedge$, $\vee$, and $\neg$. For example, consider the first statement in Example 1.1.3. Although we gave the English translation "Either John isn't stupid and he is lazy, or he's stupid," an alternative way of conveying the same information would be to say "Either John isn't stupid *but* he is lazy, or he's stupid." Often, the word *but* is used in English to mean *and*, especially when there is some contrast or conflict between the statements being combined. For a more striking example, imagine a weather forecaster ending his forecast with the statement "Rain and snow are the only two possibilities for tomorrow's weather." This is just a roundabout way of saying that it will either rain or snow tomorrow. Thus, even though the forecaster has used the word *and*, the meaning expressed by his statement is a disjunction. The lesson of these examples is that to determine the logical form of a statement you must think about what the statement means, rather than just translating word by word into symbols.

Sometimes logical words are hidden within mathematical notation. For example, consider the statement $3 \leq \pi$. Although it appears to be a simple statement that contains no words of logic, if you read it out loud you will hear the word *or*. If we let P stand for the statement $3 < \pi$ and Q for the statement $3 = \pi$, then the statement $3 \leq \pi$ would be written $P \vee Q$. In this example the statements represented by the letters P and Q are so short that it hardly seems worthwhile to abbreviate them with single letters. In cases like this we will sometimes not bother to replace the statements with letters, so we might also write this statement as $(3 < \pi) \vee (3 = \pi)$.

For a slightly more complicated example, consider the statement $3 \leq \pi < 4$. This statement means $3 \leq \pi$ *and* $\pi < 4$, so once again a word of logic has been hidden in mathematical notation. Filling in the meaning that we just worked out for $3 \leq \pi$, we can write the whole statement as $[(3 < \pi) \vee (3 = \pi)] \wedge (\pi < 4)$. Knowing that the statement has this logical form might be important in understanding a piece of mathematical reasoning involving this statement.

Exercises

*1. Analyze the logical forms of the following statements:
   (a) We'll have either a reading assignment or homework problems, but we won't have both homework problems and a test.
   (b) You won't go skiing, or you will and there won't be any snow.
   (c) $\sqrt{7} \nleq 2$.

2. Analyze the logical forms of the following statements:
   (a) Either John and Bill are both telling the truth, or neither of them is.
   (b) I'll have either fish or chicken, but I won't have both fish and mashed potatoes.
   (c) 3 is a common divisor of 6, 9, and 15.

3. Analyze the logical forms of the following statements:
   (a) Alice and Bob are not both in the room.
   (b) Alice and Bob are both not in the room.
   (c) Either Alice or Bob is not in the room.
   (d) Neither Alice nor Bob is in the room.

4. Which of the following expressions are well-formed formulas?
   (a) $\neg(\neg P \vee \neg\neg R)$.
   (b) $\neg(P, Q, \wedge R)$.
   (c) $P \wedge \neg P$.
   (d) $(P \wedge Q)(P \vee R)$.

*5. Let P stand for the statement "I will buy the pants" and S for the statement "I will buy the shirt." What English sentences are represented by the following expressions?
   (a) $\neg(P \wedge \neg S)$.
   (b) $\neg P \wedge \neg S$.
   (c) $\neg P \vee \neg S$.
6. Let S stand for the statement "Steve is happy" and G for "George is happy." What English sentences are represented by the following expressions?
   (a) $(S \vee G) \wedge (\neg S \vee \neg G)$.
   (b) $[S \vee (G \wedge \neg S)] \vee \neg G$.
   (c) $S \vee [G \wedge (\neg S \vee \neg G)]$.
7. Identify the premises and conclusions of the following deductive arguments and analyze their logical forms. Do you think the reasoning is valid? (Although you will have only your intuition to guide you in answering this last question, in the next section we will develop some techniques for determining the validity of arguments.)
   (a) Jane and Pete won't both win the math prize. Pete will win either the math prize or the chemistry prize. Jane will win the math prize. Therefore, Pete will win the chemistry prize.
   (b) The main course will be either beef or fish. The vegetable will be either peas or corn. We will not have both fish as a main course and corn as a vegetable. Therefore, we will not have both beef as a main course and peas as a vegetable.
   (c) Either John or Bill is telling the truth. Either Sam or Bill is lying. Therefore, either John is telling the truth or Sam is lying.
   (d) Either sales will go up and the boss will be happy, or expenses will go up and the boss won't be happy. Therefore, sales and expenses will not both go up.

1.2. Truth Tables

We saw in Section 1.1 that an argument is valid if the premises cannot all be true without the conclusion being true as well. Thus, to understand how words such as *and*, *or*, and *not* affect the validity of arguments, we must see how they contribute to the truth or falsity of statements containing them.

When we evaluate the truth or falsity of a statement, we assign to it one of the labels *true* or *false*, and this label is called its *truth value*. It is clear how the word *and* contributes to the truth value of a statement containing it. A statement of the form $P \wedge Q$ can only be true if both P and Q are true; if either P or Q is false, then $P \wedge Q$ will be false too. Because we have assumed that P and

P	Q	$P \wedge Q$
F	F	F
F	T	F
T	F	F
T	T	T

Figure 1

Q both stand for statements that are either true or false, we can summarize all the possibilities with the table shown in Figure 1. This is called a *truth table* for the formula $P \wedge Q$. Each row in the truth table represents one of the four possible combinations of truth values for the statements P and Q. Although these four possibilities can appear in the table in any order, it is best to list them systematically so we can be sure that no possibilities have been skipped. The truth table for $\neg P$ is also quite easy to construct because for $\neg P$ to be true, P must be false. The table is shown in Figure 2.

P	$\neg P$
F	T
T	F

Figure 2

The truth table for $P \vee Q$ is a little trickier. The first three lines should certainly be filled in as shown in Figure 3, but there may be some question about the last line. Should $P \vee Q$ be true or false in the case in which P and Q are both true? In other words, does $P \vee Q$ mean "P or Q, or both" or does it mean "P or Q but not both"? The first way of interpreting the word *or* is called the *inclusive or* (because it *includes* the possibility of both statements being true), and the second is called the *exclusive or*. In mathematics, *or* always means inclusive or, unless specified otherwise, so we will interpret $\vee$ as inclusive or. We therefore complete the truth table for $P \vee Q$ as shown in Figure 4. See exercise 3 for more about the exclusive or.

P	Q	$P \vee Q$
F	F	F
F	T	T
T	F	T
T	T	?

Figure 3

P	Q	$P \vee Q$
F	F	F
F	T	T
T	F	T
T	T	T

Figure 4

Using the rules summarized in these truth tables, we can now work out truth tables for more complex formulas. All we have to do is work out the truth values of the component parts of a formula, starting with the individual letters and working up to more complex formulas a step at a time.

Example 1.2.1. Make a truth table for the formula $\neg(P \vee \neg Q)$.

Solution

P	Q	$\neg Q$	$P \vee \neg Q$	$\neg(P \vee \neg Q)$
F	F	T	T	F
F	T	F	F	T
T	F	T	T	F
T	T	F	T	F

The first two columns of this table list the four possible combinations of truth values of P and Q. The third column, listing truth values for the formula $\neg Q$, is found by simply negating the truth values for Q in the second column. The fourth column, for the formula $P \vee \neg Q$, is found by combining the truth values for P and $\neg Q$ listed in the first and third columns, according to the truth value rule for $\vee$ summarized in Figure 4. According to this rule, $P \vee \neg Q$ will be false only if both P and $\neg Q$ are false. Looking in the first and third columns, we see that this happens only in row two of the table, so the fourth column contains an F in the second row and T's in all other rows. Finally, the truth values for the formula $\neg(P \vee \neg Q)$ are listed in the fifth column, which is found by negating the truth values in the fourth column. (Note that these columns had to be worked out in order, because each was used in computing the next.)

Example 1.2.2. Make a truth table for the formula $\neg(P \wedge Q) \vee \neg R$.

Solution

P	Q	R	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg R$	$\neg(P \wedge Q) \vee \neg R$
F	F	F	F	T	T	T
F	F	T	F	T	F	T
F	T	F	F	T	T	T
F	T	T	F	T	F	T
T	F	F	F	T	T	T
T	F	T	F	T	F	T
T	T	F	T	F	T	T
T	T	T	T	F	F	F

Note that because this formula contains three letters, it takes eight lines to list all possible combinations of truth values for these letters. (If a formula contains n different letters, how many lines will its truth table have?)

Here's a way of making truth tables more compactly. Instead of using separate columns to list the truth values for the component parts of a formula, just list those truth values below the corresponding connective symbol in the original formula. This is illustrated in Figure 5, for the formula from Example 1.2.1.

In the first step, we have listed the truth values for P and Q below these letters where they appear in the formula. In step two, the truth values for $\neg Q$ have been added under the $\neg$ symbol for $\neg Q$. In the third step, we have combined the truth values for P and $\neg Q$ to get the truth values for $P \vee \neg Q$, which are listed under the $\vee$ symbol. Finally, in the last step, these truth values are negated and listed under the initial $\neg$ symbol. The truth values added in the last step give the truth value for the entire formula, so we will call the symbol under which they are listed (the first $\neg$ symbol in this case) the *main connective* of the formula. Notice that the truth values listed under the main connective in this case agree with the values we found in Example 1.2.1.

Step 1

P	Q	$\neg(P \vee \neg Q)$
F	F	**F** **F**
F	T	**F** **T**
T	F	**T** **F**
T	T	**T** **T**

Step 2

P	Q	$\neg(P \vee \neg Q)$
F	F	F **T**F
F	T	F **F**T
T	F	T **T**F
T	T	T **F**T

Step 3

P	Q	$\neg(P \vee \neg Q)$
F	F	F **T** TF
F	T	F **F** FT
T	F	T **T** TF
T	T	T **T** FT

Step 4

P	Q	$\neg(P \vee \neg Q)$
F	F	**F** F T TF
F	T	**T** F F FT
T	F	**F** T T TF
T	T	**F** T T FT

Figure 5

Now that we know how to make truth tables for complex formulas, we're ready to return to the analysis of the validity of arguments. Consider again our first example of a deductive argument:

It will either rain or snow tomorrow.
It's too warm for snow.
Therefore, it will rain.

As we have seen, if we let P stand for the statement "It will rain tomorrow" and Q for the statement "It will snow tomorrow," then we can represent the argument symbolically as follows:

$P \vee Q$
$\underline{\neg Q}$
$\therefore P$ (The symbol $\therefore$ means *therefore*.)

We can now see how truth tables can be used to verify the validity of this argument. Figure 6 shows a truth table for both premises and the conclusion of the argument. Recall that we decided to call an argument valid if the premises cannot all be true without the conclusion being true as well. Looking at Figure 6 we see that the only row of the table in which both premises come out true is row three, and in this row the conclusion is also true. Thus, the truth table confirms that if the premises are all true, the conclusion must also be true, so the argument is valid.

		Premises		Conclusion
P	Q	$P \vee Q$	$\neg Q$	P
F	F	F	T	F
F	T	T	F	F
T	F	T	T	T
T	T	T	F	T

Figure 6

Example 1.2.3. Determine whether the following arguments are valid.

1. Either John isn't stupid and he is lazy, or he's stupid.
   John is stupid.
   Therefore, John isn't lazy.
2. The butler and the cook are not both innocent.
   Either the butler is lying or the cook is innocent.
   Therefore, the butler is either lying or guilty.

Solutions

1. As in Example 1.1.3, we let S stand for the statement "John is stupid" and L stand for "John is lazy." Then the argument has the form:

$$\begin{array}{l} (\neg S \wedge L) \vee S \\ \underline{S \qquad\qquad\quad} \\ \therefore \neg L \end{array}$$

   Now we make a truth table for both premises and the conclusion. (You should work out the intermediate steps in deriving column three of this table to confirm that it is correct.)

		Premises		Conclusion
S	L	$(\neg S \wedge L) \vee S$	S	$\neg L$
F	F	F	F	T
F	T	T	F	F
T	F	T	T	T
T	T	T	T	F

   Both premises are true in lines three and four of this table. The conclusion is also true in line three, but it is false in line four. Thus, it is possible for both premises to be true and the conclusion false, so the argument is invalid. In fact, the table shows us exactly why the argument is invalid. The problem occurs in the fourth line of the table, in which S and L are both true – in other words, John is both stupid and lazy. Thus, if John is both stupid and lazy, then both premises will be true but the conclusion will be false, so it would be a mistake to infer that the conclusion must be true from the assumption that the premises are true.

2. Let B stand for the statement "The butler is innocent," C for the statement "The cook is innocent," and L for the statement "The butler is lying." Then the argument has the form:

$$\begin{array}{l} \neg(B \wedge C) \\ \underline{L \vee C} \\ \therefore L \vee \neg B \end{array}$$

Here is the truth table for the premises and conclusion:

			Premises		Conclusion
B	C	L	$\neg(B \wedge C)$	$L \vee C$	$L \vee \neg B$
F	F	F	T	F	T
F	F	T	T	T	T
F	T	F	T	T	T
F	T	T	T	T	T
T	F	F	T	F	F
T	F	T	T	T	T
T	T	F	F	T	F
T	T	T	F	T	T

The premises are both true only in lines two, three, four, and six, and in each of these cases the conclusion is true as well. Therefore, the argument is valid.

If you expected the first argument in Example 1.2.3 to turn out to be valid, it's probably because the first premise confused you. It's a rather complicated statement, which we represented symbolically with the formula $(\neg S \wedge L) \vee S$. According to our truth table, this formula is false if S and L are both false, and true otherwise. But notice that this is exactly the same as the truth table for the simpler formula $L \vee S$! Because of this, we say that the formulas $(\neg S \wedge L) \vee S$ and $L \vee S$ are *equivalent*. Equivalent formulas always have the same truth value no matter what statements the letters in them stand for and no matter what the truth values of those statements are. The equivalence of the premise $(\neg S \wedge L) \vee S$ and the simpler formula $L \vee S$ may help you understand why the argument is invalid. Translating the formula $L \vee S$ back into English, we see that the first premise could have been stated more simply as "John is either lazy or stupid (or both)." But from this premise and the second premise (that John is stupid), it clearly doesn't follow that he's not lazy, because he might be both stupid and lazy.

Example 1.2.4. Which of these formulas are equivalent?

$$\neg(P \wedge Q), \qquad \neg P \wedge \neg Q, \qquad \neg P \vee \neg Q.$$

Solution

Here's a truth table for all three statements. (You should check it yourself!)

P	Q	$\neg(P \wedge Q)$	$\neg P \wedge \neg Q$	$\neg P \vee \neg Q$
F	F	T	T	T
F	T	T	F	T
T	F	T	F	T
T	T	F	F	F

The third and fifth columns in this table are identical, but they are different from the fourth column. Therefore, the formulas $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ are equivalent, but neither is equivalent to the formula $\neg P \wedge \neg Q$. This should make sense if you think about what all the symbols mean. For example, suppose P stands for the statement "The Yankees won last night" and Q stands for "The Red Sox won last night." Then $\neg(P \wedge Q)$ would mean "The Yankees and the Red Sox did not both win last night," and $\neg P \vee \neg Q$ would mean "Either the Yankees or the Red Sox lost last night"; these statements clearly convey the same information. On the other hand, $\neg P \wedge \neg Q$ would mean "The Yankees and the Red Sox both lost last night," which is an entirely different statement.

You can check for yourself by making a truth table that the formula $\neg P \wedge \neg Q$ from Example 1.2.4 is equivalent to the formula $\neg(P \vee Q)$. (To see that this equivalence makes sense, notice that the statements "Both the Yankees and the Red Sox lost last night" and "Neither the Yankees nor the Red Sox won last night" mean the same thing.) This equivalence and the one discovered in Example 1.2.4 are called *DeMorgan's laws.*

In analyzing deductive arguments and the statements that occur in them it is helpful to be familiar with a number of equivalences that come up often. Verify the equivalences in the following list yourself by making truth tables, and check that they make sense by translating the formulas into English, as we did in Example 1.2.4.

DeMorgan's laws

$$\neg(P \wedge Q) \text{ is equivalent to } \neg P \vee \neg Q.$$

$$\neg(P \vee Q) \text{ is equivalent to } \neg P \wedge \neg Q.$$

Commutative laws

$$P \wedge Q \text{ is equivalent to } Q \wedge P.$$

$$P \vee Q \text{ is equivalent to } Q \vee P.$$

Associative laws

$$P \wedge (Q \wedge R) \text{ is equivalent to } (P \wedge Q) \wedge R.$$

$$P \vee (Q \vee R) \text{ is equivalent to } (P \vee Q) \vee R.$$

Idempotent laws

$$P \wedge P \text{ is equivalent to } P.$$

$$P \vee P \text{ is equivalent to } P.$$

Distributive laws

$$P \wedge (Q \vee R) \text{ is equivalent to } (P \wedge Q) \vee (P \wedge R).$$

$$P \vee (Q \wedge R) \text{ is equivalent to } (P \vee Q) \wedge (P \vee R).$$

Absorption laws

$$P \vee (P \wedge Q) \text{ is equivalent to } P.$$

$$P \wedge (P \vee Q) \text{ is equivalent to } P.$$

Double Negation law

$$\neg\neg P \text{ is equivalent to } P.$$

Notice that because of the associative laws we can leave out parentheses in formulas of the forms $P \wedge Q \wedge R$ and $P \vee Q \vee R$ without worrying that the resulting formula will be ambiguous, because the two possible ways of filling in the parentheses lead to equivalent formulas.

Many of the equivalences in the list should remind you of similar rules involving $+$, $\cdot$, and $-$ in algebra. As in algebra, these rules can be applied to more complex formulas, and they can be combined to work out more complicated equivalences. Any of the letters in these equivalences can be replaced by more complicated formulas, and the resulting equivalence will still be true. For example, by replacing P in the double negation law with the formula $Q \vee \neg R$, you can see that $\neg\neg(Q \vee \neg R)$ is equivalent to $Q \vee \neg R$. Also, if two formulas are equivalent, you can always substitute one for the other in any expression and the results will be equivalent. For example, since $\neg\neg P$ is equivalent to

P, if $\neg\neg P$ occurs in any formula, you can always replace it with P and the resulting formula will be equivalent to the original.

Example 1.2.5. Find simpler formulas equivalent to these formulas:

1. $\neg(P \vee \neg Q)$.
2. $\neg(Q \wedge \neg P) \vee P$.

Solutions

1. $\neg(P \vee \neg Q)$

is equivalent to	$\neg P \wedge \neg\neg Q$	(DeMorgan's law),
which is equivalent to	$\neg P \wedge Q$	(double negation law).

You can check that this equivalence is right by making a truth table for $\neg P \wedge Q$ and seeing that it is the same as the truth table for $\neg(P \vee \neg Q)$ found in Example 1.2.1.

2. $\neg(Q \wedge \neg P) \vee P$

is equivalent to	$(\neg Q \vee \neg\neg P) \vee P$	(DeMorgan's law),
which is equivalent to	$(\neg Q \vee P) \vee P$	(double negation law),
which is equivalent to	$\neg Q \vee (P \vee P)$	(associative law),
which is equivalent to	$\neg Q \vee P$	(idempotent law).

Some equivalences are based on the fact that certain formulas are either always true or always false. For example, you can verify by making a truth table that the formula $Q \wedge (P \vee \neg P)$ is equivalent to just Q. But even before you make the truth table, you can probably see why they are equivalent. In every line of the truth table, $P \vee \neg P$ will come out true, and therefore $Q \wedge (P \vee \neg P)$ will come out true when Q is also true, and false when Q is false. Formulas that are always true, such as $P \vee \neg P$, are called *tautologies*. Similarly, formulas that are always false are called *contradictions*. For example, $P \wedge \neg P$ is a contradiction.

Example 1.2.6. Are these statements tautologies, contradictions, or neither?

$$P \vee (Q \vee \neg P), \quad P \wedge \neg(Q \vee \neg Q), \quad P \vee \neg(Q \vee \neg Q).$$

Solution

First we make a truth table for all three statements.

P	Q	$P \vee (Q \vee \neg P)$	$P \wedge \neg(Q \vee \neg Q)$	$P \vee \neg(Q \vee \neg Q)$
F	F	T	F	F
F	T	T	F	F
T	F	T	F	T
T	T	T	F	T

From the truth table it is clear that the first formula is a tautology, the second a contradiction, and the third neither. In fact, since the last column is identical to the first, the third formula is equivalent to P.

We can now state a few more useful laws involving tautologies and contradictions. You should be able to convince yourself that all of these laws are correct by thinking about what the truth tables for the statements involved would look like.

Tautology laws

$P \wedge$ (a tautology) is equivalent to P.

$P \vee$ (a tautology) is a tautology.

$\neg$(a tautology) is a contradiction.

Contradiction laws

$P \wedge$ (a contradiction) is a contradiction.

$P \vee$ (a contradiction) is equivalent to P.

$\neg$(a contradiction) is a tautology.

Example 1.2.7. Find simpler formulas equivalent to these formulas:

1. $P \vee (Q \wedge \neg P)$.
2. $\neg(P \vee (Q \wedge \neg R)) \wedge Q$.

Solutions

1. $P \vee (Q \wedge \neg P)$
   is equivalent to $(P \vee Q) \wedge (P \vee \neg P)$ (distributive law),
   which is equivalent to $P \vee Q$ (tautology law).
   The last step uses the fact that $P \vee \neg P$ is a tautology.
2. $\neg(P \vee (Q \wedge \neg R)) \wedge Q$
   is equivalent to $(\neg P \wedge \neg(Q \wedge \neg R)) \wedge Q$ (DeMorgan's law),
   which is equivalent to $(\neg P \wedge (\neg Q \vee \neg\neg R)) \wedge Q$ (DeMorgan's law),
   which is equivalent to $(\neg P \wedge (\neg Q \vee R)) \wedge Q$ (double negation law),
   which is equivalent to $\neg P \wedge ((\neg Q \vee R) \wedge Q)$ (associative law),
   which is equivalent to $\neg P \wedge (Q \wedge (\neg Q \vee R))$ (commutative law),
   which is equivalent to $\neg P \wedge ((Q \wedge \neg Q) \vee (Q \wedge R))$ (distributive law),
   which is equivalent to $\neg P \wedge (Q \wedge R)$ (contradiction law).

The last step uses the fact that $Q \wedge \neg Q$ is a contradiction. Finally, by the associative law for $\wedge$ we can remove the parentheses without making the formula ambiguous, so the original formula is equivalent to the formula $\neg P \wedge Q \wedge R$.

Exercises

*1. Make truth tables for the following formulas:
 (a) $\neg P \vee Q$.
 (b) $(S \vee G) \wedge (\neg S \vee \neg G)$.
2. Make truth tables for the following formulas:
 (a) $\neg[P \wedge (Q \vee \neg P)]$.
 (b) $(P \vee Q) \wedge (\neg P \vee R)$.
3. In this exercise we will use the symbol $+$ to mean *exclusive or*. In other words, $P + Q$ means "P or Q, but not both."
 (a) Make a truth table for $P + Q$.
 (b) Find a formula using only the connectives $\wedge$, $\vee$, and $\neg$ that is equivalent to $P + Q$. Justify your answer with a truth table.
4. Find a formula using only the connectives $\wedge$ and $\neg$ that is equivalent to $P \vee Q$. Justify your answer with a truth table.
*5. Some mathematicians use the symbol $\downarrow$ to mean *nor*. In other words, $P \downarrow Q$ means "neither P nor Q."
 (a) Make a truth table for $P \downarrow Q$.
 (b) Find a formula using only the connectives $\wedge$, $\vee$, and $\neg$ that is equivalent to $P \downarrow Q$.
 (c) Find formulas using only the connective $\downarrow$ that are equivalent to $\neg P$, $P \vee Q$, and $P \wedge Q$.
6. Some mathematicians write $P \mid Q$ to mean "P and Q are not both true." (This connective is called *nand*, and is used in the study of circuits in computer science.)
 (a) Make a truth table for $P \mid Q$.
 (b) Find a formula using only the connectives $\wedge$, $\vee$, and $\neg$ that is equivalent to $P \mid Q$.
 (c) Find formulas using only the connective $\mid$ that are equivalent to $\neg P$, $P \vee Q$, and $P \wedge Q$.
*7. Use truth tables to determine whether or not the arguments in exercise 7 of Section 1.1 are valid.
8. Use truth tables to determine which of the following formulas are equivalent to each other:
 (a) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.
 (b) $\neg P \vee Q$.
 (c) $(P \vee \neg Q) \wedge (Q \vee \neg P)$.
 (d) $\neg(P \vee Q)$.
 (e) $(Q \wedge P) \vee \neg P$.
*9. Use truth tables to determine which of these statements are tautologies, which are contradictions, and which are neither: