Asymptotic Spectral Flow

Xianzhe Dai, Yihan Li

Abstract: In this paper we studied the asymptotic behavior of the spectral flow of a one-parameter family $\{D_s\}$ of Dirac operators acting on the spinor bunldle S twisted by a vector bundle E of rank k, with the parameter $s \in [0, r]$ when r gets sufficiently large. Our method uses the variation of eta invariant and local index theory technique. The key is a uniform estimate of the eta invariant $\bar{\eta}(D_r)$ which is established via local index theory technique and heat kernel estimate.

1 Introduction

Spectral flow of a one-parameter family of Dirac operators is first introduced by Atiyah-Patodi-Singer in their index theorem for Dirac operators over manifolds with boundary[1] [2] [3]. It is closely related to the η-invariant, which is defined for a Dirac operator in the same theorem as the boundary correction term. Both spectral flow and η-invariant has been found significant application in diverse fields in mathematics and physics.

In this paper, we will consider a closed manifold M of an odd dimension n equipped with its spinor bundle S and an hermitian vector bundle E of rank k. Let ∇^E be a unitary connection on E and a a End$E$-valued one-form on M which is unitary with respect to the hermitian metric. Define a one-parameter family $\{\nabla_s = \nabla \otimes 1 + 1 \otimes \nabla^E + 1 \otimes sa\}_{s\in[0,r]}$ of unitary connctions on the bundle $\mathscr{E} = S \otimes E$, where ∇ denotes the connection on S induced by the Levi-Civita connection. Then this family of connetions induces a one-parameter family $\{D_s\}$ of Dirac operators on $\mathscr{E}$. For simplicity we write the family of connection as $\{\nabla_s = \nabla_0 + s\hat{a}\}_{s\in[0,1]}$ by setting $\hat{a} = ra$, and accordingly $D_s = D_0 + sc(\hat{a})$, and denote by F_s the curvature of ∇_s (note the rescaled s). The main problem discussed in this paper is the asymptotic behavior of the spectral flow of this family. This problem is initiated by Taubes in his proof of Weinstein conjecture in dimension 3 [13] and is later discussed for the general cases in [12]. Our main result, stated below, provides improvement of the estimate in [12] for the general cases.

Theorem 1. *Let M be an odd dimensional compact spin manifold, and D_0 be a Dirac operator on it, and $D_s = D_0 + sc(\hat{a})$, $0 \leqslant s \leqslant 1$, be the smooth curve of Dirac operators, where $\hat{a} = ra$ is a Lie algebra $\mathfrak{u}(k)$-valued 1-form on M with parameter $r > 0$. Denote by*

$R = \sup_M\{|F_1|\}$, *then there exists a constant* $C > 0$, *such that the spectral flow satisfies*

$$|\mathrm{sf}\{D_s, [0,1]\} - (\frac{1}{2\pi\sqrt{-1}})^{\frac{n+1}{2}} \int_0^1 \int_M tr_{\mathbb{C}^k}[\hat{a} \wedge \hat{A}(M) \wedge e^{F_s}]_n ds| \leqslant cR^{\frac{n}{2}}$$

Generally speaking, for a one-parameter family $\{D_s\}_{s\in[0,1]}$ of Dirac operators, spectral flow is the net number of eigenvalues that change sign while the parameter s varies from 0 to 1. On the other hand, the η-invariant $\eta(D_s)$ for each single Dirac operator is a measure of its spectral asymmetry. From this point of view, spectral flow is naturally related to the change of η-invariant. To be more precise, the relation between spectral flow and the change of (reduced) eta invariant over the interval $[0,1]$, is as follows:

$$\mathrm{sf}\{D_s\} = -\int_0^1 \frac{d\bar{\eta}(D_s)}{ds} ds + \bar{\eta}(D_1) - \bar{\eta}(D_0), \tag{1}$$

where $\bar{\eta}(D_s) = \frac{1}{2}(\eta(D_s) + dimKer(D_s))$ is called the reduced η-invariant in the sense of Atiyah-Patodi-Singer as in[1]. Based on this relation, the proof of Theorem 1 consists of the explicit calculation of the variation $\frac{d}{ds}\bar{\eta}(D_s)$ via local index theory technique and the uniform estimate of $\eta(D_1)$.

The relation (1) can also be viewed as a special case of the Atiyah-Patodi-Singer index theorem by applying it on $M \times [0,1]$. This viewpoint indicates that the integrand in the first term on its right-hand side, the variation of eta invariant, can be calculated explicitly using the technique of local index theorem, Getzler rescaling introduced in [7]. The section 3 provides the detail of the calculation which is stated without proof in [12]. Furthermore the property of Getzler rescaling also plays a crucial role in the estimate of eta invariant.

The second part of the estimate, done in Section 4, is actually focusing on $\bar{\eta}(D_1)$ and its dependence on the parameter r, which is considered to be large enough. Based on the formula

$$\eta(D_1) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} Tr[D_1 e^{tD_1^2}] dt,$$

the estimate is separated into two parts: the small-time part $\int_0^{t_0} t^{-\frac{1}{2}} Tr[D_1 e^{D_1^2}]dt$, and the large-time part $\int_{t_0}^\infty t^{-\frac{1}{2}} Tr[D_1 e^{D_1^2}]dt$. In the first part, the basic idea is applying Getzler rescaling to approximate the trace $Tr[D_1 e^{tD_1^2}]$ for small time t by some rescaled heat kernel at the fixed "time" 1. Inspired by the work of Dai-Liu-Ma [5], we developed a unifrom estimate for such a family of "rescaled" heat kernel and therefore provide the estimate of the small-time part. On the other hand, the large-time part, together with $dimKerD_1$, can be controlled by $Tr[e^{-t_0 D_1^2}]$, whose estimate could be provided by the heat kernel estimate in [12]. The main result of this part can be summarized as

Theorem 2. *Let* M *be an odd dimensional compact spin manifold, and* D *be a Dirac operator acting on the bundle* $S \otimes E$, *a be a Lie algebra* $\mathfrak{u}(k)$*-valued one-form on* M, *and* $r > 0$. *Denote by* $R = \sup_M\{|F_1|\}$. *Then there exists a constant* $C' > 0$, *such that*

$$|\bar{\eta}(D + rc(a))| \leqslant C' R^{\frac{n}{2}}$$

when $r > 0$ is sufficiently large.

This problem, the asymptotic of eta invariant, has also been discussed recently by Savale in [10] and [11], where he discussed in detail the asymptotic of the eta invariant of the Dirac operators D_s acting on $S \otimes L$ where L is a line bundle. In [10], it is shown that $\bar{\eta}(D+rc(a)) \sim o(r^{\frac{n}{2}})$. In 2018, he has improved the estimate of the eta invariant in [11] to $\eta(D+rc(a)) \sim O(r^{\frac{n-1}{2}})$ under some extra assumptions. In this case, the result of Theorem 2 implies that $\bar{\eta}(D+rc(a)) \sim O(r^{\frac{n}{2}})$.

2 Preliminaries

This chapter is a short review of some well-known results on Clifford algebras and the spin geometry that is involved in the proof of the main result

2.1 Clifford Algebra over odd-dimensional vector space

Definition 2.1. *Let V be a vector space over a field $\mathbf{K}$ of dimension n endowed with a non-degenerate bilinear form g. The Clifford algebra $Cl(V,g)$ associated to g on V is an associative algebra with unit, defined by*

$$Cl(V,g) := T(V)/I(V,g)$$

where $T(V) = \bigoplus_{r=1}^{\infty}(\otimes^r V)$ is the tensor algebra of V, and $I(V,g)$ is the ideal generated by all the elements of the form $v \otimes v + g(v,v)1$.

Remark 2.2.

1. *Given $x, y \in V$ viewed as elements in the Clifford algebra, they satisfy the relation*

$$x \cdot y + y \cdot x = -2g(x,y)1$$

2. *Given a g-orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of V, the following set*

$$\{e_{i_1} \cdot e_{i_2} \cdot \cdots \cdot e_{i_k} | 1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_k \leqslant n, 0 \leqslant k \leqslant n\}$$

is a basis of $Cl(V,g)$, therefore $\dim Cl(V,g) = 2^n$.

Notice from the remark above, Clifford algebra has the same dimension as the exterior algebra $\wedge^* V$. Furthermore, there is a canonical isomorphism of vector spaces between them, which provides an essential preliminary for Getzler rescaling.

Propostion 2.3. *The Clifford algebra $Cl(V,g)$ can be identified with the exterior algebra by the canonical isomorphism of vector spaces given by:*

$$\wedge^* V \to Cl(V,g)$$
$$e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_k} \mapsto e_{i_1} \cdot e_{i_2} \cdot \cdots \cdot e_{i_k}$$

Furthermore, the Clifford multiplication can be determined by the following formula: for all $v \in V$ and $\varphi \in Cl(V,g)$,

$$v \cdot \varphi = v^* \wedge \varphi - i(v)\varphi \tag{2}$$

2.2 Spinor Bundle and Dirac Operator

Another essential relation is between the complex Clifford algebra $\mathbb{C}l(n)$ and endomorphism group $End(S_n)$ of the complex hermitian space S_n of spinors defined as follows.

Definition 2.4. *The vector space of complex n-spinors is defined to be*

$$S_n = \mathbb{C}^{2^{[\frac{n}{2}]}}$$

where $[\frac{n}{2}]$ denotes the integer part of $\frac{n}{2}$. Elements of S_n are called complex spinors.

Propostion 2.5. *The space of complex spinors are characterized by*

$$\mathbb{C}l(n) = \begin{cases} End(S_n) & \textit{if } n \textit{ is even;} \\ End(S_n) \oplus End(S_n) & \textit{if } n \textit{ is odd;} \end{cases} \tag{3}$$

Because the main object in this paper, our discussion is in particular for n odd. One can easily verify that the trace Tr behaves on the oodd elements of $\mathbb{C}l(n)$ in exactly the same way as supertrace Tr_s on the even elements of $\mathbb{C}l(n)$ for n even, i.e.

$$tr[1] = 2^{[\frac{n}{2}]}, \qquad tr[c(e_1)\dots c(e_n)] = 2^{[\frac{n}{2}]}(-\sqrt{-1})^{\frac{n+1}{2}} \tag{4}$$

and that the trace of the other monomials in $\mathbb{C}l(n)$ is 0. This is an essential observation related with Getzler rescaling.

2.3 Spectral Flow and η-invariant

The concept of spectral flow for a smooth family of Dirac operators is first introduced by Atiyah-Patodi-Singer in their study of index theory on manifolds with boundary in [1][3].

Definition 2.6. *If D_s, $0 \leqslant s \leqslant 1$ is a curve of self-adjoint Fredholm operators, the spectral flow* $\mathrm{sf}\{D_s\}$ *counts the net number of eigenvalues of D_s which change sign when s varies from 0 to 1.*

It was summarized in [6] the following properties of spectral flow.

Propostion 2.7. *The spectral flow has the following properties:*

(1) If D_s, $0 \leqslant s \leqslant 1$, is a curve of self-adjoint Fredholm operators, and $\tau \in [0,1]$, then

$$\mathrm{sf}\{D_s,\ [0,1]\} = \mathrm{sf}\{D_s,\ [0,\tau]\} + \mathrm{sf}\{D_s,\ [\tau,1]\}. \tag{5}$$

(2) If D_s, $0 \leqslant u \leqslant 1$, is a smooth curve of self-adjoint elliptic pseudodifferential operators on a closed manifold, and $\bar{\eta}(D_s) = \frac{1}{2}(\eta(D_s) + dim\, Ker D_s)$ is the reduced η invariant of D_s in the sense of Atiyah-Patodi-Singder, then η is smooth mode $\mathbb{Z}$ and

$$\mathrm{sf}\{D_s\} = -\int_0^1 \frac{d\bar{\eta}(D_s)}{ds} ds + \bar{\eta}(D_1) - \bar{\eta}(D_0). \tag{6}$$

(3) If D_s, $0 \leqslant s \leqslant 1$, is a periodic one-parameter family of self adjoint Dirac-type operators on a closed manifold, and $\tilde{D}$ is the corresponding Dirac-type operator on the mapping torus, then

$$\mathrm{sf}\{D_s\} = \mathrm{ind}\tilde{D} \tag{7}$$

2.4 Heat Kernel and η-invariant

First of all, for any single Dirac operator, we first define the corresponding η-function as

$$\begin{aligned}\eta_{D_s}(z) &= \sum_{\lambda \in Spec\{D_s\}} sgn(\lambda)|\lambda|^{-z} = \sum_{\lambda \in Spec\{D_s\}} \frac{\lambda}{|\lambda|}|\lambda|^{-z} \\ &= \sum_{\lambda \in Spec\{D_s\}} \lambda|\lambda|^{-z-1},\end{aligned}$$

which can be written as

$$\eta_{D_s}(z) = \sum_{\lambda \in Spec\{D_s\}} \lambda(\lambda^2)^{-\frac{z+1}{2}}.$$

It can be shown that this function has a meromorphic extension on the complex plane for $z \in \mathbb{C}$, and, in particular, it is analytic at $z = 0$ which allows us to define the η-invariant by evaluating the η-function at $z = 0$ as

$$\eta(D_s) = \eta_{D_s}(0).$$

Note that for any real number $\lambda \neq 0$, one has

$$\lambda^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-t\lambda} dt$$

We can write

$$\begin{aligned}\eta_{D_s}(z) &= \sum_{\lambda_s \in Spec(D_s)} \lambda_s \frac{1}{\Gamma(\frac{z+1}{2})} \int_0^\infty t^{\frac{z-1}{2}} e^{-t\lambda_s^2} dt \\ &= \frac{1}{\Gamma(\frac{z+1}{2})} \int_0^\infty t^{\frac{z-1}{2}} \sum_{\lambda_s \in Spec(D_s)} \lambda_s e^{-t\lambda_s^2} dt \\ &= \frac{1}{\Gamma(\frac{z+1}{2})} \int_0^\infty t^{\frac{z-1}{2}} Tr[D_s e^{-tD_s^2}] dt.\end{aligned}$$

It follows immediately that the η-invariant $\eta(D_s)$ can be written as

$$\eta(D_s) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} Tr[D_s e^{-tD_s^2}] dt.$$

And, also, one has

$$\frac{d}{ds}\eta_{D_s}(z) = -\frac{z}{\Gamma(\frac{z+1}{2})} \int_0^\infty t^{\frac{z-1}{2}} Tr[D_s' e^{-tD_s^2}] dt \tag{8}$$

These equations naturally involves the heat operator e^{-tD^2} and allow us to use heat kernel.

Our motivation to look at the heat kernel comes from the Lidskii's Theorem which identified the trace of an operator with the integral of the trace of its kernel.

Theorem 2.8. *If $T : L^2 \to L^2$ is defined by a continuous kernel function $T(x;y)$ by $(Tf)(x) = \int_M T(x,y)f(y)dy$, then*

$$Tr(T) = \int_M trT(x,x)dx$$

This two formulas relates the η-invariant closely to the heat operator $e^{-tD_u^2}$ and the kernel $K_{D_u^2}(t;x,y)$ of it defined by the following equaiton.

$$(e^{-tD_u^2}s)(x) = \int_M K_{D_u^2}(t;x,y)s(y)dvol_y \tag{9}$$

Therefore, in the following sections, it will mainly involves the estimate of $Tr[D_s' e^{-tD_s^2}]$ and $Tr[D_1 e^{-tD_1}]$, which will be done by dealing with their corresponding kernels.

3 Variation of η-invariant

As mentioned above, the first term $\int_{0^1} \frac{d\bar{\eta}(D_s)}{ds} ds$ (6) can be viewed as the integral term in the APS index theorem ([1]) where the variation of $\bar{\eta}(D_s)$ corresponds to the integrand. This observation motivates us to apply the technique for the local index theorem introduced by Getzler in [7] to provide an explicit calculation which is done as follows.

3.1 Variation and the Trace of Heat Kernels

In this section we proved the following lemma which related the problem to the trace of a heat kernel. Furthermore, this shows an essential observation that the variation of η-invariant is actually a local invariant, which allows us to apply the technique used in local index theorem.

Lemma 3.1. *If $\{D_s\}_{s\in[0,1]}$ is a one-parameter family of Dirac operators and $Tr[D_s' e^{-tD_s^2}]$ has the asymptotic expansion for t near 0 as*

$$Tr[D_s' e^{-tD_s^2}] = t^{-\frac{n}{2}}\Phi_{-\frac{n}{2}} + t^{-\frac{n}{2}+1}\Phi_{-\frac{n}{2}+1} + \cdots + t^{-\frac{1}{2}}\Phi_{-\frac{1}{2}} + O(t^{\frac{1}{2}}), \tag{10}$$

then

$$\frac{d}{ds}\eta(D_s) = -\frac{\Phi_{-\frac{1}{2}}}{\sqrt{\pi}} \tag{11}$$

Proof. Starting from the formula (8)

$$\frac{d}{ds}\eta_{D_s}(u) = -\frac{u}{\Gamma(\frac{u+1}{2})}\int_0^\infty t^{\frac{u-1}{2}} Tr[D_s' e^{-tD_s^2}]dt,$$

the integral can be separated as

$$\int_0^\infty t^{\frac{u-1}{2}} Tr[D_s' e^{-tD_s^2}]dt = (\int_0^1 + \int_1^\infty) t^{\frac{u-1}{2}} Tr[c(\hat{a})e^{-tD_s^2}]dt.$$

The second term $\int_1^\infty t^{\frac{u-1}{2}} Tr[D_s' e^{-tD_s^2}]dt$ is analytic as a function of u. Whereas for $t \in (0,1)$, applying the asymptotic expansion

$$Tr[D_s' e^{-tD_s^2}] = t^{-\frac{n}{2}}\Phi_{-\frac{n}{2}} + t^{-\frac{n}{2}+1}\Phi_{-\frac{n}{2}+1} + \cdots + t^{-\frac{1}{2}}\Phi_{-\frac{1}{2}} + O(t^{\frac{1}{2}}),$$

This allows us to write

$$\begin{aligned} \int_0^1 t^{\frac{u-1}{2}} Tr[c(\hat{a})e^{-tD_s^2}]dt &= \sum_{i=0}^{\frac{n-1}{2}} \Phi_{-\frac{n}{2}+i}\int_0^1 t^{u-\frac{n}{2}+i-1} \\ &= \sum_{i=0}^{-\frac{n-1}{2}} \frac{\Phi_{-\frac{n}{2}+i}}{u-\frac{n-1}{2}+i} + \phi(u) \end{aligned} \tag{12}$$

for $Re(u)$ large enough, and the function $\phi(u)$ is holomorphic for $Re(u) > -\frac{1}{2}$. This formula also gives a meromorphic extension with only simple poles at $-\frac{n-1}{2}, -\frac{n-3}{2}, \ldots, 0$. Therefore, setting $u = 0$, it follows that

$$\frac{d}{ds}\eta(D_s) = \frac{\Phi_{-\frac{1}{2}}}{\sqrt{\pi}} \tag{13}$$

$\square$

The next thing to be shown is

Propostion 3.2. *The coefficient of $t^{-\frac{1}{2}}$ in the asymptotic expansion can be computed explicitly by*

$$\begin{aligned}\Phi_{-\frac{1}{2}} &= \lim_{t\to 0} t^{\frac{1}{2}} Tr[c(\hat{a})e^{-tD_s^2}] \\ &= \int_M \lim_{t\to 0} t^{\frac{1}{2}} c(\hat{a}) K_{D_s^2}(t;x,x) dvol_x\end{aligned}$$

Remark 3.3. *The second equality above follows from Lidskii's Theorem 2.8. Once the limit $\lim_{t\to 0} t^{\frac{1}{2}} c(\hat{a}) K_{D_s^2}(t;x,x)$ has been computed, the first equality would automatically hold.*

3.2 Localization of the Problem

It now follows from the construction of the asymptotic expansion of heat kernel that the coefficients $\Phi_{-\frac{n}{2}}, \Phi_{-\frac{n}{2}+1}, \ldots,$ are determined by local information [9]. This allows us to localize the problem as follows.

Fix a point x and let $X \in \mathbb{R}^n$ be the normal coordinates around x with local orthonormal frame $\{e_i\}_{i=0}^n$. Let $a > 0$ be the injectivity radius of the manifold (M, g), and $\delta \in (0, \frac{a}{4})$. We denote by $B^M(x, \delta)$ and $B^{T_xM}(0, \delta)$ the open balls in M and T_xM wieh center x_0 and radius δ, respectively. Then the exponential map $\exp_x : T_xM \to M$ is a diffeomorphism from $B^{T_xM}(0, \delta)$ to $B^M(x, \delta)$. From now on, we identify $B^{T_xM}(0, \delta)$ with $B^M(x, \delta)$. Thus $X = 0$ at x. Since M is compact, there exists $\{x_i\}_{i=0}^k$ s.t. $\{B_M(x_i, \delta)\}_{i=0}^k$ is an open covering of M. We can also identify $B^M(x_i, \delta)$ with $B^{T_{x_i}M}(0, \delta)$.

We now fix a point x_0 and let $X \in \mathbb{R}^n$ be the normal coordinates around x_0. Furthermore, we trivialize the bundle $\mathscr{E}$ in the normal neighborhood by parallel translation along radial geodesics from x_0. In fact, by extending everything trivially outside the normal neighborhood, we can assume that $M = \mathbb{R}^n$ with a metric which is Euclidean outside a compact set. The bundle is now trivialized as $\mathbb{R}^n \times S_n \otimes \mathbb{C}^k$.

From now on, we consider heat kernel as an element in $C^\infty(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n; End(S_n) \otimes End(\mathbb{C}^k))$ defined for $s \in C^\infty(\mathbb{R}^n, S_n \otimes \mathbb{C}^k)$ by

$$(e^{-tD_s^2} s)(X) = \int_{\mathbb{R}^n} K(t; X, Y) s(Y) dvol. \tag{14}$$

Since $End(S_n) = \mathbb{C}l(n)$, it allows us to write

$$K(t; X, Y) s(Y) = \sum_I (a_I(t; X, Y) c(e_I) s)(X) \tag{15}$$

In particular on the diagonal, we have

$$K(t; X, X) = \sum_I a_I(t; X) c(e_I) \in End(S_n) \otimes End(\mathbb{C}^k) \tag{16}$$

This expression will allow us to apply Getzler rescaling and provides an essential observation in the Lemma 3.4.

3.3 Getzler Rescaling and the McKean-Singer Cancellation

Let $\{e_i\}_{i=1}^n$ be a local orthonormal frame around x_0, and $I = (i_1, i_2, \ldots, i_k)$ be multi-index. The heat kernel $K(t, X, X)$ as an endomorphism on the fibre $\mathscr{S}_x$, it can be written as

$$K_{D_s^2}(t, X, X) = \sum_{|I| \text{ even}} a_I(t; X)c(e_I), \tag{17}$$

where $a_I(t, X) \in End(\mathbb{C}^k)$

Hence, for the operator $D_s' e^{-tD_s^2}$, whose kernel is exactly $c(\hat{a})K_{D_s^2}(t; X, Y)$. It follows from the formula (17) that

$$\begin{aligned} t^{\frac{1}{2}}\sqrt{-1}c(\hat{a})K_{D_s^2}(t; X, X) &= c(\hat{a}) \sum_{|I| \text{ even}} a_I(t; X)c(e_I) \\ &= \sum_{|I| \text{ odd}} b_I(t; X)c(e_I) \end{aligned} \tag{18}$$

In the Clifford algebra $C(\mathbb{R}^n)$ with n odd, there is a property of essential importance for us shown in [4]:

$$tr[1] = 2^{\frac{n-1}{2}}, tr[e_1 \ldots e_n] = 2^{\frac{n-1}{2}}(-\sqrt{-1})^{\frac{n+1}{2}} \tag{19}$$

while the trace of the other monomials in $\mathbb{C}l(n)$ is 0. Let's denote the multi-index $(1, 2, \ldots, n)$ by I_n

So it follows that only the top degree term contribute to the trace of the kernel, i.e.

$$tr[c(\hat{a})K_{D_s^2}(t; X, X)] = tr_{\mathbb{C}^k}[b_{I_n}(t; X)]tr[c(e_{I_n})] = 2^{\frac{n-1}{2}}(-\sqrt{-1})^{\frac{n+1}{2}} tr_{\mathbb{C}^k}[b_{I_n}(t; X)] \tag{20}$$

The Getzler rescaling could also possess a similar property, capturing the McKean-Singer "fantastic cancellation". In addition to its action on function on the coordinate components for a function f on $\mathbb{R}_+ \times \mathbb{R}^n$ as $(\delta_\varepsilon f)(t, x) = f(\varepsilon t, \varepsilon^{\frac{1}{2}}x)$, it also applies a rescaling on the elements of Clifford algebra, which incorporates the Clifford degrees.

Applying the identification between Clifford algebra and exterior algebra, we can As mentioned above, Clifford algebra could be identified with the exterior algebra $\wedge^* V$, with the Clifford multiplication defined in 2.3. From now on, we can identify $\mathbb{C}l(T_{x_0}M)$ with $\wedge^*(T_{x_0}M)$ and the effect of Getzler rescaling on Clifford algebra could be determined by

$$\delta_\varepsilon(e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p}) = \varepsilon^{-\frac{p}{2}} e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p} \tag{21}$$

This will induce its effect on Clifford multiplication

$$c_\varepsilon(e_i) = \delta_\varepsilon c(e_i)\delta_\varepsilon^{-1} = \varepsilon^{-\frac{1}{2}} e^i \wedge -\varepsilon^{\frac{1}{2}}\iota(e_i) \tag{22}$$

Taking limit of the a rescaled Clifford multiplication, we have

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{n}{2}} \delta_\varepsilon(c(e_I))\delta_\varepsilon^{-1} = \begin{cases} 0, & \text{if } |I| < n \\ e^1 \wedge e^2 \wedge \cdots \wedge e^n, & \text{if } I = (1, 2, \ldots, n) \end{cases} \tag{23}$$

This provides the property of Getzler rescaling that captures the McKean-Singer cancellation, stated as following:

Lemma 3.4. *For any* $a(t;x) = \sum_{|I|\ odd} a_I(t;x)c(e_I)$,

$$\lim_{t\to 0} Tr(a(t;0))dvol = 2^{(n-1)/2}(-\sqrt{-1})^{(n+1)/2} \lim_{\varepsilon\to 0} \varepsilon^{n/2}[(\delta_\varepsilon a_I)(t;x)c_\varepsilon(e_I)]_{(n)};$$

where $[]_{(n)}$ *means taking the degree* n *term (the top degree term), and* $c_\varepsilon(e_I) = \delta_\varepsilon(c(e_I))\delta_\varepsilon^{-1}$.

3.4 Variation of η-invariant

Apply the lemma above, we have

Lemma 3.5. *The trace of the kernel function* $c(\hat{a})K_{D_s^2}$ *can be calculated by the following equation*

$$Tr[c(\hat{a})K_{D_s^2}] = 2^{(n-1)/2}(-\sqrt{-1})^{(n+1)/2} tr_{\mathbb{C}^k}[\lim_{\varepsilon\to 0} \varepsilon^{n/2} t^{\frac{1}{2}} c_\varepsilon(\hat{a})(\delta_\varepsilon K_{D_s^2})(t;X,X)] \tag{24}$$

On the other hand, $\varepsilon^{\frac{n}{2}}\delta_\varepsilon(c(\hat{a})K_{D_s^2})(t;X,X)$ is the kernel of the rescaled operator $\varepsilon\delta_\varepsilon D_s^2\delta_\varepsilon^{-1}$, whose limit can be computed explicitly as follows.

As beginning, we have the Lichnerowicz formula for the operator D_s.

Propostion 3.6 (Lichnerowicz Formula)**.** *For the Dirac operators* D_s, $s \in [0,1]$

$$D_s^2 = \nabla_s^* \nabla_s + c(F_s) + \frac{K}{4},$$

where F_s *is the curvature of the connection* A_s *on the vector bundle* E *and* $c(F_s) = \sum_{i<j} F_s(e_i,e_j)c(e_i)c(e_j)$.

Based on this formula, it can be shown that

Propostion 3.7. *The limit of the rescaled operator* $\varepsilon\delta_\varepsilon(D_s^2)\delta_\varepsilon^{-1}$ *as* ε *approached to 0 is:*

$$\lim_{\varepsilon\to 0} \varepsilon\delta_\varepsilon(D_s^2)\delta_\varepsilon^{-1} = \mathscr{L} + F_s, \tag{25}$$

where

$$\mathscr{L} = -(\partial_{x^i} + \frac{1}{4}\Omega_{ij}x^j)^2$$

is the generalized harmonic oscillator.

Proof. Applying Getzler rescaling on the operator D_s^2, it follows from the Lichnerowicz formula that

$$\delta_\varepsilon(D_s^2)\delta_\varepsilon^{-1} = \varepsilon\delta_\varepsilon(\nabla_s^*\nabla_s)\delta_\varepsilon^{-1} + \frac{\varepsilon R}{4} + \varepsilon c_\varepsilon(F)$$

It is not hard to see that

$$\lim_{\varepsilon\to 0}\varepsilon c_\varepsilon(F_s) = F_s \wedge .$$

On the other hand, noticing that the curvature of the connection ∇_s is

$$\begin{aligned}\nabla_s^2 &= \nabla^2 \otimes 1 + 1 \otimes F_s \\ &= \frac{1}{2}\sum_{i<j,k<l} R_{ijkl}c(e^k)c(e^l) + F_s,\end{aligned}$$

the action of the covariant derivative ∇_{s,e_i} can be expressed in local coordinate as

$$\nabla_{s,e_i} = \partial_{x^i} + \frac{1}{4}\sum_{j,k<l} R_{ijkl}x^j + g(x) + O(|x|^2),$$

where $R_{jikl} = (R(e_i,e_j)e_k, e_l)$ is the Riemann curvature tensor on M at the point x_0, and $g(x) = O(|x|) \in \mathbb{C}^\infty(\mathbb{R}^n, End(E))$.

Thus

$$\lim_{\varepsilon\to 0}\varepsilon^{\frac{1}{2}}\delta_\varepsilon(\nabla_{s,e_i})\delta_\varepsilon^{-1} = \partial_{x^i} + \frac{1}{4}\sum_{j,k<l} R_{ijkl}x^j + O(\varepsilon^{\frac{1}{2}})$$

Therefore,

$$\begin{aligned}\lim_{\varepsilon\to 0}\varepsilon\delta_\varepsilon(\nabla_s^*\nabla_s)\delta_\varepsilon^{-1} &= \lim_{\varepsilon\to 0}\varepsilon\delta_\varepsilon(\nabla_{s,e_i}\nabla_{s,e_i} - \nabla_{s\nabla_{s,e_i}e_i})\delta_\varepsilon^{-1} \\ &= -(\partial_{x^i} + \frac{1}{4}\sum_{j,k<l} R_{ijkl}x^j)^2\end{aligned}$$

Therefore, we have

$$\lim_{\varepsilon\to 0}\varepsilon\delta_\varepsilon(D_s^2)\delta_\varepsilon^{-1} = \mathscr{L} + F_s,$$

where

$$\mathscr{L} = -(\partial_{x^i} + \frac{1}{4}\Omega_{ij}x^j)^2, \text{ with } \Omega_{ij} = \sum_{k<l} R_{ijkl}e^k \wedge e^l$$

is the generalized harmonic oscillator. $\square$

Using the lemma

Lemma 3.8. *The heat kernel of the generalized harmonic oscillator is given by*

$$K_{\mathscr{L}}(t;X,0) = \frac{1}{(4\pi t)^{\frac{n}{2}}}det^{\frac{1}{2}}(\frac{t\Omega/2}{\sinh\Omega/2})exp(-\frac{1}{4t}(\frac{t\Omega/2}{\tanh t\Omega/2})_{ij}x^ix^j)$$

Thus the heat kernel of the limit operator is

$$K_0(t;x,0) = K_{\mathscr{L}}(t;x,0)e^{-tF_s}$$

Thus we have

Propostion 3.9. *The variation of η-invariant is*

$$\frac{d}{ds}\bar{\eta}(D_s) = (\frac{1}{2\pi\sqrt{-1}})^{\frac{n+1}{2}} \int_M [\hat{a} \wedge \hat{A}(\frac{\Omega}{2\pi}) \wedge exp(F_s)]_{(n)} \tag{26}$$

Proof. Firstly, it follows from the lemma above that

$$\begin{aligned}
\lim_{t\to 0} t^{\frac{1}{2}} c(\hat{a})K_{D_s^2}(t;0,0) &= (-2\sqrt{-1})^{\frac{n-1}{2}}[\lim_{\varepsilon\to 0} t^{\frac{1}{2}}\varepsilon^{\frac{n}{2}}(\delta_\varepsilon(c(\hat{a})K_{D_u^2}))(t;x,0)]_{(n)} \\
&= (\sqrt{-1})^{-1}(-2\sqrt{-1})^{\frac{n-1}{2}}[\lim_{t\to 0} t^{\frac{1}{2}}\hat{a} \wedge K_0(t,0,0)]_{(n)} \\
&= -\sqrt{-1}(-2\sqrt{-1})^{\frac{n-1}{2}}[\lim_{t\to 0} \frac{1}{(4\pi)^{\frac{n}{2}} t^{\frac{n-1}{2}}}\hat{a} \wedge det^{\frac{1}{2}}(\frac{t\Omega/2}{\sinh \Omega/2})e^{(tF_s)}]_{(n)} \\
&= \sqrt{\pi}(\frac{1}{2\pi\sqrt{-1}})^{\frac{n+1}{2}}[\hat{a} \wedge \hat{A}(\frac{\Omega}{2\pi}) \wedge exp(F_s)]_{(n)}
\end{aligned}$$

On the other hand, according to the lemma 3.1 that

$$\begin{aligned}
\frac{d}{ds}\bar{\eta}(D_s) &= \frac{1}{\sqrt{\pi}} \int_M tr_{\mathbb{C}^k}[\lim_{t\to 0} t^{\frac{1}{2}} c(\hat{a})K_{D_s^2}(t;0,0)] \\
&= (\frac{1}{2\pi\sqrt{-1}})^{\frac{n+1}{2}} \int_M tr_{\mathbb{C}^k}[\hat{a} \wedge \hat{A}(\frac{\Omega}{2\pi}) \wedge exp(F_s)]_{(n)}
\end{aligned}$$

□

When the twisted part E is a line bundle, i.e. $k = 1$, $F_s = srda$. It has been pointed out in [12] that

Remark 3.10. *When the twisted part E is a line bundle, i.e. $k = 1$, the 1-form $\hat{a}$ is purely imaginary-valued and it has been asserted in [12] that the leading order term in this part is given by*

$$(\frac{1}{4\pi\sqrt{-1}})^{\frac{n+1}{2}} \frac{1}{(\frac{n+1}{2})!} r^{\frac{n+1}{2}} \int_M a \wedge (da)^{\frac{n-1}{2}}, \tag{27}$$

which also gives the leading order term of the asymptotic spectral flow.

When $k > 1$, the 1- form $\hat{a}$ is $\mathfrak{u}(k)$-valued. First write A_0 as $A_0 = d + \omega$, then $A_s = d + \omega + ra$ and therefore

$$F_s = r^2(a \wedge a) + r(da + \omega \wedge a) + (d\omega + \omega \wedge \omega). \tag{28}$$

So, similarly, we have

Remark 3.11. *When the twisted part E is vector bundle of rank $k > 1$, the 1-form $\hat{a}$ is $\mathfrak{u}(k)$-valued and the leading order term in this part is given by*

$$(\frac{1}{4\pi\sqrt{-1}})^{\frac{n+1}{2}}\frac{1}{(\frac{n+1}{2})!}r^n\int_M tr_{\mathbb{C}^k}[a^n]. \tag{29}$$

4 The Estimate of η-Invariant

The second part of the estimate is the asymptotic of η-invariant i.e. $\eta(D_1) = \eta(D + c(\hat{a}))$. As mentioned before, we can relate η-invariant with heat operator by

$$\eta(D_1) = \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-\frac{1}{2}}Tr[D_1e^{-tD_1^2}] \tag{30}$$

Based on the following observation, we will separate the estimate into two parts as following for each part, we have a strategy to deal with it.

$$\eta(D_1) = \frac{1}{\sqrt{\pi}}(\int_{t_0}^\infty t^{-\frac{1}{2}}Tr[D_1e^{-tD_1^2}]dt + \int_0^{t_0} t^{-\frac{1}{2}}Tr[D_1e^{-tD_1^2}]dt) \tag{31}$$

For the consistency of the notation for the cases that E is a line bundle or a vector bundle, we introduce another parameter R to replace r, defined as $R = \sup_M\{|F_1|\}$, its dependence on r is as follows

$$R \sim \begin{cases} r, & \text{for } k = 1; \\ r^2 & \text{for } k > 1. \end{cases} \tag{32}$$

The choice of the constant t_0 depends on R as $t_0 = \frac{1}{2R}$. Since the case that E is a line bundle has been discussed by Savale in [10], we will mainly focus on the case E is a vector bundle of rank $k > 1$. However all the following estimate also works for that case and provides an estimate that is weaker than that in [10].

Short-time

In this part, we need an estimate of the kernel of the operator $D_1e^{-tD_1^2}$ for small time $t \in (0, t_0]$, especially for its dependence on the parameter $r > 0$. It has been pointed out in [4] that its asymptotic expansion starts with the term $t^{\frac{1}{2}}$. This guaranteed the regularity of this part. In order to get a better observation on the dependence of the parameter r, we apply Getzler rescaling to covert problem to a uniform estimate of the heat kernel of a family of rescaled Dirac operators at a fixed time 1. This kind of estimate has been done in [5]. Combining these approaches together, we can get the desired estimate.

We first introduced an auxiliary Grassmann variable to make the operator $D_1e^{-uD_1^2}$ as a part of a heat operator and then use Getzler rescaling to convert the estimate of heat kernel for small time to a uniform estimate of heat kernel at fixed time for a family of Dirac operators with a small rescaling parameter t. And the estimate is given by applying the approached introduced in [5]. An exponential is also applied to guarantee the convergence of the rescaled operators.

4.1 Localization of the Problem

Fix a point x and let $X \in \mathbb{R}^n$ be the normal coordinates around x with local orthonormal frame $\{e_i\}_{i=0}^n$. Let $a > 0$ be the injectivity radius of the manifold (M, g), and $\delta \in (0, \frac{a}{4})$. We denote by $B^M(x, \delta)$ and $B^{T_xM}(0, \delta)$ the open balls in M and T_xM wieh center x_0 and radius δ, respectively. Then the exponential map $\exp_x : T_xM \to M$ is a diffeomorphism from $B^{T_xM}(0, \delta)$ to $B^M(x, \delta)$. From now on, we identify $B^{T_xM}(0, \delta)$ with $B^M(x, \delta)$. Thus $X = 0$ at x. Since M is compact, there exists $\{x_i\}_{i=0}^k$ s.t. $\{U_i = B_M(x_i, \delta)\}_{i=0}^k$ is an open covering of M. We can also identify $B^M(x_i, \delta)$ with $B^{T_{x_i}M}(0, \delta)$.

Before the estimate, we first use the finite propagation speed to localize the problem.

Let $f : \mathbb{R} \to [0, 1]$ be a smooth even function such that:

$$f(v) = \begin{cases} 1 & \text{for } |v| \leqslant \frac{\delta}{2} \\ 0 & \text{for } |v| > \delta \end{cases} \tag{33}$$

We then define

Definition 4.1. *For $u > 0$, $z \in \mathbb{C}$, set*

$$\begin{aligned} G_u(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\sqrt{-1}vz} \exp(-\frac{v^2}{2}) f(\sqrt{u}v) dv \\ H_u(z) &= \frac{1}{\sqrt{2\pi u}} \int_{-\infty}^{+\infty} e^{\sqrt{-1}vz} \exp(-\frac{v^2}{2u})(1 - f(v)) dv \end{aligned} \tag{34}$$

Clearly,

$$G_u(\sqrt{u}D_1) + H_u(D_1) = \exp(-uD_1^2) \tag{35}$$

Let $G(u; x, y)$ and $H(u; x, y)$ be the smooth kernels associated to $G_u(\sqrt{u}D_1)$ and $H_u(D_1)$.

First we have

Propostion 4.2. *For any $m \in \mathbb{N}$, $u_0 > 0$, $\varepsilon > 0$, there exists $C > 0$ such that for any $x, y \in M$, $u \geqslant 0$,*

$$|H(u; x, y)|_{\mathcal{C}^m} \leqslant C r^{2m+2n+2} \exp(-\frac{\delta^2}{10u}) \tag{36}$$

Proof. First we have, for any $m \in \mathbb{N}$, there exists $C_m > 0$ depending on δ such that

$$\sup_{z \in \mathbb{R}} |z|^m |H_u(z)| \leqslant C_m \exp(-\frac{\delta^2}{10u}) \tag{37}$$

We now prove that for differential operators P, Q of order m, m' with compact support in U_i, U_j respectively, there exists $C > 0$ such that for $u \geqslant u_0$,

$$\|PH_u(D_1)Qs\|_{L^2} \leqslant Cr^{m+m'} \exp(-\frac{\delta^2}{10u})\|s\|_{L^2} \tag{38}$$

For this purpose we first show that

$$\|s\|_{H^{m+1}} \leqslant C'_m \sum_{j=0}^{m+1} r^{m+1-j}\|D_1^j s\|_{L^2} \tag{39}$$

Let Q be a differential operator of order $m \in \mathbb{N}$, notice that

$$[D_1, Q] = [D, Q] + [c(\hat{a}), Q] \tag{40}$$

where the first part is a differential operator of order m and the second is of order $m-1$. Also, we have

$$\|s\|_{H^1} \leqslant C(\|D_1 s\|_{L^2} + r\|s\|_{L^2}) \tag{41}$$

Apply this inequality on Qs, we have

$$\begin{aligned}\|Qs\|_{H^1} &\leqslant C(\|D_1 Qs\|_{L^2} + r\|Qs\|_{L^2}) \\ &\leqslant C(\|QD_1 s\|_{L^2} + \|[D_1, Q]s\|_{L^2} + r\|Qs\|_{L^2}) \\ &\leqslant C(\|QD_1 s\|_{L^2} + r\|Qs\|_{H^m}) \\ &\leqslant C(\|D_1 s\|_{H^m} + r\|Qs\|_{H^m})\end{aligned} \tag{42}$$

This implies that

$$\|s\|_{H^m} \leqslant C_m(\|D_1 s\|_{H^{m-1}} + r\|s\|_{H^{m-1}}) \tag{43}$$

Repeating this inequality on $\|D_1 s\|_{H^{m-1}}$, we have

$$\begin{aligned}\|s\|_{H^m} &\leqslant C_m(\|D_1 s\|_{H^{m-1}} + r\|s\|_{H^{m-1}}) \\ &\leqslant C_m((\|D_1^2 s\|_{H^{m-2}} + r\|D_1 s\|_{H^{m-2}}) + r(\|D_1 s\|_{H^{m-2}} + r\|s\|_{H^{m-2}})) \\ &\leqslant \cdots \leqslant C_m \sum_{j=0}^{m} r^{m-j}\|D_1^j s\|_{L^2}\end{aligned} \tag{44}$$

Apply this to $PH_u(D_1)Qs$, there exists a constant $C > 0$

$$\begin{aligned}\|PH_u(D_1)Qs\|_{L^2} &\leqslant C\|H_u(D_1)Qs\|_{H^m} \\ &\leqslant C\sum_{j=0}^{m} r^{m-j}\|D_1^j H_u(D_1)Qs\|_{L^2} \\ &\leqslant C\sum_{j=0}^{m} r^{m-j}\|Q^* D_1^j H_u(D_1)s\|_{L^2}\end{aligned} \tag{45}$$

And once again, we have for each $0 \leqslant j \leqslant m$

$$\begin{aligned} \|Q^* D_1^j H_u(D_1)s\|_{L^2} &\leqslant C \sum_{j=0}^{m} \|D_1^j H_u(D_1)s\|_{H^{m'}} \\ &\leqslant C \sum_{j=0}^{m} \sum_{k=0}^{m'} r^{m'-k} \|D_1^j D_1^k H_u(D_1)s\|_{L^2} \end{aligned} \tag{46}$$

Combining the inequalities above, one has

$$\|PH_u(D_1)Qs\|_{L^2} \leqslant C \sum_{j=0}^{m+m'} r^{m+m'-j} \|D_1^j H_u(D_1)s\|_{L^2} \tag{47}$$

Finally, we have that for differential operators P, Q of order m, m' with compact support in U_i, U_j respectively, there exists $C > 0$ such that for $u \geqslant u_0$,

$$\|PH_u(D_1)Qs\|_{L^2} \leqslant Cr^{m+m'} \exp{(-\frac{\delta^2}{10u})} \|s\|_{L^2} \tag{48}$$

Applying Sobolev inequality we have

$$\|H_u(D_1)s\|_{\mathscr{C}^m} \leq Cr^{2n+2+m} \exp{(-\frac{\delta^2}{u})} \tag{49}$$

□

Using finite propagation speed [8], it is clear that for $x, y \in M$, $G(u; x, y)$ only depends on the restriction of D_1 to $B^M(x, \delta)$, and is zero if $d(x, y) > \delta$.

Once again, fix a point x_0 and let $X \in \mathbb{R}^n$ be the normal coordinates around x_0. Furthermore, we trivialize the bundle $\mathscr{E}$ in the normal neighborhood by parallel translation along radial geodesics from x_0 via the Levi-Civita connection ∇ and A_1 respectively. Furthermore, by extending everything trivially outside the normal neighborhood, we can assume that $M = \mathbb{R}^n$ with a metric which is Euclidean outside a compact set. The bundle is now trivialized as $\mathbb{R}^n \times (S_n \otimes \mathbb{C}^k)$.

5 Auxiliary Grassmann Variable

As in [4], we first introduce a auxiliary Grassmann variable z which anticommutes with $e_1, \dots e_n$, considered as elements of $c(TM)$. It allows us to write

$$exp(-tD_1^2 + \frac{z\sqrt{t}}{2} D_1) = e^{-tD_1^2} + z\sqrt{t} D_1 e^{-tD_1^2}$$

whose kernel is

$$P(t; x, y) = P^0(t; x, y) + z\sqrt{t} K_{D_1^2}(t; x, y) \tag{50}$$

where $P^0(t;x,y)$ is the heat kernel of D_1^2 and $K_{D_1^2}(t;x,y)$ is the kernel of the operator $D_1e^{-tD_1^2}$.

It follows from Lidskii Theorem that

$$Tr[D_1e^{-tD_1^2}] = \int_M tr[P^1(t;x,x)]dvol_x \tag{51}$$

Thus, for this kind of operators, we define Tr_z as following

Definition 5.1. *For A, $B \in c(TM)$, we define*

$$Tr_z[A+zB] = Tr[B] \tag{52}$$

Thus we have

$$Tr_z[P(t;x,x)] = \sqrt{t}Tr[P^1(t;x,x)] \tag{53}$$

On the other hand, Bismut and Freed shown in [4] that there is a C^∞ function $b_{1/2}(x)$ on M such that as $t \to 0$,

$$tr[P^1(t;x,x)] = b_{1/2}(x)t^{1/2} + O(t^{3/2},x) \tag{54}$$

and $O(t^{3/2},x)$ is uniform on M.

For our problem, we need to look further for its uniform dependence on the parameter r.

Furthermore, by introducing z, we can apply the method introduced in [5] to the operator $exp(-tD_1^2 + z\sqrt{t}D_1) = e^{-tD_1^2} + z\sqrt{t}D_1e^{-tD_1^2}$ and finally give the estimate of η-invariant. It is guaranteed by the following propositions.

Propostion 5.2. *For an operator $T = A+zB$, where A commutes with z, it is invertible if and only if A is invertible and its inverse is*

$$(A+zB)^{-1} = A^{-1} - zA^{-1}BA^{-1} \tag{55}$$

Proof. If T is invertible, assume $T^{-1} = U + zV$. Then

$$\begin{aligned} TT^{-1} &= (A+zB)(U+zV) \\ &= AU + (zBU + A(zV)) \end{aligned} \tag{56}$$

Then $AU = I$ and $zBU + A(zV) = 0$ and therefore, A is invertible. Now we can write,

$$\begin{aligned} T^{-1} &= (A+zB)^{-1} = [A(I+zA^{-1}B)]^{-1} \\ &= (I+zA^{-1}B)^{-1}A^{-1} \\ &= A^{-1} - zA^{-1}BA^{-1} \end{aligned} \tag{57}$$

Conversely, if A is invertible, the equation (5.2) gives T^{-1} explicitly. ∎

Apply this proposition to the operator $D_1^2 - \frac{z}{\sqrt{t}} D_1$ and take higher powers, we have

$$(\lambda - D_1^2 + \frac{z}{\sqrt{t}} D_1)^{-1} = (\lambda - D_1^2)^{-1} - \frac{z}{\sqrt{u}} D_1 (\lambda - D_1^2)^{-2}$$

Taking k-th power of it, it follows that

Corollary 5.3. *Let k be any positive integer,*

$$(\lambda - D_1^2 + \frac{z}{\sqrt{u}} D_1)^{-k} = (\lambda - D_1^2)^{-k} - \frac{z}{\sqrt{u}} k D_1 (\lambda - D_1^2)^{-k-1} \tag{58}$$

It then follows as in [5], that

Propostion 5.4. *For any $k \in \mathbb{N}^*$, we have*

$$e^{-uD_1^2 + z\sqrt{u}D_1} = \frac{(-1)^{k-1}(k-1)!}{2\pi\sqrt{-1}u^{k-1}} \int_\Gamma e^{-u\lambda} (\lambda - D_1^2 + \frac{z}{\sqrt{u}} D_1)^{-k} d\lambda \tag{59}$$

where $\Gamma = \{x + \varepsilon\sqrt{-1} | x \geqslant -\varepsilon\} \cup \{-\varepsilon + y\sqrt{-1} | -\varepsilon \leqslant y \leqslant \varepsilon\} \cup \{x - \varepsilon\sqrt{-1} | x \geqslant -\varepsilon\}$ is a contour with $\varepsilon > 0$

This formula is generalized from [5] for the operators with the auxiliary Grassmann variable z.

5.1 Getzler Rescaling and Exponential Transformation

As in [4], an alternative description of $tr_z[P(t; x, x)]$ is

$$P(t; x, x) = \sum_{|I| \text{ even}} a_I(t, x) c(c_I) + \sum_{|I| \text{ odd}} b_I(t, x) c(e_I) \tag{60}$$

where $I = \{i_1, i_2, \ldots, i_k\}$ is multi-index. This gives an important observation from the property of Clifford multiplication:

$$tr_z[P(t; x, x)] = 2^{\frac{n-1}{2}} (-\sqrt{-1})^{\frac{n+1}{2}} tr_{\mathbb{C}^k}[b_{1\ldots n}(t, x)] \tag{61}$$

5.2 Getzler's rescaling

Combining with Getzler rescaling, we have

Lemma 5.5.

$$tr[t^{\frac{n}{2}} (\delta_t P^1)(1, 0, 0)] = 2^{-\frac{n-1}{2}} (-\sqrt{-1})^{-\frac{n+1}{2}} tr_z[P(t, 0, 0)] e_1 \wedge e_2 \wedge \cdots \wedge e_n + O(t) \tag{62}$$

On the other hand, if we denote by $D_{1,t}$ the rescaled Dirac operator

$$D_{1,t} = t^{\frac{1}{2}} \delta_t(D_1) \delta_t^{-1} = D_t + t^{\frac{1}{2}} c_t(\hat{a}) \tag{63}$$

we have,

Lemma 5.6. *The kernel of the rescaled heat operater* $exp(-uD_{1,t}^2 + \frac{z\sqrt{u}}{2}D_{1,t})$ *is*

$$t^{\frac{n}{2}}(\delta_t P)(u;x,y) = t^{\frac{n}{2}}(\delta_t P^0)(u;x,y) + z\sqrt{u}t^{\frac{n+1}{2}}(\delta_t P^1)(u;x,y) \tag{64}$$

where the $t^{\frac{n}{2}}(\delta_t P^0)(u;x,y)$ *is the heat kernel of the operator* $D_{1,t}^2$ *and* $t^{\frac{n+1}{2}}(\delta_t P^1)(u;x,y)$ *is the kernel of the operator* $D_{1,t}e^{-uD_{1,t}^2}$

With Getzler rescaling we have

1. related the trace of the kernel function with the kernel of the rescaled operater
2. converted the estimate of heat kernel in small time to a uniform estimate of the heat kernel of a family of rescaled operators with a small rescaling parameter t.

5.3 Conjugation by the Exponential Transformation

Before doing that, we still have another obstruction, the convergence of the operator $t\delta_t(exp(-uD_1^2 + \frac{z\sqrt{u}}{2}D_1))\delta_t^{-1}$. This is solved by the exponential transformation introduced as following.

Noticing that conjugation preserves the trace of operators, we first conjugate the operator by the exponential transformation

$$e^{\frac{zx_i c(e_i)}{2}} = 1 + \frac{zx_i c(e_i)}{2} \tag{65}$$

Which gives the operator

$$L = e^{\frac{zx_i c(e_i)}{2}}(-D_1^2 + \frac{z}{\sqrt{u}}D_1)e^{-\frac{zx_i c(e_i)}{2}} = -D_1^2 + zL^1$$

where

$$L^1 = \frac{1}{\sqrt{u}}[(\langle\frac{\partial}{\partial x^j}, e_i\rangle c(e_j)\nabla_{1,e_i} - D_1) + x_i c(\nabla_{e_j} e_i)\nabla_{1,e_j} - \frac{1}{2}c(\nabla^*\nabla(x_i e_i))] \tag{66}$$

Applying Getzler rescaling now on the operator L gives

$$L_t = t\delta_t(L)\delta_t^{-1} = D_{1,t}^2 + \frac{z}{\sqrt{u}}L_t^1 \tag{67}$$

It follows that there exists a constant $C > 0$ independent of the parameters r, t.

$$\|L^t\|_t^{0,1} \leqslant Crt \tag{68}$$

From now on, we'll keep focus on the operator e^{-uL_t} with the corresponding kernel

$$K_t(u;x,y) = K_t^0(u;x,y) + z\sqrt{u}K_t^1(u;x,y)$$

where $K_t^0(u;x,y) = t^{\frac{n}{2}}(\delta_t K^0)(u;x,y)$ is the kernel of the operator $e^{-uD_{1,t}^2}$ and $K_t^1(u;x,y) = t^{\frac{n+1}{2}}(\delta_t K^1)(u;x,y)$ is the kernel of the operator

$$H^1 = D_{1,t}e^{-uD_{1,t}^2} + [\frac{zx_i c(e_i)}{2}, e^{-uD_{1,t}^2}].$$

It can be seen from this observation that the kernel $K_t^1(u,;x,y)$ has the same trace as $P_t^1(u;x,y)$. Furthermore, the operator L_t is convergent as $t \to 0$ and $tr[K_t^1(u;x,y)] = tr[P_t^1(u,x,x)]$.

To apply the approach introduced in [5], we have the corresponding lemmas for the operator L_t by applying the conjugation and Getzler rescaling.

Lemma 5.7. *For any $\lambda \in \Gamma$, where the contour Γ is defined as above, we have that the operator $\lambda - L_t$ is invertible and*

$$(\lambda - L_t)^{-k} = (\lambda - D_{1,t}^2)^{-k} + \frac{z}{\sqrt{u}}\sum_{i=1}^{k}(\lambda - D_{1,t}^2)^{-i}L_t^1(\lambda - D_{1,t}^2)^{-k+i-1} \tag{69}$$

Thus we have

Lemma 5.8. *For any $k \in \mathbb{N}^*$, one has*

$$\begin{aligned} e^{-uL_t} =& \frac{(-1)^{k-1}(k-1)!}{2\pi\sqrt{-1}u^{k-1}}\int_\Gamma e^{-u\lambda}(\lambda - L_t)^{-k}d\lambda \\ =& \frac{(-1)^{k-1}(k-1)!}{2\pi\sqrt{-1}u^{k-1}}\int_\Gamma e^{-u\lambda}(\lambda - D_{1,t}^2)^{-k}d\lambda \\ &+ \frac{z(-1)^{k-1}(k-1)!}{2\pi\sqrt{-1}u^{k-\frac{1}{2}}}\int_\Gamma e^{-u\lambda}\sum_{i=1}^{k}(\lambda - D_{1,t}^2)^{-i}L_t^1(\lambda - D_{1,t}^2)^{-k+i-1}d\lambda \end{aligned} \tag{70}$$

where $\Gamma = \{x + \varepsilon\sqrt{-1}|x \geqslant -\varepsilon\} \cup \{-\varepsilon + y\sqrt{-1}| -\varepsilon \leqslant y \leqslant \varepsilon\} \cup \{x - \varepsilon\sqrt{-1}|x \geqslant -\varepsilon\}$ is a contour with $\varepsilon > 0$ as we defined before.

So fix $x \in M$, apply the Getzler rescaling δ_t under the normal coordinate around x, when the rescale parameter t is small enough, $t^{-1}K_t^1(1;x,x) = t^{-1/2}tr[K^1(t;x,x)] + O(t^{1/2})$. So from the estimate of $K_t^1(1;x,x)$ we can give an estimate of $t^{-1/2}tr[K^1(t;x,x)]$.

5.4 A Uniform Heat Kernel Estimate near Diagonal

The setup above allowed us to use the method introduce by Dai-Liu-Ma in [5]. In this part we are going to prove

Theorem 5.9. *For any $u_0 > 0$, there exists a $C > 0$ and $N \in \mathbb{N}$ such that for any $t \in (0, t_0]$ and $u \geqslant u_0$ and $x \in T_{x_0}X = \mathbb{R}^n$,*

$$\sup_{X,Y\in\mathbb{R}^n}|K_t^1(u;X,Y)| \leqslant CrR^{\frac{n-1}{2}}t(1 + |X| + |Y|)^N exp(\varepsilon u) \tag{71}$$

$$\sup_{X,Y\in\mathbb{R}^n}|K_t^0(u;0,x)| \leqslant CR^{\frac{n}{2}}(1 + |X| + |Y|)^N exp(\varepsilon u) \tag{72}$$

The estimate given by (71) will immediately give an estimate of η-invariant.

The idea is to give an estimate to a properly chosen Sobolev norm H^{2n+2} for $K_t^0(u;x,y)$ and $K_t^1(u;x,y)$, which can provide a bound for their L^∞-norm.

5.5 The Sobolev Norm H_t^m

Recall as in the chapeter 3, we defined the rescaled connection by

$$\begin{aligned} \nabla_1 &= \nabla \otimes 1 + 1 \otimes A_1 \\ \nabla_{1,t} &= t^{\frac{1}{2}} \delta_t \nabla_1 \delta_t^{-1} = \nabla_t \otimes 1 + 1 \otimes A_{1,t} \end{aligned} \tag{73}$$

We first introduce the rescaled Sobolev norms for the sections, operators and kernels.

Definition 5.10. *For $s \in C^\infty(M, \mathscr{E})$, set*

$$\|s\|_{t,m}^2 = \sum_{l=0}^{m} \sum_{i_1,\ldots,i_l=1}^{n} \|\nabla_{1,t,e_{i_1}}, \ldots, \nabla_{1,t,e_{i_l}} s\|_{L^2} \tag{74}$$

In order to guarantee that our Sobolev constant is independent of the parameter r, we need to compare this Sobolev norm is equivalent to the "usual" Sobolev norm for sections induced by the Sobolev norm in the following sense. For smooth section s of the bundle $\mathscr{E}$, we define its H^m Sobolev $\|s\|_m$ as the H^m Sobolev norm of $|s|$ as a smooth function. This is provided by the following lemma.

Lemma 5.11. *The Sobolev norm defined in (74) is equivalent to the usual Sobolev norm on any closed ball $B^{\mathbb{R}^n}(0,q)$, i.e. for $t \in [0,t_0]$ there exists a constant $C > 0$ independent of the parameters r,t such that*

$$\frac{1}{C(1+q)^m} \|s\|_{t,m} \leqslant \|s\|_m \leqslant C(1+q)^m \|s\|_{t,m} \tag{75}$$

Proof. Given any $s \in C^\infty(\mathbb{R}^n, S_n \otimes \mathbb{C}^k)$,

$$\|\nabla_{1,t,e_i} s\|_{L^2} = \|(\nabla_{e_i} \otimes 1 + 1 \otimes A_{1,t}(e_i))s\|_{L^2}. \tag{76}$$

From the choice of the orthonormal frame and the trivialization of the bundle $\mathscr{E}$, the connection A_1 can be written as

$$A_1 = d + \sum_{i=1}^{n} \alpha_i dx^i,$$

where $\{\alpha_i\}$ is $\mathfrak{u}(k)$-valued function that can be determined by the curvature F_1 via

$$\alpha_i|_x = \int_0^1 \rho x^j F_1(\partial_j, \partial_i)(\rho x) d\rho$$

As a result, given $t \in [0, t_0]$, we have

$$\begin{aligned} t^{\frac{1}{2}}\delta_t(A_1(e_i)) &= \sum_{j=1}^{n} \delta_t(\alpha_j\langle\partial_j, e_i\rangle) \\ &= \sum_{j,k=1}^{n} t^{\frac{1}{2}}\delta_t(\int_0^1 \rho x^k F_1(\partial_k, \partial_j)\langle\partial_j, e_i\rangle(\rho x)d\rho) \\ &= \sum_{j,k=1}^{n} t^{\frac{1}{2}}(\int_0^1 \rho t^{\frac{1}{2}} x^k F_1(\partial_k, \partial_j)\langle\partial_j, e_i\rangle(\rho t^{\frac{1}{2}} x)d\rho) \\ &= \sum_{j,k=1}^{n} t(\int_0^1 \rho x^k F_1(\partial_k, \partial_j)\langle\partial_j, e_i\rangle(\rho t^{\frac{1}{2}} x)d\rho). \end{aligned} \tag{77}$$

Therefore, given any $q > 0$, there exists a constant $C' > 0$ independent of the parameters r, t, and q, such that for $s \in C^\infty(B^{\mathbb{R}^n}(0, q), S_n \otimes \mathbb{C}^k)$,

$$\frac{1}{C'(1+q)}\|s\|_1 \leqslant \|\nabla_{1,t,e_i}s\|_{L^2} \leqslant C'(1+q)\|s\|_1. \tag{78}$$

Thus there exists a constant $C > 0$ independent of the parameters r, t, q such that

$$\frac{1}{C'(1+q)}\|s\|_1 \leqslant \|s\|_{1,t} \leqslant C'(1+q)\|s\|_1, \tag{79}$$

This implies that the Sobolev norm defined in (74) is equivalent to the usual Sobolev norm on any closed ball $B^{TM}(0, q)$, i.e. there exists a constant $C > 0$ independent of the parameters r, t such that

$$\frac{1}{C(1+q)^m}\|s\|_{t,m} \leqslant \|s\|_m \leqslant C(1+q)^m\|s\|_{t,m}.$$

$\square$

This norm also induced the inner-product $\langle s, s'\rangle$ on $\mathscr{C}^\infty(\mathbb{R}^n, \mathscr{E}_{x_0})$. Denote by H_t^m the Sobolev space of order m with norm $\|\cdot\|_{t,m}$. And let H_t^{-1} be the Sobolev space of order -1 and let $\|\cdot\|_{t,-1}$ be the norm on H_t^{-1} defined by

$$\|s\|_{t,-1} = \sup_{\|s'\|_{t,1}=1} |\langle s, s'\rangle| \tag{80}$$

This norms of sections induce the corresponding norms of operators.

Definition 5.12. *If $A \in \mathscr{L}(H^m, H^{m'})$ for some integers m, m', we denote by $\|A\|_t^{m,m'}$ the norm of A induced by $\|\cdot\|_{t,m}$ and $\|\cdot\|_{t,m'}$, i.e.*

$$\|A\|_t^{m,m'} = \sup_{\|s\|_{t,m}=1} \|As\|_{t,m'} \tag{81}$$

5.6 The Uniform Heat Kernel Estimate

With these norms, the estimate of the operators is given via the estimate to the norm $\|Q_xQ'_yK^0_t(u;x,y)\|$ and $\|Q_xQ'_yK^1_t(u;x,y)\|$ with $Q, Q' \in \mathcal{Q} = \{\nabla_{t,e_{i_1}} \dots \nabla_{t,e_{i_j}}\}_{j\leqslant n+1}$.

$Q_xQ'_yK^1_t(u;x,y)$ and $Q_xQ'_yK^1_t(u;x,y)$ are the kernels of the operator

$$QH^1Q' = -\frac{(-1)^{k-1}(k-1)!}{2\pi\sqrt{-1}u^k}\int_\Gamma e^{-u\lambda}\sum_{i=1}^{k} Q(\lambda - D^2_{1,t})^{-i}L^1_t(\lambda - D^2_{1,t})^{-k+i-1}Q'd\lambda. \tag{82}$$

Noticing that for $Q_1, Q_2, \dots, Q_m \in \{\nabla_{t,e_i}\}_{i=1}^n$ the operators

$$Q_{i_1}Q_{i_2}\dots Q_{i_l}(\lambda - D^2_{1,t})^{-i}L^1_t(\lambda - D^2_{1,t})^{-k+i-1}$$

are linear combination of the following two types of operators:

$$(\lambda - D^2_{1,t})^{-1}R_1(\lambda - D^2_{1,t})^{-1}R_2\dots(\lambda - D^2_{1,t})^{-1}L^1_t(\lambda - D^2_{1,t})^{-1}\dots R_{m'}(\lambda - D^2_{1,t})^{-1}$$

$$(\lambda - D^2_{1,t})^{-1}R_1(\lambda - D^2_{1,t})^{-1}R_2\dots(\lambda - D^2_{1,t})^{-1}R(\lambda - D^2_{1,t})^{-1}\dots R_{m'}(\lambda - D^2_{1,t})^{-1}$$

where the operator R is of the form $[Q_{j_1}, [Q_{j_2}, \dots, [Q_{j_l}, L^1_t]]]$, and R_i is of the form $[Q_{j_1}, [Q_{j_2}, \dots, [Q_{j_l}, D^2_{1,t}]]]$.

We will give a uniform estimate for both these two types of the operators above.

Recall from the Lichnerowicz formula (3.6), $D^2_{1,t} = \nabla^*_{1,t}\nabla_{1,t} + c(F_1) + \frac{t\delta_tK}{4}$, where

$$\nabla_1 = \nabla\otimes 1 + 1\otimes A_1 \qquad \nabla_{1,t} = t^{\frac{1}{2}}\delta_t\nabla_1\delta_t^{-1}$$

In order to achieve the uniform estimate of its dependence on the parameter r, the most important observation is to let:

$$M_t = \nabla^*_{1,t}\nabla_{1,t} + \frac{t\delta_tK}{4} \tag{83}$$

This allows us to write $D^2_{1,t} = M_t + tc_t(F_s)$ with the following property.

Lemma 5.13. *For $t \leqslant t_0$ the spectrum of M_t would be the same as that of L^0_t which is contained in $[0, +\infty)$. And for $\lambda \in \Gamma$, we have*

$$(\lambda - D^2_{1,t})^{-1} = (\lambda - M_t)^{-1}\sum_{k=0}^{\infty}(tc_t(F_1)(\lambda - M_t)^{-1})^k \tag{84}$$

Remark 5.14. *The series in (84) is uniformly convergent and the leading order term in r is given by the power of $F_1\wedge$. Since $F_1\wedge$ is nilpotent, the dependence on R contributed by it is up to $R^{\frac{n-1}{2}}$-th power. With this observation, we can start the estimate by giving estimate to the operators of the form QM_tQ'. As in [5], this is done by giving estimate to all the operators involved in it as stated as following.*

Theorem 5.15. *There exists constants C_1, C_2, $C_3 > 0$ such that for $t \in (0, t_0]$ and any $s, s' \in C_0^\infty(\mathbb{R}^n, S_{x_0})$, we have*

$$\langle M_t s, s\rangle_{t,0} \geqslant C_1 \|s\|_{t,1}^2 - C_2 \|s\|_{t,0}^2 \tag{85}$$

$$|\langle M_t s, s'\rangle_{t,0}| \leqslant C_3 \|s\|_{t,1} \|s'\|_{t,1} \tag{86}$$

Proof. It follows from the Lichnerowicz formula (3.6) that

$$\begin{aligned} \langle M_t s, s\rangle_{t,0} &= \langle \nabla_{1,t}^* \nabla_{1,t} s, s\rangle + \langle \frac{tK}{4} s, s\rangle \\ &= \langle \nabla_{1,t} s, \nabla_{1,t} s\rangle + \langle \frac{tK}{4} s, s\rangle \end{aligned} \tag{87}$$

The equation (85) and (86) now follows directly from it.

□

Then we have

Theorem 5.16. *There exists $C > 0$ such that for $t \in (0, t_0]$ and $\lambda \in \Gamma$, we have*

$$\|(\lambda - M_t)^{-1}\|_t^{0,0} \leqslant C \tag{88}$$

$$\|(\lambda - M_t)^{-1}\|_t^{-1,1} \leqslant C(1 + |\lambda|^2) \tag{89}$$

Proof. The inequality (88) follows directly from the Lemma5.13.

For $\lambda_0 \in \mathbb{R}$, $\lambda_0 \leqslant -2C_2 t$, $(\lambda_0 - M_t)^{-1}$ exists and we have

$$\|(\lambda_0 - M_t)^{-1}\|_t^{-1,1} \leqslant \frac{1}{C_1} \tag{90}$$

Now for any $\lambda \in \Gamma$, we have

$$(\lambda - M_t)^{-1} = (\lambda_0 - M_t)^{-1} - (\lambda - \lambda_0)(\lambda - M_t)^{-1}(\lambda_0 - M_t)^{-1} \tag{91}$$

and thus we have

$$\|(\lambda - M_t)^{-1}\|_t^{-1,0} \leqslant \frac{1}{C_1}(1 + \frac{1}{\varepsilon}|\lambda - \lambda_0|) \tag{92}$$

Therefore, we have

$$\|(\lambda - M_t)^{-1}\|_t^{-1,1} \leqslant C(1 + |\lambda|^2) \tag{93}$$

□

Propostion 5.17. *Take $m \in \mathbb{N}^*$, there exists $C_m > 0$, such that for $t \in (0, t_0]$ and $Q_1, Q_2, \ldots, Q_m \in \{\nabla_{t,e_i}, X_i\}_{i=1}^n$ and $s, s' \in C^\infty(\mathbb{R}^n, S_{x_0})$, we have*

$$|\langle [Q_1, [Q_2, \ldots, [Q_m, M_t]] \ldots]s, s'\rangle_{t,0}| \leqslant C_m \|s\|_{t,1} \|s'\|_{t,1} \tag{94}$$

Proof. Noticing that

$$[\nabla_{1,t,e_i}, M_t] = [\nabla_{1,t,e_i}, \nabla^*_{1,t}\nabla_{1,t}] + t\delta_t(e_i(K))$$

The last term $t\delta_t(e_i(K))$ is obviously small for t small. The first term are differential operators of order 2 and 1 respectively, with some curvature terms, as follows

$$[\nabla_{1,t,e_i}, \nabla^*_{1,t}\nabla_{1,t}] = t^{\frac{3}{2}}\delta_t([\nabla_{1,e_i}, \nabla^*_1\nabla_1])\delta_t^{-1}.$$

It follows from our discussion in the previous section that this operator consists only of differential operators of at most 2 with some curvature terms independent of r.

By iteration, we know that $[Q_1, [Q_2, \dots, [Q_m, M_t] \dots]$ has the same structure and we get the desired estimate.

Therefore, the inequality holds. □

Now with the estimates above, we first get.

Theorem 5.18. *For any $t \in (0, t_0]$, $\lambda \in \Gamma$ and $m \in \mathbb{N}^*$, $(\lambda - M_t)^{-1}$ maps H_t^m into H_t^{m+1}. Moreover, for any $\alpha \in \mathbb{Z}^n$, there exist $N \in \mathbb{N}$ and $C_{\alpha,m} > 0$, such that*

$$\|Z^\alpha(\lambda - M_t)^{-1}s\|_{t,m+1} \leqslant C_{\alpha,m}(1 + |\lambda|^2)^N \sum_{\alpha' \leqslant \alpha} \|Z^{\alpha'}s\|_{t,m} \tag{95}$$

Proof. For $Q_1, Q_2, \dots, Q_m \in \{\nabla_{t,e_i}\}_{i=1}^n$, and $Q_{m+1}, Q_{m+2}, \dots, Q_{m+|\alpha|} \in \{x_i\}_{i=1}^n$, $Q_1Q_2 \dots Q_{m+|\alpha|}(\lambda - M_t)^{-1}$ can be written as a linear combination of the operators of the type

$$[Q_1, [Q_2, \dots [Q_{m'}, (\lambda - M_t)^{-1}]] \dots]Q_{m'+1} \dots Q_{m'+|\alpha|}$$

Let $\mathscr{R}_t = \{[Q_{j_1}, [Q_{j_2}, \dots [Q_{j_l}, M_t]] \dots]\}$. Clearly, any commutator $[Q_1, [Q_2, \dots [Q_{m'}, (\lambda - M_t)^{-1}]] \dots]$ is a linear combination of operators of the form

$$(\lambda - M_t)^{-1}R_1(\lambda - M_t)^{-1}R_2 \dots R_{m'}(\lambda - M_t)^{-1} \tag{96}$$

with $R_1, R_2, \dots, R_{m'} \in \mathscr{R}_t$. By the proposition above, the norm $\|R_j\|_t^{1,-1}$ is uniformly bounded. And by 5.16 we find that there exists a constant $C > 0$ and $N \in \mathbb{N}$ such that the norm $\|\|_t^{0,1} < C(1 + |\lambda|^2)^N$. □

Now, taking the extra term $rtc_t(da)$ into account, we have

Propostion 5.19. *For any $m \in \mathbb{N}^*$, $Q \in \mathcal{Q}^m$, there exist $C_m > 0$ and $M \in \mathbb{N}^*$ such that for any $\lambda \in \Gamma$, we have*

$$\|Q(\lambda - D^2_{1,t})^{-m}\|_t^{0,0} \leqslant C_m r^{\frac{n-1}{2}}(1 + |\lambda|^2)^M \tag{97}$$

Furthermore, we have

Propostion 5.20. *For any $m \in \mathbb{N}^*$, $Q \in \mathcal{Q}^m$, there exist $C_m > 0$ and $M \in \mathbb{N}^*$ such that for any $\lambda \in \Gamma$ and $s \in C^\infty(\mathbb{R}^n, S_{x_0})$, we have*

$$\|Q(\lambda - D_{1,t}^2)^{-i} L_t^1 (\lambda - D_{1,t}^2)^{-k+i-1} s\|_{0,t} \leqslant C_m r^{\frac{n+1}{2}} t^{\frac{1}{2}} (1 + |\lambda|^2)^M \tag{98}$$

Proof. (Theorem 5.9)

It now follows from 5.8 and 5.20 that

$$\|QH^1\|_t^{0,0} \leqslant C_m r^{\frac{n+1}{2}} t^{\frac{1}{2}} e^{\varepsilon u} \tag{99}$$

In addition to that, the rescaled Sobolev norm is equivalent to the usual Sobolev norm within a closed ball, i.e. there exists $C > 0$ such that for $s \in \mathcal{C}^\infty(X)(\mathbb{R}^n, S_{x_0})$, $supp\{s\} \subset B_{\mathbb{R}^n}(0, q)$, $m > 0$,

$$\frac{1}{C}(1+q)^{-m}\|s\|_{t,m} \leqslant \|s\|_m \leqslant C(1+q)^m \|s\|_{t,m} \tag{100}$$

Together with Sobolev inequality, it follows that

$$\sup_{|X|,|Y|<q} |K^1(u; X, Y)| \leqslant CrR^{\frac{n-1}{2}} t^{\frac{1}{2}} (1+q)^{n+2} e^{\varepsilon u} \tag{101}$$

$$\sup_{|X|,|Y|<q} |K^0(u; X, Y)| \leqslant CR^{\frac{n-1}{2}} (1+q)^{n+2} e^{\varepsilon u} \tag{102}$$

Thus we get the desired estimate in Theorem 5.9. □

Large time

While the previous part has actually provided the estimate for $\int_0^{t_0} t^{-\frac{1}{2}} Tr[D_1 e^{-tD_1^2}]dt$, this part is focusing on $\int_{t_0}^{+\infty} t^{-\frac{1}{2}} Tr[D_1 e^{-tD_1^2}]dt$.

First of all, for the integral $\int_{t_0}^{\infty} t^{-\frac{1}{2}} Tr[D_1 e^{-tD_1^2}]dt$, we can show that

Lemma 5.21.

$$\begin{aligned} |\int_{t_0}^{\infty} t^{-\frac{1}{2}} Tr[D_1 e^{-tD_1^2}]dt| &\leqslant \int_{t_0}^{\infty} t^{-\frac{1}{2}} |Tr[D_1 e^{-tD_1^2}]|dt \\ &\leqslant \int_{t_0}^{\infty} t^{-\frac{1}{2}} \sum_{\lambda \in Spec\{D_1\}} |\lambda e^{-t\lambda^2}|dt \\ &< \frac{\sqrt{\pi}}{2} Tr[e^{-\frac{t_0}{2}D_1^2}]. \end{aligned} \tag{103}$$

Proof. For a single eigenvalue λ of D_1 and $t \geqslant t_0$, $e^{-t\lambda^2} \leqslant e^{-\frac{(t+t_0)\lambda^2}{2}}$, thus

$$\begin{aligned}\int_{t_0}^{\infty} t^{-\frac{1}{2}}|\lambda e^{-t\lambda^2}|dt &\leqslant \int_{t_0}^{\infty} t^{-\frac{1}{2}}|\lambda|e^{-\frac{(t+t_0)\lambda^2}{2}}dt \\ &= e^{-\frac{t_0\lambda^2}{2}}\int_{t_0}^{\infty} t^{-\frac{1}{2}}|\lambda e^{-\frac{t\lambda^2}{2}}|dt \\ &\leqslant \frac{\sqrt{\pi}}{2}e^{-\frac{t_0\lambda^2}{2}}.\end{aligned}$$

Taking the sum over the spectrum of D_1, it follows that

$$\int_{t_0}^{\infty} \sum_{\lambda\in Spec\{D_1\}} |\lambda|e^{-t\lambda^2} < \frac{\sqrt{\pi}}{2}Tr[e^{-\frac{t_0}{2}D_1^2}],$$

and therefore the entire inequality above holds. □

The estimate given by Taubes in [12], which is stated as follows will provide the first-step estimate of $Tr[e^{-\frac{t_0 D_1^2}{2}}]$.

Propostion 5.22. *There exists a constant $c > 0$*

$$|K_{D_1^2}(t,x,0)| < c(\frac{1}{4\pi t})^{\frac{n}{2}}e^{cRt}$$

Proof. Firstly, note that

$$d\langle s_1, s_2\rangle = \langle \nabla_1 s_1, s_2\rangle + \langle s_1, \bar{\nabla}_1 s_2\rangle \tag{104}$$

Take an arbitrary $s \in S_0$, such that $|s| = 1$

Then we have

$$\frac{d}{du}K_{D_1^2}(u;x,0)s = -\hat{D}^2 K_{D_1^2}(u;x,0)s$$

Thus we have

$$\begin{aligned}\frac{d}{du}|K_{D_1^2}(u;x,0)s| =& \frac{d}{du}\langle K_{D_1^2}(u;x,0)s, K_{D_1^2}(u;x,0)s\rangle_x^{1/2} \\ =&\langle K_{D_1^2}(u;x,0)s, K_{D_1^2}(u;x,0)s\rangle_x^{-1/2}\langle \frac{d}{du}K_{D_1^2}(u;x,0)s, K_{D_1^2}(u;x,0)s\rangle_x^{1/2} \\ =&|K_{D_1^2}(u;x,0)s|\langle -\hat{D}^2 K_{D_1^2}(u;x,0)s, K_{D_1^2}(u;x,0)s\rangle_x^{1/2} \\ =&|K_{D_1^2}(u;x,0)s|(\langle -\nabla^*\nabla K_{D_1^2}(u;x,0)s, K_{D_1^2}(u;x,0)s\rangle_x \\ &- \langle \frac{K}{4}K_{D_1^2}(u;x,0)s, K_{D_1^2}(u;x,0)s\rangle_x \\ &- \langle c(F_1)K_{D_1^2}(u;x,0)s, K_{D_1^2}(u;x,0)s\rangle_x)^{\frac{1}{2}}\end{aligned}$$

Then it follows from the fact $|a| \geqslant 0$ and R, F_1 are both bounded on the compact manifold M that

$$\begin{aligned}\frac{d}{du}|K_{D_1^2}(u;x,0)s| \leqslant & |K_{D_1^2}(u;x,0)s|^{-1}(\langle -\nabla^*\nabla K_{D_1^2}(u;x,0)s, K_{D_1^2}(u;x,0)s\rangle_x^{\frac{1}{2}} \\ & - \langle c(F_1)K_{D_1^2}(u;x,0)s, K_{D_1^2}(u;x,0)s\rangle_x^{\frac{1}{2}}) \\ \leqslant & |K_{D_1^2}(u;x,0)s|^{-1}(\langle -\nabla^*\nabla K_{D_1^2}(u;x,0)s, K_{D_1^2}(u;x,0)s\rangle_x^{\frac{1}{2}} \\ & + C(1+R)\langle K_{D_1^2}(u;x,0)s, K_{D_1^2}(u;x,0)s\rangle_x)\end{aligned}$$

where $C > 0$ is a constont.

On the other hand,

$$\begin{aligned}d|K_{D_1^2}(u;x,0)s| &= \langle K_{D_1^2}(u;x,0)s, K_{D_1^2}(u;x,0)s\rangle_x^{-1/2}\langle \nabla K_{D_1^2}(u;x,0)s, K_{D_1^2}(u;x,0)s\rangle_x \\ &= |K_{D_1^2}(u;x,0)s|^{-1}\langle \nabla K_{D_1^2}(u;x,0)s, K_{D_1^2}(u;x,0)s\rangle_x\end{aligned}$$

Then

$$\begin{aligned}d^*d|K_{D_1^2}(u;x,0)s| =& - tr(\nabla d|K_{D_1^2}(u;x,0)s|) \\ =& - |K_{D_1^2}(u;x,0)s|^{-1}\langle \nabla^*\nabla K_{D_1^2}(u;x,0)s, K_{D_1^2}(u;x,0)s\rangle \\ & + |K_{D_1^2}(u;x,0)s|^{-3}\langle \nabla K_{D_1^2}(u;x,0)s, K_{D_1^2}(u;x,0)s\rangle^2 \\ & - |K_{D_1^2}(u;x,0)s|^{-1}\langle \nabla K_{D_1^2}(u;x,0)s, \nabla K_{D_1^2}(u;x,0)s\rangle \\ \leqslant & |K_{D_1^2}(u;x,0)s|^{-1}\langle \nabla^*\nabla K_{D_1^2}(u;x,0)s, K_{D_1^2}(u;x,0)s\rangle\end{aligned}$$

Therefore we have the inequality

$$\frac{\partial}{\partial u}|K_{D_1^2}(u;x,0)| \leqslant -d^*d|K_{D_1^2}(u;x,0)s| + CR\langle K_{D_1^2}(u;x,0)s, K_{D_1^2}(u;x,0)s\rangle_x^{1/2}) \quad (105)$$

Set $f(u,x) = e^{-C(1+R)u}|K_{D_1^2}(u;x,0)|$, then

$$\frac{\partial}{\partial u}f(u,x) \leqslant -d^*df(u,x)$$

In Taubes' paper it set up this inequality for the heat kernel $K_{D_1^2}(u;x,y)$, where y is fixed. For $g(u,x) = e^{-C(1+R)u}|K^1_{D_1^2}(u;x,y)|$, we have

$$\frac{\partial}{\partial u}g(u,x) \leqslant -d^*dg(u,x)$$

From the fact

$$\lim_{u\to 0} K_{D_1^2}(u;x,y) = Id_x\delta_y(x)$$

it follows that $g(u,x) \leqslant (\frac{1}{4\pi u})^{n/2} e^{-\frac{d^2(x,y)^2}{4u}}$ and thus

$$|K_{D_1^2}(u;x,y)| \leqslant \kappa(\frac{1}{4\pi u})^{n/2} e^{\kappa R u} e^{-\frac{d^2(x,y)}{4u}}$$

□

It now follows that

$$\int_{t_0}^{\infty} t^{-\frac{1}{2}} Tr[D_1 e^{-tD_1^2}]dt < C_1(\frac{1}{2\pi t_0})^{\frac{n}{2}} e^{C_1 R t_0} \tag{106}$$

From our choice of $t_0 = \frac{1}{2R}$, it now follows that

Propostion 5.23. *There exists a constant $C > 0$ independent of r, such that*

$$Tr[e^{-t_0 D_1^2}]dt < CR^{\frac{n}{2}}. \tag{107}$$

This induces the estimate of the rest part of the reduced η-invariant.

Theorem 5.24. *There exists a constant $C > 0$, such that*

$$|dimKer(D_1) + \int_{t_0}^{\infty} t^{-\frac{1}{2}} Tr[D_1 e^{-tD_1^2}]dt| \leqslant CR^{\frac{n}{2}}$$

5.7 The Estimate of η-invariant

Now the proof of Theorem 2 follows from the uniform estimate of $K^1(u;X,Y)$, the equation (31) and the Proposition 5.23

Theorem 2. *Let M be an odd dimensional compact spin manifold, and D be a Dirac operator acting on the bundle $S \otimes E$, a be a Lie algebra $\mathfrak{u}(k)$-valued one-form on M, and $r > 0$. Denote by $R = \sup_M\{|F_1|\}$. Then there exists a constant $C' > 0$, such that*

$$|\bar{\eta}(D + rc(a))| \leqslant C' R^{\frac{n}{2}}$$

when $r > 0$ is sufficiently large.

Proof. Note that

$$\begin{aligned}
\bar{\eta}(D_1) &= \frac{1}{2}(dimKer(D_1) + \eta(D_1)) \\
&= \frac{1}{2}(dimKer(D_1) + \int_0^{\infty} t^{-\frac{1}{2}} Tr[D_1 e^{-tD_1^2}])dt \\
&= \frac{1}{2}(dimKer(D_1) + \int_{t_0}^{\infty} t^{-\frac{1}{2}} Tr[D_1 e^{-tD_1^2}]dt + \int_0^{t_0} t^{-\frac{1}{2}} Tr[D_1 e^{-tD_1^2}]dt).
\end{aligned}$$

From the estimate achieved above, it follows that there exists constants $C_1, C_2 > 0$ such that

$$|dimKer(D_1) + \int_{t_0}^{\infty} t^{-\frac{1}{2}} Tr[D_1 e^{-tD_1^2}]dt| \leqslant C_1 R^{\frac{n}{2}},$$
$$|\int_0^{t_0} t^{-\frac{1}{2}} Tr[D_1 e^{-tD_1^2}]dt| \leqslant C_2 R^{\frac{n-1}{2}}.$$

Combining the estimates above provides the desired estimate for the reduced η-invariant. □

In conclusion, suming up all the results above, it eventually provides our estimate of the asymptotic spectral flow.

Theorem 1. *Let M be an odd dimensional compact spin manifold, and D_0 be a Dirac operator on it, and $D_s = D_0 + sc(\hat{a})$, $0 \leqslant s \leqslant 1$, be the smooth curve of Dirac operators, where $\hat{a} = ra$ is a Lie algebra $\mathfrak{u}(k)$-valued 1-form on M with parameter $r > 0$. Denote by $R = \sup_M\{|F_1|\}$, then there exists a constant $C > 0$, such that the spectral flow satisfies*

$$|\mathrm{sf}\{D_s, [0,1]\} - (\frac{1}{2\pi\sqrt{-1}})^{\frac{n+1}{2}} \int_0^1 \int_M tr_{\mathbb{C}^k}[\hat{a} \wedge \hat{A}(M) \wedge e^{F_s}]_n ds| \leqslant cR^{\frac{n}{2}}$$

References

[1] M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and riemannian geometry. I. *Mathematical Proceedings of the Cambridge Philosophical Society*, 77(1):43–69, 1975.

[2] M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and riemannian geometry. II. *Mathematical Proceedings of the Cambridge Philosophical Society*, 78(3):405–432, 1975.

[3] M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and riemannian geometry. III. *Mathematical Proceedings of the Cambridge Philosophical Society*, 79(1):71–99, 1976.

[4] J.-M. Bismut and D. S. Freed. The analysis of elliptic families: II. dirac operators, eta invariants, and the holonomy theorem. *Communications in Mathematical Physics*, 107:103–163, 1986.

[5] X. Dai, K. Liu, and X. Ma. On the asymptotic expansion of bergman kernel. *Journal of Differential Geometry*, 72(1):1–41, 2006.

[6] X. Dai and W. Zhang. Higher spectral flow. *Mathematical Research Letters*, 3:93–102, 1996.

[7] E. Getzler. A short proof of the local atiyah-singer index theorem. *Topology*, 25(1):111–117, 1986.

[8] X. Ma and G. Marinescu. *Holomorphic Morse inequalities and Bergman kernels*, volume 254. Springer Science & Business Media, 2007.

[9] S. Rosenberg. *The Laplacian on a Riemannian manifold: an introduction to analysis on manifolds*, volume 31. Cambridge University Press, 1997.

[10] N. Savale. Asymptotics of the eta invariant. *Communications in Mathematical Physics*, 332(2):847–884, 2014.

[11] N. Savale. A gutzwiller type trace formula for the magnetic dirac operator. *Geometric and Functional Analysis*, 28(5):1420–1486, 2018.

[12] C. H. Taubes. Asymptotic spectral flow for dirac operators. *Communications in Analysis and Geometry*, 15(3):569–587, 2007.

[13] C. H. Taubes. The seiberg–witten equations and the weinstein conjecture. *Geometry & Topology*, 11(4):2117–2202, 2007.