# MAXIMAL VOLUME ENTROPY RIGIDITY FOR $\mathsf{RCD}^*(-(N-1), N)$ SPACES

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Abstract. For $n$-dimensional Riemannian manifolds $M$ with Ricci curvature bounded below by $-(n-1)$, the volume entropy is bounded above by $n-1$. If $M$ is compact, it is known that the equality holds if and only if $M$ is hyperbolic [LW]. We extend this result to $\mathsf{RCD}^*(-(N-1), N)$ spaces. While the upper bound follows quickly, the rigidity case is quite involved due to the lack of a smooth structure in $\mathsf{RCD}^*$ spaces. As an application we obtain an almost rigidity result which partially recovers a result in [CRX] for manifolds.

## 1. Introduction

Volume entropy is a fundamental geometric invariant, related to the topological entropy of geodesic flows, minimal volume, simplicial volume, bottom spectrum of the Laplacian of the universal cover, among others. For a compact Riemannian manifold $(M^n, g)$, the volume entropy is defined as,

$$h(M,g) = \lim_{R\to\infty} \frac{\ln \mathrm{Vol}(B(\tilde{x}, R))}{R}.$$

Here $B(\tilde{x}, R)$ is a ball in the universal cover $\tilde{M}$ of $M$. For $M$ compact, the limit exists and is independent of the base point $\tilde{x} \in \tilde{M}$ [Mann]. Thus, the volume entropy measures the exponential growth rate of the volume of balls in the universal cover. It is nonzero if and only if the fundamental group $\pi_1(M)$ has exponential growth.

When $\mathrm{Ric}_M \geq -(n-1)$, Bishop volume comparison gives the upper bound $h(M,g) \leq n-1$, which is the volume entropy of any hyperbolic $n$-manifold. Ledrappier-Wang [LW] showed that if $h(M,g) = n-1$, then $M$ is isometric to a hyperbolic manifold. This is called the maximal volume entropy rigidity. Liu found a simpler proof [Liu], and recently Chen-Rong-Xu gave a quantitative version of this rigidity result [CRX].

In this paper we will show the same kind of maximal entropy rigidity holds for a class of metric measure spaces—known by now as $\mathsf{RCD}^*(K,N)$ *spaces*—that is of interest in both optimal transport and in the theory of limits of Riemannian manifolds with bounded Ricci curvature (known as *Ricci limit spaces*).

Alexandrov geometry can be seen as a synthetic approach to the spaces that occur as limits of smooth manifolds with sectional curvature bounded below. In this same spirit, $\mathsf{RCD}^*(K,N)$ spaces can be thought as the synthetic analog to Ricci curvature being bounded below by $K$, for dimension at most $N$. These spaces include Ricci limit spaces and Alexandrov spaces [Pet], and have been studied extensively, see Section 2 for details.

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The last-named author joint with Mondino proved that the universal cover of an $\mathsf{RCD}^*(K,N)$ space with $1 < N < \infty$ exists and is also an $\mathsf{RCD}^*(K,N)$ space [MW]. This allows us to define the volume entropy similarly for compact $\mathsf{RCD}^*(K,N)$ spaces.

That is, let $(X,d,m)$ be a compact $\mathsf{RCD}^*(K,N)$ space, and $(\widetilde{X},\tilde{d},\tilde{m})$ its universal cover. We define the *volume growth entropy* of $(X,d,m)$ as

$$h(X,d,m) := \limsup_{R\to\infty} \frac{1}{R} \ln \widetilde{m}(B_{\widetilde{X}}(x,R)).$$

The volume growth entropy is well defined, and it is independent of $x$ and the measure $m$, (see [Rev, BCGS]). Observe that if $(M,g)$ is a Riemannian manifold then with the induced distance $d = d_g$ and the volume measure $m = d\mathrm{vol}_g$, both definitions coincide.

Our main results are:

**Theorem 1.1.** *Let $1 < N < \infty$ and $(X,\mathsf{d},m)$ be a compact $\mathsf{RCD}^*(-(N-1),N)$ space. Then $h(X) \le N-1$, and the equality holds if and only if $N$ is an integer and the universal cover of $X$ is isometric to the $N$-dimensional real hyperbolic space.*

As in the smooth case the compactness of $X$ is essential here. For $N > 1$ the well known smooth metric measure space $((0,\infty), |\cdot|, \sinh^{N-1}(x)\,dx)$ is an $\mathsf{RCD}^*(-(N-1),N)$ space with volume entropy $= N-1$. This example does not contradict our theorem as it is not the universal cover of any compact $\mathsf{RCD}^*(-(N-1),N)$ space.

The key step in proving the above theorem is the following result, which is of independent interest. In the statement we use the language of differential calculus developed by Gigli (see Section 2). We refer to the Preliminaries section for definitions and more details.

**Theorem 1.2.** *Let $1 < N < \infty$ and $(X,\mathsf{d},m)$ be a complete $\mathsf{RCD}^*(-(N-1),N)$ space. If there exists a function $u$ in $D_{loc}(\Delta)$ such that $|\nabla u| = 1$ $m$-a.e. and $\Delta u = N-1$, then $X$ is isometric to a warped product space $\mathbb{R}\times_{e^t} X'$, where $X'$ is an $\mathsf{RCD}^*(-(N-1),N-1)$ space.*

An immediate consequence is the rigidity of the bottom of the spectrum of the Laplace operator.

**Corollary 1.3.** *Let $1 < N < \infty$ and $(X,\mathsf{d},m)$ be a compact $\mathsf{RCD}^*(-(N-1),N)$ space. If*

$$\lambda_{(\tilde{X},d,m)} := \inf\left\{ \frac{\int_{\tilde{X}} |\nabla f|^2\,\mathrm{d}m}{\int_{\tilde{X}} f^2\,\mathrm{d}m} \mid f \in \mathrm{Lip}(\tilde{X},d)\cap L^2(\tilde{X},m),\ \int_{\tilde{X}} f^2\,\mathrm{d}m \ne 0 \right\} = \frac{(N-1)^2}{4},$$

*then $\tilde{X}$ is isometric to the $N$-dimensional real hyperbolic space.*

The previous corollary follows from the inequality $\sqrt{\lambda_{(\tilde{X},d,m)}} \le \frac{1}{2}\limsup_{R\to\infty}\frac{1}{R}\ln m(B_{\tilde{X}}(x,R))$ ([Stu3, Theorem 5]).

In all above statements we prove in fact that the isometries are measure preserving.

The corresponding results for Alexandrov spaces have very recently been proved by Jiang [Jiang].

Rigidity results for $\mathsf{RCD}^*$ spaces often imply stability results directly as $\mathsf{RCD}^*$ spaces are closed under measured Gromov-Hausdorff convergence. Theorem 1.1 implies an almost rigidity assuming the volume entropy is continous under measured Gromov-Hausdorff convergence, which is true for non-collapsed sequences. As a result we have

**Theorem 1.4.** *Let $1 < N < \infty$, $v > 0$, $D > 0$. There exists $\epsilon(N,v,D) > 0$ such that for $0 < \epsilon < \epsilon(N,v,D)$, if $(X,\mathsf{d},m)$ is a compact $\mathsf{RCD}^*(-(N-1),N)$ space satisfying $\mathrm{diam}(X) \le D$, $h(X) \ge N-1-\epsilon$, $m(X) \ge v$, then $X$ is $\Psi(\epsilon|N,v,D)$ measured Gromov-Hausdorff close to an $N$-dimensional hyperbolic manifold.*

We conjecture that the non-collapsing condition is not necessary. In fact when $X$ is a manifold a diameter upper bound and almost maximal volume entropy imply non-collapsing as proved in [CRX].

The strategy and techniques used in proving our results somewhat resemble those of Gigli's Splitting Theorem in the non-smooth context [Gig4], as well as the "volume cone implies metric cone" Theorem by De Philippis-Gigli [DePG]. The key idea for proving these results is to work at the level of the Sobolev space. In this way we overcome obstacles that appear due to the lack of analytical tools available in the smooth category. Once a result is obtained at this level it can be *transported* to a statement at the level of the metric measure space itself.

We now present a summary of our strategy. In order to show that the universal cover $(\tilde{X}, \tilde{d}, \tilde{m})$ of an $\mathsf{RCD}^*(-(N-1), N)$ space $(X, d, m)$ with maximal volume entropy is isomorphic—isometric via a measure preserving isometry—to a real hyperbolic space, it is sufficient to show that $\tilde{X}$ is isomorphic to a warped product space of the form $X' \times_{e^t} \mathbb{R}$, and then show that $X'$ is regular enough. At this point an analogy with [DePG] becomes clear, as now our problem can be considered as a warped splitting theorem under the assumption of maximality of volume entropy.

To obtain a metric measure space which is a candidate for the role of $X'$, we reconstruct in our context Liu's ideas [Liu] and build a Busemann-type function $u : \tilde{X} \to \mathbb{R}$ in $D_{loc}(\Delta)$, which is regular enough to admit a Regular Lagrangian Flow $F : \mathbb{R} \times \tilde{X} \to \tilde{X}$ associated to $\nabla u$ (in the sense of Ambrosio-Trevisan [AT]). The trajectories $F_{(\cdot)}(x)$ of our flow induce a partition of $\tilde{X}$. The high regularity of $u$ provides useful information on how the reference measure $\tilde{m}$ changes under the flow. Moreover, an analysis of how the Cheeger energy of Sobolev functions changes once composed with the flow shows that a representative of $F$ can be chosen such that the maps $F_t$ are bi-Lipschitz. Then we proceed to obtain estimates of the local Lipschitz constants of F.

Therefore, the natural candidate for $X'$ is $u^{-1}(0)$, the slice at time 0 of the partition induced by $F$, endowed with the natural intrinsic metric and an appropriately defined measure which agrees with the data provided by $F$. Given that $X'$ is non compact the measure defined on it is written in a similar way to [Gig4] and not as in [DePG]. The proof that it is a complete, separable and geodesic space is more involved than in [Gig4] and [DePG]. In [Gig4], the distance in $X'$ can be seen as the restriction of the metric of $\tilde{d}$ and in [DePG] $X'$ is compact. It is also shown that $X'$ is locally doubling and not doubling as in [DePG].

At this point we need to show that the natural maps from and into $\tilde{X}$ and $\mathbb{R} \times_{e^t} X'$ are isomorphisms of metric measure spaces. As mentioned above, we obtain this at the level of the Sobolev spaces. The relation between the Sobolev spaces $W^{1,2}(\tilde{X})$ and $W^{1,2}(\mathbb{R} \times_{e^t} X')$ is explained by studying the metric speeds of curves in $X$ in relation with those in $X'$. This leads to a relationship between the minimal weak upper gradients of Sobolev functions in $X'$ and $\tilde{X}$. Gathering everything together, and combining them with the work of Gigli-Han [GH] on the structure of Sobolev spaces of warped products, the task can be finished.

Finally, the structure of a warped product space naturally implies via Bochner's inequality that $X'$ is an $\mathsf{RCD}^*(-(N-1), N-1)$ space. To complete our proof, we apply Chen-Rong-Xu's argument [CRX], which shows that $\mathbb{R} \times_{e^t} X'$ is isomorphic to the $N$-dimensional hyperbolic space.

The article is organized as follows. In §2, we review definitions and properties of metric measure spaces and, in particular, $\mathsf{RCD}^*$ spaces that will be needed in the paper. In §3, using the Bishop-Gromov volume comparison theorem we provide the upper estimate of the volume entropy for $\mathsf{RCD}^*(-(N-1), N)$ spaces. For the rigidity case, we construct the Busemann function $u$, calculate its Hessian and construct a Regular Lagrangian Flow associated to $\nabla u$. In §4 we estimate the minimal weak upper gradient of functions of the form $f \circ F_t$ for $f \in W^{1,2}(\tilde{X}, \tilde{d}, \tilde{m})$. In the next section we use this to improve the regularity of the Regular Lagrangian Flow $F$, define the metric measure space $(X', d', m')$ and estimate the minimal weak upper gradients of functions $g \in W^{1,2}(X')$ in terms of functions in $W^{1,2}(\tilde{X})$. Moreover, we prove that $(X', d', m')$ is an infinitesimally Hilbertian space. In §6 we use Gigli's Contraction By Local Duality Lemma, and his proposition on isomorphisms via duality with Sobolev norms, to show that the warped product space $\mathbb{R} \times_{e^t} X'$ is isometric to

$(\tilde{X}, \tilde{d}, \tilde{m})$. In §7 we prove that $(X', d', m')$ is an $\mathsf{RCD}^*(-(N-1), N-1)$ space. In the final section we see that $N \in \mathbb{N}$ and $\mathbb{R} \times_{e^t} X'$ is isometric to the hyperbolic space $\mathbb{H}^N$.

On a complementary direction, the work of Besson-Courtois-Gallot [BCG1, BCG2] treated the *minimal* entropy of smooth manifolds and established major rigidity results for locally symmetric spaces of negative curvature. Their work implies that negatively curved locally symmetric Riemannian metrics with given total volume cannot be perturbed to nonsymmetric ones without increasing the volume entropy. A number of important corollaries in geometric rigidity and applications to dynamics then follow. We have also extended these barycenter techniques to $\mathsf{RCD}^*$ spaces in [CDNPSW].

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## Contents

## 2. Preliminaries

The following is a review of the necessary definitions and results. First we recall the concepts pertaining to first order calculus on metric spaces, we refer readers to [Gig2, Gig4] for further details.

2.1. **Calculus on metric measure spaces.** We will consider a proper metric space $(X,d)$. Let $C([0,1];X)$ be the set of continuous curves in $(X,d)$. A curve $\gamma\in C([0,1];X)$ is said to be *absolutely continuous* if there exists an integrable function $f$ on $[0,1]$ such that for every $0\le t<s\le 1$,

$$d(\gamma_t,\gamma_s)\le\int_t^s f(r)\,\mathrm{d}r.$$

Absolutely continuous curves $\gamma$ have a well defined *metric speed*,

$$|\dot\gamma_t|:=\lim_{t\to 0}\frac{d(\gamma_{t+h},\gamma_t)}{|h|},$$

which is a function in $L^1([0,1])$. The set of absolutely continuous curves in $(X,d)$ will be denoted by $AC([0,1];X)$.

Let $m$ be a non-negative Radon measure and $\mathcal{P}(C([0,1];X))$ be the space of probability measures on $C([0,1];X)$. A measure $\pi\in\mathcal{P}(C([0,1];X))$ is called a *test plan* if there exists $C>0$ such that for every $t\in[0,1]$,

$$(e_t)_\#\pi\le Cm$$

and

$$\int\int_0^1|\dot\gamma_t|^2\,\mathrm{d}t\,\mathrm{d}\pi(\gamma)<\infty.$$

Here, $e_t:C([0,1];X)\to X$ is the evaluation map $e_t(\gamma)=\gamma_t$.

The *Sobolev class* $S^2(X):=S^2(X,d,m)$ (respectively $S^2_{loc}(X):=S^2_{loc}(X,d,m)$) is the space of all Borel functions $f:X\to\mathbb{R}$ such that there exists a non-negative function $G\in L^2(X):=L^2(X,m)$ (respectively $G\in L^2_{loc}(X):=L^2_{loc}(X,m)$)—called *weak upper gradient*—such that for any test plan $\pi$ the following inequality is satisfied

$$\int|f(\gamma_1)-f(\gamma_0)|\,\mathrm{d}\pi(\gamma)\le\int\int_0^1 G(\gamma_t)|\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\pi(\gamma).$$

It is possible to prove that there exists a minimal $G$, which we denote by $|\nabla f|$, called the *minimal weak upper gradient* of $f$. We now recall the following fundamental result.

**Proposition 2.1.** [AGS14, Definition 5.6, Proposition 5.7] , [Gig2, Definition B.2, Theorem B.4] *The following are equivalent,*

(i) $f\in S^2(X)$ *and* $G$ *is a weak upper gradient.*

(ii) *For every test plan $\pi$ the following holds: For $\pi$-a.e. $\gamma$ the function $t \mapsto f(\gamma_t)$ is equal at $t = 0$, $t = 1$ and almost everywhere else on $[0,1]$ to an absolutely continuous function $f_\gamma : [0,1] \to \mathbb{R}$ whose derivative for a.e. $t \in [0,1]$ satisfies $|f'_\gamma|(t) \le G(\gamma_t)|\dot\gamma_t|$.*

A local version of the Sobolev class is produced in the following manner: A function $f : \Omega \subset X \to \mathbb{R}$, with $\Omega$ an open set, is an element of $S^2_{loc}(\Omega) := S^2_{loc}(\Omega, d, m)$ if for any Lipschitz function $\chi : X \to \mathbb{R}$ with $\mathrm{supp}(\chi) \subset \Omega$ we have that $f\chi \in S^2_{loc}(X)$. In this case $|\nabla f| : \Omega \to \mathbb{R}$ is given by

$$|\nabla f| := |\nabla(f\chi)| \quad m - a.e. \text{ on } \chi = 1.$$

Then, the set $S^2(\Omega)$ is defined as the subset of $S^2_{loc}(\Omega)$ of functions $f$ such that $|\nabla f| \in L^2(\Omega, m)$.

The *Sobolev space* is defined as

$$W^{1,2}(X, d, m) := L^2(X, d, m) \cap S^2(X, d, m)$$

endowed with the norm

$$\|f\|^2_{W^{1,2}(X)} := \|f\|^2_{L^2(X)} + \||\nabla f|\|^2_{L^2(X)} = \int_X (f^2 + |\nabla f|^2)\,\mathrm{d}m$$

We say that a proper metric measure space $(X, d, m)$ is *infinitesimally Hilbertian* if $W^{1,2}(X)$ is a Hilbert space, i.e., if $\|\cdot\|^2_{W^{1,2}(X)}$ is induced by an inner product. This happens if and only if the parallelogram rule is satisfied, so the condition is that

$$\||\nabla(f+g)|\|^2_{L^2(X)} + \||\nabla(f-g)|\|^2_{L^2(X)} = 2\left(\||\nabla f|\|^2_{L^2(X)} + \||\nabla g|\|^2_{L^2(X)}\right)$$

for all $f, g \in S^2(X)$. On an infinitesimally Hilbertian metric measure space $(X, d, m)$, for $\Omega \subset X$ open and any $f, g \in S^2_{loc}(\Omega)$ the functions $D^\pm : \Omega \to \mathbb{R}$ defined $m$-a.e. by

$$D^+ f(\nabla g) = \inf_{\varepsilon > 0} \frac{|\nabla(g + \varepsilon f)|^2 - |\nabla g|^2}{2\varepsilon}$$
$$D^- f(\nabla g) = \sup_{\varepsilon > 0} \frac{|\nabla(g + \varepsilon f)|^2 - |\nabla g|^2}{2\varepsilon}$$

coincide $m$-a.e. on $\Omega$. We denote the common value by $\langle \nabla f, \nabla g\rangle$.

Let $(X, d, m)$ be an infinitesimally Hilbertian metric measure space and $\Omega \subset X$ an open set. Let $g : \Omega \to \mathbb{R}$ be a locally Lipschitz function. We say that $g$ *has a measure valued Laplacian*, provided there exists a Radon measure $\mu$ on $\Omega$ such that

$$-\int_\Omega \langle \nabla f, \nabla g\rangle\,\mathrm{d}m = \int_\Omega f\,\mathrm{d}\mu$$

for all $f : \Omega \to \mathbb{R}$ Lipschitz and compactly supported in $\Omega$. In this case $\mu$ is the measure valued Laplacian of $g$, and it is denoted by $\mathbf{\Delta} g|_\Omega$. The set of all locally Lipschitz functions $g$ admitting a measure valued Laplacian is denoted by $D(\mathbf{\Delta}, \Omega)$. A particular instance of the notation is that $D(\mathbf{\Delta}, X) = D(\mathbf{\Delta})$ and then $\mathbf{\Delta} g|_X = \mathbf{\Delta} g$.

A different definition is that of the $L^2$-Laplacian operator defined as follows. The domain $D(\Delta)$ of the $L^2$-Laplacian is the subset of $W^{1,2}(X)$ of all $g$ such that for some $h \in L^2(X)$,

$$-\int \langle \nabla f, \nabla g\rangle\,\mathrm{d}m = \int f h\,\mathrm{d}m$$

for all $f \in W^{1,2}(X)$, written as $\Delta g = h$. Both definitions agree in the sense that $g \in D(\Delta)$ if and only if $g \in W^{1,2}(X) \cap D(\mathbf{\Delta})$ and $\mathbf{\Delta} g = h\,m$ (see [Gig4, Definition 4.6]). We similarly define $D_{loc}(\Delta)$ to be the corresponding subset of $W^{1,2}_{loc}(X)$.

2.2. **Tangent and cotangent modules.** We will now give a brief account of some of the tools of the tangent and cotangent modules as defined and developed in detail by Gigli [Gig1] (see also the section on preliminaries of [DePG]).

Given an infinitesimally Hilbertian metric measure space $(X, d, m)$, recall that there is a unique couple $(L^2(T^*X), d)$ (up to isomorphism) where $L^2(T^*X)$ is an $L^2(m)$-normed $L^\infty(m)$-module (see [Gig1, Definition 1.2.10]) and $d : S^2(X) \to L^2(T^*X)$ is a linear operator such that the following two conditions hold

(i) $|df| = |\nabla f|$ $m$-a.e. for every $f \in S^2(X)$. Here $|df|$ denotes the pointwise norm of $df$ in $L^2(T^*X)$.

(ii) $L^2(T^*X)$ is spanned by $\{df \mid f \in S^2(X)\}$.

The module $L^2(T^*X)$ is called the *cotangent module of* $X$ and $d$ is the *differential.* Note that we abuse the notation slightly by using $d$ for the differential of a function and the distance of the space.

The tangent module of $X$, denoted by $L^2(TX)$ is defined as the dual module of $L^2(T^*X)$ and the *gradient* $\nabla f \in L^2(TX)$ of a function $f \in W^{1,2}(X)$ is the unique element associated to $df$ via the Riesz isomorphism.

Let $(Y, d_Y, m_Y)$ be a metric measure space. We will say that a map $F : Y \to X$ has *bounded compression* if $F_\# m_Y \le Cm$ for some $C > 0$. Given an $L^2$-normed $L^\infty$-module $\mathcal{M}$ over $X$, the *pullback module* $F^*\mathcal{M}$ is an $L^2$-normed $L^\infty$-module over $Y$ carrying a *pullback operator* $F^* : \mathcal{M} \to F^*\mathcal{M}$ defined (uniquely up to isomorphism) in the following way: $F^*$ is linear and satisfies the following,

(i) $|F^*v| = |v| \circ F$, $m_Y$-a.e. for all $v \in \mathcal{M}$,

(ii) $\{F^*v \mid v \in \mathcal{M}\}$ generates $F^*\mathcal{M}$ as a module.

Denote by $\mathcal{M}^*$ the dual module of $\mathcal{M}$. Then, we have the *unique duality relation*,

$$F^*\mathcal{M}^* \times F^*\mathcal{M} \to L^1(Y, m_Y).$$

It is $L^\infty(Y)$-bilinear, continuous and satisfies

$$F^*w(F^*v) = w(v) \circ F, \ m_Y - a.e. \text{ for all } v \in \mathcal{M}, w \in \mathcal{M}^*.$$

For $\mathcal{M} = L^2(T^*X)$ (respectively $\mathcal{M} = L^2(TX)$) the pullback is denoted by $L^2(T^*X, F, m_Y)$ (respectively $L^2(TX, F, m_Y)$). A special instance of this construction occurs when $Y = C([0,1]; X)$ equipped with the sup distance and a test plan $\pi$ as reference measure. The evaluation maps $e_t$ have bounded compression and there exists a unique element $\pi'_t \in L^2(TX, e_t, \pi)$ such that

$$L^1(\pi) - \lim_{h \to 0} \frac{f \circ e_{t+h} - f \circ e_t}{h} = (e_t^* df)(\pi'_t)$$

for all $f \in W^{1,2}(X)$. It follows from this result that for $\pi$-a.e. $\gamma$ and a.e. $t \in [0,1]$,

$$|\pi'_t|(\gamma) = |\dot{\gamma}_t|.$$

2.3. $\mathsf{CD}^*(K,N)$ **and** $\mathsf{RCD}^*(K,N)$**-spaces.** Here we briefly recall the synthetic notions of lower Ricci curvature bounds on metric measure spaces.

A notion of metric measure spaces with Ricci curvature bounded below by $K \in \mathbb{R}$ and dimension bounded above by $N \in (1, \infty]$ was first considered in the setting of Optimal Transport Theory by Lott-Sturm-Villani [LV, Stu, Stu1], resulting in the class of spaces with the *curvature dimension condition* or briefly $\mathsf{CD}(K,N)$ *spaces.* It was then proved by Ohta that smooth compact Finsler manifolds are $\mathsf{CD}$ spaces [Ohta]. In contrast, a Finsler manifold can only arise as a limit of Riemannian manifolds with Ricci curvature uniformly bounded below if and only if it is Riemannian. Recall that a Finsler manifold is Riemannian if and only if the Cheeger energy is quadratic or, equivalently, if the heat flow is linear.

To address the problem of isolating the class of Riemannian-like $\mathsf{CD}$-spaces, Gigli proposed in [Gig2] to reinforce the definition of a $\mathsf{CD}(K,N)$ space $(X,d,m)$ with the functional-analytic condition of *infinitesimal Hilbertianity*, that is, that the Sobolev space $W^{1,2}(X,d,m)$ is an Hilbert space (see Definition 2.4). This definition came out as a result of a program initiated by Gigli in [Gig3], further developed by Gigli-Kuwada-Ohta [GKO] and Ambrosio-Gigli-Savaré [AGS14], with the aim of investigating the heat flow on metric measure spaces and the introduction of $\mathsf{RCD}(K,\infty)$ spaces [AGS, AGMR]. The finite dimensional case, i.e. $\mathsf{RCD}(K,N)$ for $N\in(1,\infty)$ was then analyzed independently in [EKS] and [AMS]).

At the emergence of $\mathsf{CD}(K,N)$ spaces, it was not clear whether this class exhibits a *local-to-global* property, i.e. whether satisfying $\mathsf{CD}(K,N)$ for all subsets of a covering implies the condition on the full space. To address this issue, Bacher-Sturm introduced an a priori slightly weaker condition of Ricci curvature bounded below by $K$ with dimension at most $N$, namely the *reduced curvature-dimension condition* or $\mathsf{CD}^*(K,N)$ [BS].

To state the definitions and results in this section, we begin by recalling the so called *distortion coefficients*. Given $K,N\in\mathbb{R}$ with $N\geq 0$, for $(t,\theta)\in[0,1]\times\mathbb{R}_+$ we define

$$\sigma^{(t)}_{K,N}(\theta) := \begin{cases} \infty, & \text{if } K\theta^2\geq N\pi^2,\\ \dfrac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})} & \text{if } 0<K\theta^2<N\pi^2,\\ t & \text{if } K\theta^2<0 \text{ and } N=0, \text{ or if } K\theta^2=0,\\ \dfrac{\sinh(t\theta\sqrt{-K/N})}{\sinh(\theta\sqrt{-K/N})} & \text{if } K\theta^2\leq 0 \text{ and } N>0. \end{cases} \tag{2.1}$$

For $N\geq 1, K\in\mathbb{R}$ and $(t,\theta)\in[0,1]\times\mathbb{R}_+$ we define

$$\tau^{(t)}_{K,N}(\theta) := t^{1/N}\sigma^{(t)}_{K,N-1}(\theta)^{(N-1)/N}. \tag{2.2}$$

Let $\mathcal{P}_2(X,d,m)$ denote the family of probability measures with finite second moment, $\mathrm{Opt}(\mu_0,\mu_1)$ the set of optimal transports between $\mu_0$ and $\mu_1$ and $\mathrm{Geo}(X)$ the set of geodesics of $X$.

**Definition 2.2** ($\mathsf{CD}$ condition). *A metric measure space $(X,d,m)$ is a $\mathsf{CD}(K,N)$ space if for each pair $\mu_0,\mu_1\in\mathcal{P}_2(X,d,m)$ there exists $\pi\in\mathrm{Opt}(\mu_0,\mu_1)$ such that*

$$\rho_t^{-1/N}(\gamma_t)\geq \tau^{(1-t)}_{K,N}(d(\gamma_0,\gamma_1))\rho_0^{-1/N}(\gamma_0)+\tau^{(t)}_{K,N}(d(\gamma_0,\gamma_1))\rho_1^{-1/N}(\gamma_1), \qquad \pi\text{-}a.e.\, \gamma\in\mathrm{Geo}(X), \tag{2.3}$$

*for all $t\in[0,1]$, where $(e_t)_\sharp\pi=\rho_t m$.*

It is worth remembering here that for a Riemannian manifold $(M,g)$ of dimension $n$ and $h\in C^2(M)$ with $h>0$, the metric measure space $(M,g,h\,\mathrm{vol})$ verifies condition $\mathsf{CD}(K,N)$ with $N\geq n$ if and only if (see Theorem 1.7 of [Stu1])

$$\mathrm{Ric}_{g,h,N}\geq Kg, \qquad \mathrm{Ric}_{g,h,N} := \mathrm{Ric}_g-(N-n)\frac{\nabla_g^2 h^{\frac{1}{N-n}}}{h^{\frac{1}{N-n}}}.$$

Here one takes the convention that if $N=n$ the generalized Ricci tensor $Ric_{g,h,N}=Ric_g$ makes sense only if $h$ is constant.

The reduced $\mathsf{CD}^*(K,N)$ condition requires the same inequality (2.3) of $\mathsf{CD}(K,N)$ but with the coefficients $\tau^{(t)}_{K,N}(\mathsf{d}(\gamma_0,\gamma_1))$ and $\tau^{(1-t)}_{K,N}(d(\gamma_0,\gamma_1))$ replaced by $\sigma^{(t)}_{K,N}(\mathsf{d}(\gamma_0,\gamma_1))$ and $\sigma^{(1-t)}_{K,N}(d(\gamma_0,\gamma_1))$, respectively. Hence while the distortion coefficients of the $\mathsf{CD}(K,N)$ condition are formally obtained by imposing one direction with linear distortion and $N-1$ directions affected by curvature, the $\mathsf{CD}^*(K,N)$ condition imposes the same volume distortion in all the $N$ directions.

Now we will recall the *generalized Bishop-Gromov comparison theorem* for $\mathsf{CD}^*(K,N)$-spaces for $K<0$. Let $B(x,R)$ be the metric ball around $x$ with radius $R$ and we denote its metric closure by

$\overline{B(x,R)}$. We note that the fact that the sharp version of this result is valid for $\mathsf{CD}^*(K,N)$ spaces is a consequence of [CS, Theorem 1.1] and [Ohta1, Theorem 5.1].

**Theorem 2.3** (Generalized Bishop-Gromov volume growth inequality for $\mathsf{CD}^*(K,N)$)**.** *Assume that the metric space $(X,d,m)$ satisfies the $\mathsf{CD}^*(K,N)$-condition for some $K<0$ and $N\in\mathbb{R}$. Then for all $r\le R$,*

$$\frac{m(\overline{B(x,r)})}{m(\overline{B(x,R)})}\ge\frac{\int_0^r\sinh^{N-1}(\sqrt{-K/(N-1)}t)\ dt}{\int_0^R\sinh^{N-1}(\sqrt{-K/(N-1)}t)\ dt}.$$

*Furthermore, for the function $s_m(x,r)=\limsup_{\delta\to0}\frac{1}{\delta}m(\overline{B(x,r+\delta)}\smallsetminus B(x,r))$ the following inequality holds*

$$\frac{s_m(x,r)}{s_m(x,R)}\ge\frac{\sinh^{N-1}(\sqrt{-K/(N-1)}r)}{\sinh^{N-1}(\sqrt{-K/(N-1)}R)}.$$

We now recall the definition of the so called Riemannian curvature-dimension condition.

**Definition 2.4** ($\mathsf{RCD}^*$ condition)**.** *A metric measure space $(X,d,m)$ is a $\mathsf{RCD}^*(K,N)$ space if it is an infinitesimally Hilbertian $\mathsf{CD}^*(K,N)$ space.*

Cavalletti-Milman have shown the equivalence of the $\mathsf{CD}$ and $\mathsf{CD}^*$ conditions when the space is essentially non-branching and has finite measure [CM, Corollary 13.7]. In particular under the assumption of finite measure, $\mathsf{RCD}(K,N)$ is equivalent to $\mathsf{RCD}^*(K,N)$. It is expected that $\mathsf{RCD}(K,N)$ is equivalent to $\mathsf{RCD}^*(K,N)$ without any further assumptions.

Now we state the *Laplacian comparison for distance functions* originally proved by Gigli for $\mathsf{RCD}(K,N)$ spaces [Gig2, Corollary 5.15] and shown to hold sharply on $\mathsf{CD}^*(K,N)$ spaces (and more generally on $\mathsf{MCP}(K,N)$ spaces) in [CaMo]. We will use this result in the following section. For simplicity we only state the result for $K<0$.

**Theorem 2.5** (Laplacian comparison for distance functions)**.** *Let $K<0$, $N\in(1,\infty)$ and $(X,d,m)$ be an $\mathsf{RCD}^*(K,N)$ space. Let $r:X\to\mathbb{R}$ be the function given by $r(x)=d(x,o)$, where $o\in X$. Then $r\in D(\mathbf{\Delta},X\smallsetminus\{o\})$ and*

$$\mathbf{\Delta}r|_{X\smallsetminus\{o\}}\le\sqrt{-K(N-1)}\coth(\sqrt{-K/(N-1)}r)m. \tag{2.4}$$

In order to introduce the notion of Hessian we recall the definition of the *algebra of Test Functions*

$$\mathrm{Test}(X):=\left\{f\in D(\Delta)\cap L^\infty(X,m)\mid|\nabla f|\in L^\infty(X,m)\text{ and }\Delta f\in W^{1,2}(X)\right\}.$$

An important fact is that if $X$ satisfies $\mathsf{RCD}^*(K,N)$ then $\mathrm{Test}(X)$ is dense in $W^{1,2}(X)$. Furthermore, if $f\in\mathrm{Test}(X)$ then $|\nabla f|^2\in W^{1,2}(X)$ and by polarization, for every $f,g\in\mathrm{Test}(X)$ we have that $\langle\nabla f,\nabla g\rangle\in W^{1,2}(X)$ (see for example [Gig1, Proposition 3.1.3]). The definition of this space allows for the definition of a *Hessian*

$$\mathrm{Hess}[f]:\mathrm{Test}(X)\times\mathrm{Test}(X)\to L^2(X,m),$$

as follows. For a function $u\in\mathrm{Test}(X)$ we define the *Hessian* of $u$ as

$$\mathrm{Hess}[u](f,g):=\frac{1}{2}\left(\langle\nabla f,\nabla\langle\nabla u,\nabla g\rangle\rangle+\langle\nabla g,\nabla\langle\nabla u,\nabla f\rangle\rangle-\langle\nabla u,\nabla\langle\nabla f,\nabla g\rangle\rangle\right). \tag{2.5}$$

We note that this is a symmetric bilinear operator. The space $W^{2,2}(X)$ consists of the functions $f\in W^{1,2}(X)$ such that for any $g_1$, $g_2$, $h\in\mathrm{Test}(X)$, there exists an $A\in L^2(T^*X)\otimes L^2(T^*X)$ such that

$$2\int hA(\nabla g_1,\nabla g_2)\,\mathrm{d}m=\int-\langle\nabla f,\nabla g_1\rangle\mathrm{div}(h\nabla g_2)-\langle\nabla f,\nabla g_2\rangle\mathrm{div}(h\nabla g_1)-h\langle\nabla f,\nabla\langle\nabla g_1,\nabla g_2\rangle\rangle\,\mathrm{d}m.$$

There is a unique such $A$ in $L^2(T^*X) \otimes L^2(T^*X)$ which is denoted by $\mathrm{Hess}(f)$ (see [Gig1, Section 1.5] for details). A very important result [Gig1, Theorem 3.3.8] states that $\mathrm{Test}(X) \subset W^{2,2}(X)$ and that for every $g_1, g_2 \in \mathrm{Test}(X)$,

$$\mathrm{Hess}[f](g_1, g_2) = \mathrm{Hess}(f)(\nabla g_1, \nabla g_2). \tag{2.6}$$

2.4. **Bakry-Émery condition and Bochner's inequality.** We begin this section by recalling the weak version of Bochner's inequality obtained by Ambrosio-Mondino-Savare [AMS] and Erbar-Kuwada-Sturm [EKS].

**Theorem 2.6** (Weak Bochner's inequality [EKS, AMS])**.** *Let $(X, d, m)$ be an $\mathsf{RCD}^*(K, N)$-space. Then, for all $f \in D(\Delta)$ with $\Delta f \in W^{1,2}(X, d, m)$ and all $g \in D(\Delta) \cap L^\infty(X, m)$ non-negative with $\Delta g \in L^\infty(X.m)$ we have*

$$\frac{1}{2}\int \Delta g |\nabla f|^2 dm - \int g \langle \nabla(\Delta f), \nabla f\rangle dm \geq K \int g|\nabla f|^2 dm + \frac{1}{N}\int g(\Delta f)^2 dm. \tag{2.7}$$

A remarkable property is the equivalence of the $\mathsf{RCD}^*(K, N)$ condition and the Bochner inequality under some conditions (namely the Sobolev-to-Lipschitz property—which we recall below—and a certain volume growth estimate). The infinite dimensional case was settled in [AGS], while the (technically more involved) finite dimensional refinement was established in [EKS] and [AMS].

Let $f, g \in \mathrm{Test}(X)$ and define the measure-valued map

$$\Gamma_2(f, g) := \frac{1}{2}\mathbf{\Delta}\langle \nabla f, \nabla g\rangle - \frac{1}{2}\left(\langle \nabla f, \nabla \Delta g\rangle + \langle \nabla g, \nabla \Delta f\rangle\right) m.$$

Let $\Gamma_2(f) := \Gamma_2(f, f)$. It was shown by Ambrosio-Mondino-Savaré [AMS] and Erbar-Kuwada-Sturm [EKS] that the following non-smooth Bakry-Émery condition is satisfied on an $\mathsf{RCD}^*(K, N)$-space: for every $f \in \mathrm{Test}(X)$

$$\Gamma_2(f) \geq \left(K|\nabla f|^2 + \frac{1}{N}(\Delta f)^2\right) m. \tag{2.8}$$

Now we state a fundamental technical tool (see [Sav]) which is useful when "changing variables". In the following, $\Psi : \mathbb{R}^n \to \mathbb{R}$ is a smooth function such that $\Psi(0) = 0$. We denote $\Psi_i := \partial_i \Psi$ and $\Psi_{ij} := \partial_{ij}\Psi$. We will also let $f_1, \ldots, f_n \in \mathrm{Test}(X)$, and $\Psi(f) := \Psi(f_1, \ldots, f_n) : X \to \mathbb{R}$.

**Proposition 2.7** ([Sav])**.** *The function $\Psi(f)$ is in $\mathrm{Test}(X)$ and the following formulas hold true:*

(i) $|\nabla \Psi(f)|^2 m = \sum_{i,j}^n \Psi_i(f)\Psi_j(f)\langle \nabla f_i, \nabla f_j\rangle m$,

(ii) $\mathbf{\Delta}(\Psi(f)) = \sum_i \Psi_i(f)\mathbf{\Delta}(f_i) + \sum_{i,j}^n \Psi_{ij}(f)\langle \nabla f_i, \nabla f_j\rangle m$,

(iii) $\Gamma_2(\Psi(f)) = \sum_{ij}^n \Psi_i(f)\Psi_j(f)\Gamma_2(f_i, f_j) + 2\sum_{i,j,k}^n \Psi_i(f)\Psi_{jk}(f)H[f_i](f_j, f_k)m$
$+ \sum_{i,j,k,h}^n \Psi_{ik}(f)\Psi_{jh}(f)\langle \nabla f_i, \nabla f_j\rangle\langle \nabla f_k, \nabla f_h\rangle m$.

2.5. **Isomorphisms of metric measure spaces.** This is an account of several results in [Gig4]. We consider metric measure spaces $(X, d, m)$ such that $(X, d)$ is complete and separable and $m$ is a non-negative Radon measure on $X$. We begin by recalling the definition of isomorphism of metric measure spaces.

**Definition 2.8** (Isomorphisms between metric measure spaces)**.** *We will say that two metric measure spaces $(X_1, d_1, m_1)$ and $(X_2, d_2, m_2)$ are* isomorphic *provided there exists an isometry $T : (\mathrm{supp}(m_1), d_1) \to (\mathrm{supp}(m_2), d_2)$ such that $T_\sharp m_1 = m_2$. Any such $T$ is called an isomorphism.*

The following property will allow us to study isomorphisms between metric measure spaces in terms of isometries between their $W^{1,2}$ spaces, see Proposition 2.11.

**Definition 2.9** (Sobolev to Lipschitz property)**.** *Let $(X,d,m)$ be a metric measure space. We say that $(X,d,m)$ has the Sobolev to Lipschitz property if any $f \in W^{1,2}(X,d,m)$ with $|\nabla f| \le 1$ $m$-a.e. admits a 1-Lipschitz representative, that is, a 1-Lipschitz map $g : X \to \mathbb{R}$ such that $f = g$ $m$-a.e..*

Gigli showed (using a result of Rajala [Raj]) that $\mathsf{CD}(K,N)$-spaces have the Sobolev to Lipschitz property. Furthermore, Ambrosio-Gigli-Savaré showed that $\mathsf{RCD}(K,\infty)$-spaces also have the Sobolev to Lipschitz property (see the paragraph after [Gig4, Definition 4.9]). As $\mathsf{CD}^*(K,N)$ spaces are $\mathsf{CD}(K^*,N)$ spaces for a suitable value of $K^*$ (see [Cav] and [CS]), $\mathsf{RCD}^*(K,N)$ spaces also satisfy the Sobolev to Lipschitz property.

**Lemma 2.10** (Contractions by local duality [Gig4, Lemma 4.19])**.** *Let $(X_1,d_1,m_1)$ and $(X_2,d_2,m_2)$ be two metric measure spaces with the Sobolev to Lipschitz property where $m_2$ gives finite mass to bounded sets, and $T : X_1 \to X_2$ a Borel map such that $T_\sharp m_1 \le C m_2$ for some $C > 0$. Then the following are equivalent*

i) *$T$ is $m_1$-a.e. equivalent to a 1-Lipschitz map from $(\mathsf{supp}(m_1),d_1)$ to $(\mathsf{supp}(m_2),d_2)$*
ii) *For any $f \in W^{1,2}(X_2,d_2,m_2)$ we have $f \circ T \in W^{1,2}(X_1,d_1,m_1)$ , and moreover,*

$$|\nabla (f\circ T)| \le |\nabla f| \circ T, \qquad m_1 - a.e..$$

**Proposition 2.11** (Isomorphisms via duality with Sobolev norms [Gig4, Proposition 4.20])**.** *Let $(X_1,d_1,m_1)$ and $(X_2,d_2,m_2)$ be two metric measure spaces with the Sobolev to Lipschitz property and $T : X_1 \to X_2$ a Borel map. Assume that both $m_1$ and $m_2$ give finite mass to bounded sets. Then the following are equivalent.*

i) *Up to a modification on a $m_1$-negligible set, $T$ is an isomorphism of the metric measure spaces*
ii) *The following two are true.*
   ii-a) *There exist a Borel $m_1$-negligible set $\mathcal{N} \subset X_1$ and a Borel map $S : X_2 \to X_1$ such that $S(T(x)) = x$, $\forall x \in X_1 \setminus \mathcal{N}$.*
   ii-b) *The right composition with $T$ produces a bijective isometry of $W^{1,2}(X_2,d_2,m_2)$ in $W^{1,2}(X_1,d_1,m_1)$, i.e. $f \in W^{1,2}(X_2,d_2,m_2)$ if and only if $f\circ T \in W^{1,2}(X_1,d_1,m_1)$ and in this case $\|f\|_{W^{1,2}(X_2)} = \|f\circ T\|_{W^{1,2}(X_1)}$.*

2.6. **Warped product of metric measured spaces.** Here we review the main definitions and results concerning the warped products of metric measure spaces following Gigli-Han [GH].

Let $(X,d_X,m_X)$ and $(Y,d_Y,m_Y)$ be two complete and separable metric measured spaces and $w_d, w_m : Y \to [0,\infty)$ two continuous functions such that $\{w_d = 0\} \subset \{w_m = 0\}$. The $l_w$-length of an absolutely continuous curve $\gamma = (\gamma^Y,\gamma^X)$ in $Y \times X$ is defined by

$$l_w[\gamma] = \int_0^1 \sqrt{|\dot\gamma_t^Y|^2 + w_d^2(\gamma_t^Y)|\dot\gamma_t^X|^2}\,\mathrm{d}t.$$

The function $d_w : (Y\times X)^2 \to \mathbb{R}$ given by

$$d_w(p,q) = \inf\{l_w[\gamma] : \gamma \text{ is an absolutely continuous curve from } p \text{ to } q\}$$

is a pseudometric. Hence, it induces an equivalence relation on $Y\times X$. By taking the quotient and then its completion we obtain a metric space denoted by $Y\times_w X$ and an induced distance denoted also by $d_w$. If $w_d(y) > 0$ there is no abuse in denoting the elements of $Y\times_w X$ by $(y,x)$ with $y\in Y$ and $x \in X$, because points in the completion not coming from points in $Y\times X$ will be negligible with respect to the measure or $Y\times_w X$. The same holds for the set of $(y,x)$ such that $w_d(y) = 0$.

The measure $m_w$ on $Y\times_w X$ is defined as

$$\int f(x)g(y)\,\mathrm{d}m_w(y,x) = \int \left(\int f(x)w_m(y)\,\mathrm{d}m_X(x)\right) g(y)\,\mathrm{d}m_Y(y), \tag{2.9}$$

for any Borel non-negative functions $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$.

The warped product of $(X,d_X,m_X)$ and $(Y,d_Y,m_Y)$ via the functions $w_d$ and $w_m$, called warping functions, is the metric measure space denoted by $(Y\times_w X,d_w,m_w)$. By definition $(Y\times_w X,d_w,m_w)$ is complete, separable and is a length space.

**Definition 2.12** (Almost everywhere locally doubling space). *Let $(X,d,m)$ be a metric measure space. We say that it is an almost everywhere locally doubling space provided there exists a Borel set $B$ with $m$-negligible complement such that for every $x\in B$ there exists an open set $U$ containing $x$ and constants $C,R>0$ for which*

$$m(B_{2r}(y))\le Cm(B_r(y))$$

*for $r\in(0,R)$ and $y\in U$.*

**Definition 2.13** (Measured-length space). *Let $(X,d,m)$ be a metric measure space. We say that it is measured-length if there exists a Borel set $A\subset X$ with $m$-negligible complement that satisfies the following. For all $x_0,x_1\in A$, there exist $\varepsilon>0$ and a map $(0,\varepsilon]^2\to\mathcal{P}(X)$, $(\varepsilon_0,\varepsilon_1)\mapsto\pi^{\varepsilon_0,\varepsilon_1}$, such that*

- *For any $\varphi\in C_b(C[0,1],X)$, the map $(0,\varepsilon]^2\to\mathbb{R}$ given by*
$$(\varepsilon_0,\varepsilon_1)\mapsto\int\varphi d\pi^{\varepsilon_0,\varepsilon_1},$$
*is Borel.*
- *For every $\varepsilon_0,\varepsilon_1\in(0,\varepsilon]$ and $i=1,2$, we have*
$$(e_i)_\sharp\pi^{\varepsilon_0,\varepsilon_1}=\frac{1_{B_{\varepsilon_i}}(x_i)}{m(B_{\varepsilon_i}(x_i))}m.$$
- *We have*
$$\overline{\lim_{\varepsilon_0,\varepsilon_1\downarrow 0}}\int\int_0^1|\dot{\gamma_t}|\,dt\,d\pi^{\varepsilon_0,\varepsilon_1}(\gamma)\le d^2(x_0,x_1).$$

**Theorem 2.14** ([GH, Theorem 3.22]). *Let $(X,d,m)$ be an a.e. locally doubling and measured-length space, $I\subset\mathbb{R}$ a closed, possibly unbounded, interval and $w_d,w_m:I\to[0,\infty)$ a couple of warping functions. Assume that $w_m$ is strictly positive in the interior of $I$. Then the warped product space $(X_w,d_w,m_w)$, where $X_w=I\times_w X$, is almost everywhere doubling and a measured-length space. Hence, it has the Sobolev-to-Lipschitz property.*

The following result may be shown from the equivalence of the so-called Beppo-Levi space ([GH, Definitions 3.8, 3.9]) and the Sobolev space on warped products obtained by Gigli-Han. For simplicity, we will not restate here the precise definition of the Beppo-Levi space, rather only summarize their results in a manner suitable for our purposes (cf. [GH, Propositions 3.10, 3.13, 3.14]). Given $f:X_w\to\mathbb{R}$, let $f^{(t)}:X\to\mathbb{R}$ and $f^{(x)}:I\to\mathbb{R}$ denote the functions $f^{(t)}(x)=f(t,x)$ and $f^{(x)}(t)=f(t,x)$.

**Theorem 2.15** ([GH]). *Let $(X,d,m)$ be a metric measure space, $I\subset\mathbb{R}$ a closed, possibly unbounded, interval and $w_d,w_m:I\to[0,\infty)$ warping functions. Suppose that $\{w_m=0\}$ is finite and for some $C\in\mathbb{R}$, $w_m(t)\le C\inf_{s:w_m(s)=0}|t-s|$ for all $t\in I$, then the following two are equivalent:*

1. $f\in W^{1,2}(X_w,d_w,m_w)$
2. (i) *for $m$-a.e. $x\in X$ we have $f^{(x)}\in W^{1,2}(\mathbb{R},d_{Euc},w_m\mathcal{L}^1)$,*
   (ii) *for $w_m\mathcal{L}^1$-a.e. $t\in\mathbb{R}$ we have $f^{(t)}\in W^{1,2}(X)$,*
   (iii) *For all $(t,x)\in X_w$,*

$$|\nabla f|^2_{X_w}(t,x)=w_d^{-2}(t)|\nabla f^{(t)}|^2_X(x)+|\nabla f^{(x)}|_{L^2(\mathbb{R},d_{Eucl},w_m\mathcal{L}^1)}. \tag{2.10}$$

**Corollary 2.16.** *With the same notation and assumptions of Theorem 2.15 the following are true.*

i) *Let $f \in S^2_{\mathrm{loc}}(X_w)$. Then for $m$-a.e. $x$, $f^{(x)} \in S^2_{\mathrm{loc}}(\omega_m \mathcal{L}^1)$. For $\omega_m\mathcal{L}^1$-a.e. $t$, $f^{(t)} \in S^2_{\mathrm{loc}}(X)$. Furthermore, equation (2.10) holds in this setting.*

ii) *Let $f_1 \in S^2_{\mathrm{loc}}(w_m\mathbb{R})$ and define $f : X_w \to \mathbb{R}$ by $f(t,x) = f_1(t)$. Then $f \in S^2_{\mathrm{loc}}(X_w)$ and*

$$|\nabla f|_{X_w}(t,x) = |\nabla f_1|_{w_m\mathbb{R}}(t), \qquad m_w - a.e.\ (t,x).$$

iii) *Let $f_2 \in S^2_{\mathrm{loc}}(X)$ and define $f : X_w \to \mathbb{R}$ by $f(t,x) := f_2(x)$. Then $f \in S^2_{\mathrm{loc}}(X_w)$ and*

$$|\nabla f|_{X_w}(t,x) = w_d^{-1}(t)|\nabla f_2|_X(x), \qquad m_w - a.e.\ (t,x).$$

*Proof.* All the properties follow from the previous theorem with a truncation and cut-off argument based on the locality property of minimal weak upper gradients, see subsection 2.1. □

**Corollary 2.17.** *With the same notation and assumptions of Theorem 2.15, if $(X,d,m)$ is infinitesimally Hilbertian then the metric measure space $(X_w, d_w, m_w)$ is infinitesimally Hilbertian.*

*Proof.* Let $f,g \in S^2_{loc}(X_w)$. By Theorem 2.15 we get

$$|\nabla(f+g)|^2_{X_w} + |\nabla(f-g)|^2_{X_w} = w_d^{-2}(|\nabla(f+g)^{(t)}|^2_X + |\nabla(f-g)^{(t)}|^2_X) + (|\nabla(f+g)^{(x)}|^2_{w_m\mathbb{R}} + |\nabla(f-g)^{(x)}|^2_{w_m\mathbb{R}}).$$

Now, by Corollary 2.16 above we know that $f^{(t)}, g^{(t)} \in S^2_{loc}(X)$ and $f^{(x)}, g^{(x)} \in S^2_{loc}(w_m\mathcal{L}^1)$. As $(X,d,m)$ is infinitesimally Hilbertian,

$$|\nabla(f^{(t)} + g^{(t)})|^2_X + |\nabla(f^{(t)} - g^{(t)})|^2_X = 2\big(|\nabla f^{(t)}|^2_X + |\nabla g^{(t)}|^2_X\big) \qquad m - \text{a.e.}$$

In a similar way, because $(\mathbb{R}, d_{Euc}, \omega_m\mathcal{L}^1)$ is Hilbertian we obtain

$$|\nabla(f^{(x)} + g^{(x)})|^2_{w_m\mathbb{R}} + |\nabla(f^{(x)} - g^{(x)})|^2_{w_m\mathbb{R}} = 2\big(|\nabla f^{(t)}|^2_{w_m\mathbb{R}} + |\nabla g^{(t)}|^2_{w_m\mathbb{R}}\big) \qquad w_m\mathcal{L}^1 - \text{a.e..}$$

Putting the equations together and because the choices of $f,g \in S^2_{loc}(X_w)$ were arbitrary, we get the result. □

Now we define,

$$\mathcal{G} = \Big\{ g \in S^2_{\mathrm{loc}}(X_w) \mid g(x,t) = \tilde{g}(x) \text{ for some } \tilde{g} \in S^2(X) \cap L^\infty(X) \Big\},$$
$$\mathcal{H} = \Big\{ h \in S^2_{\mathrm{loc}}(X_w) \mid h(x',t) = \tilde{h}(t) \text{ for some } \tilde{h} \in S^2(w_m\mathbb{R}) \cap L^\infty(\mathbb{R}) \Big\},$$
$$\mathcal{A} = \text{algebra generated by } \mathcal{G} \cup \mathcal{H} \subset S^2_{\mathrm{loc}}(X_w).$$

**Proposition 2.18.** *Let $(X,d,m)$ be a metric measure space and $w_d, w_m : \mathbb{R} \to [0,\infty)$ warping functions. Suppose that $\{w_m = 0\}$ is finite and for some $C \in \mathbb{R}$,*

$$w_m(t) \le C \inf_{s:w_m(s)=0} |t-s|$$

*for all $t \in I$, then the set $\mathcal{A} \cap W^{1,2}(X_w)$ is dense in $W^{1,2}(X_w)$.*

*Proof.* Consider the algebra

$$\mathcal{A}^b_a = \text{ algebra generated by } (\mathcal{G} \cup \mathcal{H} \cap S^2_{\mathrm{loc}}(([a,b] \times_w X, d_w, m_w)).$$

By the cartesian product case proved in [Gig4, Proposition 6.6] (see also [DePG, Proposition 3.35]), $\mathcal{A}^b_a \cap W^{1,2}(X_w)$ is dense in $W^{1,2}([a,b] \times_w X, d_w, m_w)$ whenever $[a,b] \subset \mathbb{R} \setminus \{w_m = 0\}$.

It follows that $\mathcal{A} \cap W^{1,2}(X_w)$ is dense in $BL_0(X_w)$ which is the closure in $BL(X_w)$ of the space of functions which vanish in a neighborhood of $\{w_m = 0\} \cup \{\infty\}$. However, under the hypotheses, [GH, Proposition 3.14] shows that $BL_0(X_w) = BL(X_w) = W^{1,2}(X_w)$ which implies the statement. □

2.7. **Universal covers of $\mathsf{RCD}^*$ spaces.** A metric space $(Y,d_Y)$ is a *covering space* of $(X,d_X)$ if there exists a continuous map $p:Y\to X$ such that for every point $x\in X$ there exists a neighborhood $U_x\subset X$ with the property that $p^{-1}(U_x)$ is a disjoint union of open subsets of $Y$ each of which is mapped homeomorphically onto $U_x$ by $p$.

A (connected) metric space $(\tilde X, d_{\tilde X})$ is a *universal cover* of $X$, with the covering map $\tilde p$, if for any other covering space $Y$ of $X$ with the covering map $p$ there exists a continuous map $f:\tilde X\to Y$ such that $p\circ f=\tilde p$. Whenever a universal cover exists, it is unique. (Note that we do not require $X$ to be semilocally simply connected, so $\tilde X$ need not be simply connected.)

In the presence of the $\mathsf{RCD}^*$ condition, the following theorem was obtained by Mondino-Wei [MW, Theorem 1.1].

**Theorem 2.19.** *Let $(X,d,m)$ be an $\mathsf{RCD}^*(K,N)$-space for some $K\in\mathbb{R}$, $N\in(1,\infty)$. Then $(X,d,m)$ admits a universal cover $(\tilde X,\tilde d,\tilde m)$, with $\tilde m$ given by the pullback measure via the covering map, which is itself an $\mathsf{RCD}^*(K,N)$-space.*

## 3. Construction of a Busemann function

In this section we prove that the volume entropy of compact $\mathsf{RCD}^*(-(N-1),N)$ spaces is bounded above by $N-1$. In the equality case, we construct a Busemann type function $u$ defined on the universal cover of the space. Finally we show the existence and main properties of the regular Lagrangian flow of $\nabla u$.

### 3.1. Volume growth entropy estimate for $\mathsf{RCD}^*$ spaces.

**Theorem 3.1.** *Let $(X,d,m)$ be an $\mathsf{RCD}^*(K,N)$-space with $N\in(1,\infty)$ and $K<0$. Then*

$$h(X)\le\sqrt{-K(N-1)}.$$

*Proof.* By the work of Mondino-Wei [MW], the universal cover space $\widetilde X$ is also an $\mathsf{RCD}^*(K,N)$ space. In particular, it is a $\mathsf{CD}^*(K,N)$ space. Let $R>0$ and let us fix $r_0$ such that $0<r_0<R$. By Theorem 2.3,

$$\widetilde m(B_{\widetilde X}(x,R))\int_0^{r_0\sqrt{-K/(N-1)}}\sinh^{N-1}t\ dt\le\widetilde m(B_{\widetilde X}(x,r_0))\int_0^{R\sqrt{-K/(N-1)}}\sinh^{N-1}t\ dt.$$

Taking logarithms, dividing by $R$ and taking the limsup on both sides of the previous inequality we get

$$h(X)\le\lim_{R\to\infty}\frac{1}{R}\ln\left(\int_0^{R\sqrt{-K/(N-1)}}\sinh^{N-1}t\ dt\right).$$

To conclude, we use L'Hôpital's rule. □

The next corollary follows directly by taking $K=-(N-1)$.

**Corollary 3.2.** *Let $(X,d,m)$ be an $\mathsf{RCD}^*(-(N-1),N)$-space with $N\in(1,\infty)$. Then $h(X)\le N-1$.*

We remark that the previous volume entropy growth estimate holds in the more general setting of spaces which satisfy the so-called *measure contraction property* introduced by Ohta [Ohta1] and Sturm [Stu1]. Indeed, a Bishop-Gromov type inequality was obtained in [Ohta1, Theorem 5.1] and the proofs of Theorem 3.1 and Corollary 3.2 can be carried out in this setting analogously.

3.2. **Construction of a Busemann function.** In this section we will prove the following result on the existence of a Busemann-type function on the universal cover of a compact $\mathsf{RCD}^*(K,N)$ space with maximal volume entropy. We will follow the strategy developed by Liu [Liu], with the necessary adaptations (cf. [Jiang, Theorem 1.7]). More precisely, we will prove:

**Theorem 3.3.** *Let $(X,d,m)$ be a compact $\mathsf{RCD}^*(K,N)$ space with $K<0$ and $N>1$, and let $(\tilde{X},\tilde{d},\tilde{m})$ be its universal cover. If $h(X)=\sqrt{-K(N-1)}$, then there exists a function $u:\tilde{X}\to\mathbb{R}$ with $u\in D_{loc}(\Delta)$ , that satisfies $|\nabla u|=1$ $\tilde{m}$-a.e. and $\Delta u=\sqrt{-K(N-1)}$ $\tilde{m}$-a.e. .*

The theorem follows from the next technical lemma.

**Lemma 3.4.** *Let $(X,d,m)$ be a compact $\mathsf{RCD}^*(K,N)$ space with $K<0$, $N>1$, and $(\tilde{X},\tilde{d},\tilde{m})$ its universal cover. If $h(X)=\sqrt{-K(N-1)}$, then for any $y_0\in\tilde{X}$ and $R>50\,\mathrm{diam}(X)$ there exists $u_R:B_R(y_0)\to\mathbb{R}$ Lipschitz with $|\nabla u_R|=1$ $\tilde{m}$-a.e. and $\Delta u_R=\sqrt{-K(N-1)}$ $\tilde{m}$-a.e. .*

To prove the previous lemma we need the following propositions. Set $Q:=\sqrt{-K(N-1)}$. Let us recall the definition of the function $s_{\tilde{m}}$ appearing in Theorem 2.3:

$$s_{\tilde{m}}(x,r)=\limsup_{\delta\to 0}\frac{1}{\delta}\tilde{m}\left(\overline{B(x,r+\delta)}\setminus B(x,r)\right)$$

**Proposition 3.5.** *For any $o\in\tilde{X}$ we have*

$$\limsup_{r\to\infty}\frac{s_{\tilde{m}}(o,r+50R)}{s_{\tilde{m}}(o,r-50R)}=\exp^{100QR}.$$

*In particular, there is a sequence of positive numbers $r_i$ with $\lim_{i\to\infty}r_i=\infty$, such that $\frac{s_{\tilde{m}}(o,r_i+50R)}{s_{\tilde{m}}(o,r_i-50R)}$ is a monotonic increasing sequence converging to $\exp^{100QR}$.*

*Proof.* Since $h(X)=N-1>0$, $\tilde{X}$ has infinite diameter. Recall that by Mondino-Wei [MW], $(\tilde{X},\tilde{d},\tilde{m})$ is an $\mathsf{RCD}^*(K,N)$ space. By Theorem 2.3,

$$\frac{s_{\tilde{m}}(o,r+50R)}{s_{\tilde{m}}(o,r-50R)}\le\frac{\sinh^{N-1}(Q(r+50R))}{\sinh^{N-1}(Q(r-50R))}.$$

Notice that

$$\lim_{r\to\infty}\frac{\sinh^{N-1}(Q(r+50R))}{\sinh^{N-1}(Q(r-50R))}=\exp^{100QR}.$$

We will show that

$$\limsup_{r\to\infty}\frac{s_{\tilde{m}}(o,r+50R)}{s_{\tilde{m}}(o,r-50R)}=\exp^{100QR}.$$

By contradiction, suppose that there exist $r_0>100R$ and $\varepsilon>0$ such that for any $r\ge r_0$,

$$\frac{s_{\tilde{m}}(o,r+50R)}{s_{\tilde{m}}(o,r-50R)}\le(1-\varepsilon)\exp^{100QR}.$$

Therefore, for any $r>r_0$ big enough we have that

$$s_{\tilde{m}}(o,r)\le(1-\varepsilon)\exp^{100QR}s_{\tilde{m}}(o,r-100R).$$

Iterating this inequality $\lfloor\frac{r-r_0}{100R}\rfloor$ times, where $\lfloor\frac{r-r_0}{100R}\rfloor$ is the largest integer smaller than or equal to $\frac{r-r_0}{100R}$, we get

$$s_{\tilde{m}}(o,r)\le\left((1-\varepsilon)\exp^{100QR}\right)^{\lfloor\frac{r-r_0}{100R}\rfloor}s_{\tilde{m}}(o,r-\lfloor\tfrac{r-r_0}{100R}\rfloor100R).$$

Now, $r-\lfloor \frac{r-r_0}{100R} \rfloor 100R = r_0+t$ for some $t\in[0,100R)$. Hence, by Theorem 2.3 and as the hyperbolic sine is an increasing function:

$$\begin{aligned} s_{\tilde m}(o, r-\lfloor \tfrac{r-r_0}{100R} \rfloor 100R) &\leq s_{\tilde m}(o,r_0)\frac{\sinh^{N-1}(Q(r-\lfloor \frac{r-r_0}{100R} \rfloor 100R))}{\sinh^{N-1}(Qr_0)} \\ &\leq s_{\tilde m}(o,r_0)\frac{\sinh^{N-1}(Q(r_0+100R))}{\sinh^{N-1}(Qr_0)}. \end{aligned}$$

Thus, for $r\geq r_0$

$$s_{\tilde m}(o,r)\leq c(N,K,r_0,R)\left((1-\varepsilon)\exp^{100QR}\right)^{\frac{r-r_0}{100R}},$$

where we used that $\lfloor \frac{r-r_0}{100R} \rfloor \leq \frac{r-r_0}{100R}$. Integrating $s_{\tilde m}(o,\cdot)$ from $r_0$ to $r$ and using the previous inequality, we get an upper bound of $\tilde m(B(o,r)\smallsetminus B(o,r_0))$. Using this bound, we obtain

$$h(X)=\limsup_{r\to\infty}\frac{1}{r}\ln\tilde m(B(o,r))<Q.$$

This contradicts $h(X)=Q$, and concludes the proof. □

For the following proposition let us recall that any distance function $r(x):=\tilde d(o,x)$ on $\tilde X$ has a well-defined measure valued Laplacian on $\tilde X\smallsetminus\{o\}$. Then, we have the following.

**Proposition 3.6.** *Let $r:\tilde X\to\mathbb{R}$ be the function given by $r(y)=\tilde d(y,o)$. Then*

$$\int_{B(o,t)\smallsetminus\{o\}}\mathbf{\Delta} r = s_{\tilde m}(o,t),$$

*for $s_{\tilde m}$ the same as in Theorem 2.3.*

*Proof.* Let $\{\delta_i\}_{i\in\mathbb{N}}$ be a decreasing sequence such that $\delta_i\to 0$. For each $\delta_i$ define a function $f_{\delta_i}:\tilde X\to\mathbb{R}$ by

$$f_{\delta_i}(x):=\begin{cases}\delta_i & \text{if } x\in B(o,t)\\ -r(x)+\delta_i+t & \text{if } x\in A(o,t,t+\delta_i)\\ 0 & \text{otherwise,}\end{cases}$$

where $A(o,t,t+\delta_i):=\left\{x\in\tilde X \mid t<\tilde d(o,x)<t+\delta_i\right\}$. We observe that $f_{\delta_i}\in W^{1,2}(\tilde X,\tilde d,\tilde m)$ for all $i\in\mathbb{N}$. Then,

$$\int_{B(o,t+\delta_i)\smallsetminus\{o\}} f_{\delta_i}\mathbf{\Delta} r=\int_{B(o,t)\smallsetminus\{o\}}\delta_i\mathbf{\Delta} r+\int_{A(o,t,t+\delta_1)} f_{\delta_i}\mathbf{\Delta} r.$$

By the definition of $\mathbf{\Delta} r$ and $f_{\delta_i}$ we now have that,

$$\begin{aligned}\int_{B(o,t+\delta_i)\smallsetminus\{o\}} f_{\delta_i}\mathbf{\Delta} r &= -\int_{B(o,t+\delta_i)\smallsetminus\{o\}}\langle\nabla f_{\delta_i},\nabla r\rangle\,\mathrm{d}\tilde m\\ &= \int_{A(o,t,t+\delta_i)}\langle\nabla r,\nabla r\rangle\,\mathrm{d}\tilde m\\ &= \tilde m(A(o,t,t+\delta_i)).\end{aligned}$$

Hence,

$$\int_{B(o,t)\smallsetminus\{o\}}\delta_i\mathbf{\Delta} r+\int_{A(o,t,t+\delta_i)} f_{\delta_i}\mathbf{\Delta} r=\tilde m(A(o,t,t+\delta_i)).$$

Now choose $\delta_i$ to a specific sequence achieving the lim sup in the definition of $s_{\tilde m}$. Dividing the previous equality by $\delta_i$ and taking the limit when $i\to\infty$, we get:

$$\int_{B(o,t)\smallsetminus\{o\}}\mathbf{\Delta} r+\lim_{i\to\infty}\int_{A(o,t,t+\delta_1)}\tfrac{f_{\delta_i}}{\delta_i}\mathbf{\Delta} r=s_{\tilde m}(o,t)$$

Notice that $0 \le \frac{f_{\delta_i}}{\delta_i} \le 1$ and $\lim_{i\to\infty} \frac{f_{\delta_i}}{\delta_i}(x) = 0$ in $A(o,t,t+\delta_i)$. Thus, it follows from the dominated convergence theorem that

$$\lim_{i\to\infty} \int_{A(o,t,t+\delta_i)} \frac{f_{\delta_i}}{\delta_i} \mathbf{\Delta} r = 0.$$

The result follows. □

**Remark 3.7.** *The final part of the above proof shows that $s_{\tilde{m}}(x,t)$ is actually a limit,*

$$s_{\tilde{m}}(x,r) = \lim_{\delta\to 0} \frac{1}{\delta} \tilde{m}\left(\overline{B(x,r+\delta)} \setminus B(x,r)\right)$$

Let $A \subset \tilde{X}$. In the following proposition, we will use the notation $\fint_A \mathbf{\Delta} r := \frac{\int_A \mathbf{\Delta} r}{\tilde{m}(A)}$.

**Proposition 3.8.** *Set $A_i = \{y \in \tilde{X} \,|\, r_i - 50R \le \tilde{d}(o,y) \le r_i + 50R\}$. Then,*

$$\fint_{A_i} \mathbf{\Delta} r \ge Q - \Psi(i|K,N,R)$$

*where* $\lim_{i\to\infty} \Psi(i|K,N,R) = 0$ *and $r : \tilde{X} \to R$ is the function $r(y) = \tilde{d}(y,o)$.*

*Proof.* We now prove that $\fint_{A_i} \mathbf{\Delta} r \ge Q - \Psi(i|K,N,R)$. By the previous result and the definition of $A_i$,

$$\int_{A_i} \mathbf{\Delta} r = s_{\tilde{m}}(o, r_i + 50R) - s_{\tilde{m}}(o, r_i - 50R).$$

Recall that, as $i$ goes to infinity,

$$\frac{s_{\tilde{m}}(o, r_i + 50R)}{s_{\tilde{m}}(o, r_i - 50R)} \uparrow \exp^{100QR},$$

and therefore, there exist $\Psi(i|K,N,R) > 0$ such that $\lim_{i\to\infty} \Psi(i|K,N,R) = 0$ and

$$\frac{s_{\tilde{m}}(o, r_i + 50R)}{s_{\tilde{m}}(o, r_i - 50R)} + \Psi(i|K,N,R) \ge \exp^{100QR}.$$

Thus,

$$\begin{aligned}\fint_{A_i} \mathbf{\Delta} r &= \frac{s_{\tilde{m}}(o, r_i + 50R)}{\tilde{m}(A_i)} - \frac{s_{\tilde{m}}(o, r_i - 50R)}{\tilde{m}(A_i)} \\ &\ge \frac{s_{\tilde{m}}(o, r_i - 50R)}{\tilde{m}(A_i)}(\exp^{100QR} - 1) - \frac{s_{\tilde{m}}(o, r_i - 50R))}{\tilde{m}(A_i)} \Psi(i|K,N,R).\end{aligned}$$

Hence we only need to show that

$$\lim_{i\to\infty} \frac{s_{\tilde{m}}(o, r_i - 50R)}{\tilde{m}(A_i)} = \frac{Q}{\exp^{100QR} - 1}.$$

This would imply the existence of $\Psi(i|K,N,R) > 0$ that satisfies the claim.

By the Bishop-Gromov Comparison Theorem 2.3 we have that for $t \in [r_i - 50R, r_i + 50R]$,

$$\begin{aligned}\frac{\tilde{m}(A_i)}{s_{\tilde{m}}(o, r_i - 50R)} &= \int_{r_i-50R}^{r_i+50R} \frac{s_{\tilde{m}}(o,t)}{s_{\tilde{m}}(o, r_i - 50R)} dt \\ &\le \int_{r_i-50R}^{r_i+50R} \frac{\sinh^{N-1}(Qt)}{\sinh^{N-1}(Q(r_i - 50R))} dt \\ &= \frac{\int_{r_i-50R}^{r_i+50R} \sinh^{N-1}(Qt)\, dt}{\sinh^{N-1}(Q(r_i - 50R))}.\end{aligned}$$

Using L'Hôpital's rule we conclude,

$$\begin{aligned}
\lim_{i\to\infty} \frac{\tilde{m}(A_i)}{s_{\tilde{m}}(o, r_i - 50R)} &\leq \lim_{i\to\infty} \frac{\int_{r_i-50R}^{r_i+50R} \sinh^{N-1}(Qt)\,dt}{\sinh^{N-1}(Q(r_i - 50R))} \\
&= \lim_{i\to\infty} \frac{-\sinh^{N-1}(Q(r_i-50R)) + \sinh^{N-1}(Q(r_i+50R))}{(N-1)Q\sinh^{N-1}(Q(r_i-50R))\cosh(Q(r_i-50R))} \\
&= \frac{-1+\exp^{100QR}}{Q}.
\end{aligned}$$

□

Recall that $A_i = \{y \in \tilde{X} \,|\, r_i - 50R \leq \tilde{d}(o,y) \leq r_i + 50R\}$. Let $\pi : \tilde{X} \to X$ be the quotient by the action of $\Gamma$, given by the universal covering map, and set

$$A_i(y_0) = \{y \in \tilde{X} \,|\, \pi(y) = \pi(y_0),\ B(y,R) \subset A_i\}.$$

**Proposition 3.9.** *For every $i \in \mathbb{N}$, there exists $y_i \in A_i(y_0)$ such that*

$$\fint_{B(y_i,R)} \boldsymbol{\Delta} r \geq Q - \Psi(i|K,N,R).$$

*Proof.* Let $E_i$ be the maximal set of $A_i(y_0)$ such that $B(y_1,R) \cap B(y_2,R) = \varnothing$ for distinct points $y_1, y_2$ in $E_i$. Set $F_i = \bigcup_{y\in E_i} B(y,R)$. Using Proposition 3.8 we will show that

$$\fint_{F_i} \boldsymbol{\Delta} r \geq Q - \Psi(i|K,N,R).$$

As $F_i = \bigcup_{y\in E_i} B(y,R)$ is the union of mutually disjoint balls it will follow then that there a point $y_i \in E_i$ such that

$$\fint_{B(y_i,R)} \boldsymbol{\Delta} r \geq Q - \Psi(i|K,N,R).$$

To this goal, first we estimate a lower bound for $\frac{\tilde{m}(F_i)}{\tilde{m}(A_i)}$. Let $G_i = \bigcup_{y\in E_i} B(y,5R)$. The cardinality of $E_i$ is finite, all of its elements are preimages of the same point under the covering map $\pi$, and $\tilde{m}$ is locally equal to $m$, from which we obtain $\tilde{m}(F_i) = \sum_{y\in E_i} \tilde{m}(B(y,R)) = \mathrm{card}(E_i)\tilde{m}(B(y',R)))$ and

$$\tilde{m}(G_i) \leq \sum_{y\in E_i} \tilde{m}(B(y,5R))) = \mathrm{card}(E_i)\tilde{m}(B(y',5R)))$$

for $y' \in E_i$. Thus,

$$\frac{\tilde{m}(F_i)}{\tilde{m}(G_i)} \geq \frac{\mathrm{card}(E_i)\tilde{m}(B(y',R))}{\mathrm{card}(E_i)\tilde{m}(B(y',5R))} \geq \frac{v_{K,N}(R)}{v_{K,N}(5R)},$$

by applying Theorem 2.3 with $v_{K,N}(r) = \int_0^r \sinh^{N-1}(Qt)\ dt$.

Now we will find a bound for $\tilde{m}(A_i)$. We will prove that

$$A(o, r_i - 10R, r_i + 10R) = \{y \in \tilde{X} \,|\, r_i - 10R < \tilde{d}(o,y) < r_i + 10R\} \subset G_i.$$

Let $z \in A(o, r_i - 10R, r_i + 10R)$, we will show $z \in G_i$. As $z \in \tilde{X}$ there exists a point $y \in \pi^{-1}(\pi(y_0))$ such that $\tilde{d}(z,y) \leq \mathrm{diam}(X)$. Then, by the triangle inequality,

$$r_i - 10R - \mathrm{diam}(X) \leq \tilde{d}(o,y) \leq r_i + 10R + \mathrm{diam}(X).$$

The previous inequality implies $y \in A_i(y_0)$. From the definition of $E_i$ there exists a point $y' \in E_i$ such that $\tilde{d}(y,y') \leq R$. By the triangle inequality, $\tilde{d}(z,y') \leq \mathrm{diam}(X) + R$. Recalling that $R > 50\,\mathrm{diam}(\tilde{X})$ we deduce that $\tilde{d}(z,y') \leq 5R$. Hence, $z \in G_i$. This proves $A(o, r_i - 10R, r_i + 10R) \subset G_i$.

From the previous paragraph, $\tilde{m}(G_i) \geq m(A(o, r_i - 10R, r_i + 10R))$. Recall that $A_i = A[o, r_i - 50R, r_i + 50R]$. Hence, by the generalized Bishop-Gromov volume comparison for annular regions we obtain:

$$\frac{\tilde{m}(G_i)}{\tilde{m}(A_i)} \geq \frac{\tilde{m}(A(o, r_i - 10R, r_i + 10R))}{\tilde{m}(A_i)} \geq \frac{\int_{r_i-10R}^{r_i+10R} \sinh^{N-1}(Qt)\, dt}{\int_{r_i-50R}^{r_i+50R} \sinh^{N-1}(Qt)\, dt}$$

As

$$\lim_{i\to\infty} \frac{\int_{r_i-10R}^{r_i+10R} \sinh^{N-1}(Qt)\, dt}{\int_{r_i-50R}^{r_i+50R} \sinh^{N-1}(Qt)\, dt} \geq \frac{\exp^{-60QR}}{5},$$

we can write

$$\frac{\tilde{m}(G_i)}{\tilde{m}(A_i)} \geq c(K, N, R).$$

Therefore,

$$\frac{\tilde{m}(F_i)}{\tilde{m}(A_i)} = \frac{\tilde{m}(F_i)}{\tilde{m}(G_i)} \frac{\tilde{m}(G_i)}{\tilde{m}(A_i)} \geq \frac{v_{K,N}(R)}{v_{K,N}(5R)} c(K, N, R).$$

The Laplacian comparison theorem for $\mathsf{RCD}^*(K, N)$-spaces (2.4) then yields

$$\mathbf{\Delta} r|_{\tilde{X}\setminus\{o\}} \leq Q \coth(Qr)\tilde{m}.$$

Observe that $\mathbf{\Delta} r \leq (Q + \delta(i, K, N))\,\tilde{m}$ on $A_i$, because $\lim_{r\to\infty} \coth(r) = 1$ and $\coth(r) \geq 1$, here $\lim_{i\to\infty} \delta(i, K, N) = 0$. Therefore $(Q + \delta(i, K, N))\,\tilde{m} - \mathbf{\Delta} r$ is a non-negative measure. As $F_i \subset A_i$ we compute

$$0 \leq \int_{F_i} [(Q + \delta(i, K, N))\,\tilde{m} - \mathbf{\Delta} r] \leq \int_{A_i} [(Q + \delta(i, K, N))\,\tilde{m} - \mathbf{\Delta} r].$$

Changing sign in the above equation and taking the average integral we find,

$$\begin{aligned} ⨍_{F_i} [\mathbf{\Delta} r - (Q + \delta(i, K, N))\,\tilde{m}] &\geq \frac{\tilde{m}(A_i)}{\tilde{m}(F_i)} ⨍_{A_i} [\mathbf{\Delta} r - (Q + \delta(i, K, N))\,\tilde{m}] \\ &\geq \frac{\tilde{m}(A_i)}{\tilde{m}(F_i)} (Q - \varepsilon(i, K, N, R) - Q - \delta(i, K, N)) \\ &\geq -\frac{\varepsilon(i, K, N, R) + \delta(i, K, N)}{C(K, N, R)}. \end{aligned}$$

From the first to the second line above we used $⨍_{A_i} \mathbf{\Delta} r \geq Q - \varepsilon(i, K, N, R)$, and from the second to the third, $\frac{\tilde{m}(F_i)}{\tilde{m}(A_i)} \geq C(K, N, R)$. Thus,

$$⨍_{F_i} \mathbf{\Delta} r \geq Q + \delta(i, K, N) - \frac{\varepsilon(i, K, N, R) + \delta(i, K, N)}{C(K, N, R)}.$$

□

We are now ready to prove Lemma 3.4, in essentially the same way as the corresponding part of [Jiang, Theorem 1.7].

*Proof of Lemma 3.4.* Let $y_i \in \tilde{X}$ be as in Proposition 3.9. Then there exists a deck transformation (measure-preserving metric isometry) $\varphi_i : \tilde{X} \to \tilde{X}$ such that $\varphi_i(y_0) = y_i$. Define $u_i : B_R(y_0) \to \mathbb{R}$ by $u_i(y) = r(\varphi_i(y)) - \tilde{d}(o, y_i)$. As $B_R(y_0)$ is precompact and the $u_i$ are 1-Lipschitz, by the Arzelà-Ascoli Theorem there is a subsequence of $u_i$ that converges to a 1-Lipschitz function $u_R$. Moreover, the sequence $u_i$ is uniformly bounded in $W^{1,2}(B_R(y_0))$, so $u_R \in W^{1,2}(B_R(y_0))$ (with $|\nabla u_R| = 1$ $\tilde{m}$-a.e.) and

$$\int_{\tilde{X}} \psi \mathbf{\Delta} u_R = \lim_{i\to\infty} \int_{\tilde{X}} \psi \circ \varphi_i \mathbf{\Delta} u_i.$$

Here $\psi$ is a compactly supported Lipschitz function on $B_R(y_0)$.

The Laplacian comparison (2.4) implies

$$\boldsymbol{\Delta} u_i(y) = \boldsymbol{\Delta} r(\varphi_i(y)) \le Q + \Psi(i|K,N,R),\ \ y \in B_R(y_0). \tag{3.1}$$

On the other hand, Proposition 3.9 gives,

$$\fint_{B_R(y_0)} \boldsymbol{\Delta} u_i = \fint_{B_R(y_i)} \boldsymbol{\Delta} r \ge Q - \Psi(i|K,N,R).$$

It follows then that

$$\fint_{B_R(y_0)} |\boldsymbol{\Delta} u_i - Q\mathrm{d}\tilde{m}| \le \Psi(i|K,N,R).$$

From these observations we obtain:

$$\begin{aligned}\int_{\tilde{X}} \psi \boldsymbol{\Delta} u_R &= \lim_{i\to\infty} \int_{B_R(y_0)} \psi\circ\varphi_i \boldsymbol{\Delta} u_i \\ &= \int_{B_R(y_0)} \psi\circ\varphi\, Q\, \mathrm{d}\tilde{m} \\ &= \int_{B_R(y_0)} \psi\, Q\, \mathrm{d}\tilde{m}\end{aligned}$$

Whence, $u_R \in D(\boldsymbol{\Delta}, B_R(y_0))$ and $\boldsymbol{\Delta} u_R = (N-1)\tilde{m}$.

Take a sequence of radii $R_i \uparrow \infty$ and the corresponding sequence of functions $u_{R_i}$. Then, up to passing to a subsequence, the $u_{R_i}$ converge to a 1-Lipschitz function $u : \tilde{X} \to \mathbb{R}$. It is immediate that $u \in D_{loc}(\boldsymbol{\Delta})$ and that $\boldsymbol{\Delta} u = (N-1)\tilde{m}$. Moreover, since the Laplacian of $u$ is constant, $u \in D_{loc}(\Delta)$ and $\Delta u = N-1$ $\tilde{m}$-a.e.. □

3.3. **The Hessian of $u$.** Throughtout this section we maintain the assumption that $(X,d,m)$ is an $\mathsf{RCD}^*(K,N)$ space with $K<0$ and $N \in (1,\infty)$. Let us recall that we denote the universal cover of $X$ by $(\tilde{X},\tilde{d},\tilde{m})$ and that by the results of [MW], $\tilde{X}$ is an $\mathsf{RCD}^*(K,N)$-space. In this section we will compute the Hessian of the function $u : \tilde{X} \to \mathbb{R}$ constructed in Section 3.2. The strategy and computations follow along the lines of [Ket, Theorem 3.7], which in turn draws from [Stu2], originally formulated in the language of Gamma Calculus.

Let us fix a point $x \in \tilde{X}$ and let $t \in \mathbb{R}$. For each pair of functions $f,g \in \mathrm{Test}(\tilde{X})$, we consider the function $\tilde{u} = \Psi(u,f,g) = \frac{1}{2}u^2 + (1-u(x))u + t\left(fg - f(x)g - g(x)f\right)$. Observe that $\Psi(0,0,0) = 0$. The partial derivatives of $\Psi$ at $x$ are given by

$$\begin{array}{lll}\Psi_1|_x = (u + (1-u(x)))|_x = 1 & \Psi_{11}|_x = 1 & \Psi_{22}|_x = 0 \\ \Psi_2|_x = t(g-g(x))|_x = 0 & \Psi_{12}|_x = \Psi_{21}|_x = 0 & \Psi_{23}|_x = \Psi_{32}|_x = t \\ \Psi_3|_x = t(f-f(x))|_x = 0 & \Psi_{13}|_x = \Psi_{31}|_x = 0 & \Psi_{33}|_x = 0\end{array}$$

Let $\gamma_2$ be the absolutely continuous part of $\Gamma_2$. Now, following the same strategy as in [Ket, Theorem 3.7], by Equation (2.8) and Proposition 2.7 we have that for every $x \in \tilde{X}$,

$$\begin{aligned}0 &\le \gamma_2(\tilde{u}) - K|\nabla\tilde{u}|^2 + \frac{1}{N}(\Delta\tilde{u})^2 \\ &= \gamma_2(u) + 4t\mathrm{Hess}[u](f,g) + |\nabla u|^4 + 4t\langle\nabla u,\nabla f\rangle\langle\nabla u,\nabla g\rangle + 2t^2|\nabla f|^2|\nabla g|^2 \\ &\quad +2t^2(\langle\nabla f,\nabla g\rangle)^2 - K|\nabla u|^2 - \frac{(\Delta u)^2}{N} - \frac{|\nabla u|^4}{N} - \frac{4t^2}{N}(\langle\nabla f,\nabla g\rangle)^2 \\ &\quad -\frac{4t\Delta u}{N}\langle\nabla f,\nabla g\rangle - \frac{2\Delta u}{N}|\nabla u|^2 - \frac{4t}{N}|\nabla u|^2\langle\nabla f,\nabla g\rangle .\end{aligned}$$

Grouping terms we obtain,

$$
\begin{aligned}
(3.2)\qquad 0 \;\le\; & \gamma_2(u) - K|\nabla u|^2 - \frac{(\Delta u)^2}{N} + \frac{N-1}{N}|\nabla u|^2 - \frac{2\Delta u}{N}|\nabla u|^2 \\
& +4t\left(\mathrm{Hess}[u](f,g) + \langle \nabla u, \nabla f\rangle \langle \nabla u, \nabla g\rangle - \left(\frac{\Delta u + |\nabla u|^2}{N}\right)\langle \nabla f, \nabla g\rangle\right) \\
& +2t^2\left(|\nabla f|^2|\nabla g|^2 + \frac{N-2}{N}(\langle \nabla f, \nabla g\rangle)^2\right).
\end{aligned}
$$

The last term of the previous inequality (3.2), namely $|\nabla f|^2|\nabla g|^2 + \frac{N-2}{N}(\langle \nabla f, \nabla g\rangle)^2$, is non-negative. Hence, the discriminant of the right hand side of (3.2) as a polynomial in $t$ is $\le 0$. That is,

$$
\begin{aligned}
\frac{2\left(\mathrm{Hess}[u](f,g) + \langle \nabla u, \nabla f\rangle \langle \nabla u, \nabla g\rangle - \left(\frac{\Delta u+|\nabla u|^2}{N}\right)\langle \nabla f, \nabla g\rangle\right)^2}{|\nabla f|^2|\nabla g|^2 + \frac{N-2}{N}(\langle \nabla f, \nabla g\rangle)^2} \;\le\; & \gamma_2(u) - K|\nabla u|^2 - \frac{(\Delta u)^2}{N} \\
& + \frac{N-1}{N}|\nabla u|^2 - \frac{2\Delta u}{N}|\nabla u|^2.
\end{aligned}
$$

**Corollary 3.10.** *Let $u : \tilde{X} \to \mathbb{R}$ be a function in $D_{loc}(\Delta)$ such that $|\nabla u|^2 = 1$ $\tilde{m}$-a.e. and $\Delta u = N-1$ $\tilde{m}$-a.e.. Then for all functions $f, g \in \mathrm{Test}(\tilde{X})$,*

$$
\mathrm{Hess}[u](f,g) = \langle \nabla f, \nabla g\rangle - \langle \nabla u, \nabla f\rangle \langle \nabla u, \nabla g\rangle. \tag{3.3}
$$

*Proof.* Let $\{D_i\}$ be a countable collection of pairwise disjoint bounded sets such that $\tilde{X} = \bigcup_i D_i$ up to a negligible set. Note that $\Gamma_2(u) = 0$ and therefore $\gamma_2(u) = 0$. Plugging this in our previous analysis and using that $|\nabla u|^2 = 1$, $\tilde{m}$-a.e., $\Delta u = N-1$ and $K = -(N-1)$ we have that

$$
\int_{D_i} \frac{2}{a}\left(\mathrm{Hess}[u](f,g) + \langle \nabla u, \nabla f\rangle \langle \nabla u, \nabla g\rangle - \left(\frac{\Delta u + |\nabla u|^2}{N}\right)\langle \nabla f, \nabla g\rangle\right)^2 \mathrm{d}\tilde{m}
$$

is less than or equal to

$$
\int_{D_i} -K|\nabla u|^2 - \frac{(\Delta u)^2}{N} + \frac{N-1}{N} - \frac{2\Delta u}{N}\,\mathrm{d}\tilde{m} = \left(N-1-\frac{(N-1)^2}{N} + \frac{N-1}{N} - \frac{2(N-1)}{N}\right)\tilde{m}(D_i) = 0.
$$

Therefore, $\mathrm{Hess}[u](f,g)|_{D_i} = \langle \nabla f, \nabla g\rangle - \langle \nabla u, \nabla f\rangle \langle \nabla u, \nabla g\rangle$. Since this is the case for all $D_i$ we have the result. $\square$

3.4. **Regular Lagrangian flow of $\nabla u$.** In this section we will show the existence of a *Regular Lagrangian Flow* of the Busemann-type function $u : \tilde{X} \to \mathbb{R}$ constructed in the previous section, via the work developed by Ambrosio-Trevisan [AT]. To do so, we will make use of the formulation obtained by Gigli-Rigoni [GR]. This formulation depends on the language of Differential Calculus developed by Gigli [Gig1]. Let us recall the definition of a Regular Lagrangian Flow, following [GR].

**Definition 3.11.** *Let $(X_t) \in L^2([0,1], L^2_{loc}(TX))$. We say that*

$$
F^{(X_t)} : [0,1] \times X \to X
$$

*is a Regular Lagrangian Flow for $(X_t)$ provided that:*

i) *There exists $C > 0$ such that*

$$
(F_s^{(X_t)})_{\#} m \le Cm, \quad \forall s \in [0,1]. \tag{3.4}
$$

ii) *For $m$-a.e. $x \in X$ the curve $[0,1] \ni s \mapsto F_s^{(X_t)}(x) \in X$ is continuous and such that*

$$
F_0^{(X_t)}(x) = x.
$$

iii) *For every* $f \in W^{1,2}(X)$ *we have that for* $m$*-a.e.* $x \in X$ *the function* $s \mapsto f(F_s^{(X_t)}(x))$ *belongs to* $W^{1,1}(0,1)$ *and satisfies*

$$\frac{d}{ds} f(F_s^{(X_t)}(x)) = df(X_s)(F_s^{(X_t)}(x)), \quad m \times \mathcal{L}^1|_{[0,1]} - a.e.(x,s). \tag{3.5}$$

With this definition in hand, we will now recall the main result of [AT] on the existence and uniqueness of Regular Lagrangian Flows as expressed in [GR, Theorem 2.8]. The space of Sobolev vector fields $W^{1,2}_{C,loc}(TX)$ is the space of $V \in L^2_{loc}(TX)$ for which there is $T$ in the tensor product of $L^2(TX)$ with itself such that

$$\int hT(\nabla g, \nabla \tilde{g})\,\mathrm{d}m = \int \langle V, \nabla \tilde{g}\rangle\,\mathrm{div}(h\nabla g) + h\mathrm{Hess}(\tilde{g})(V, \nabla g)\,\mathrm{d}m$$

for every $h, g, \tilde{g} \in \mathrm{Test}(X)$ with bounded support. In this case $T$ is the covariant derivative of $V$ and we will denote it by $\nabla V$.

**Theorem 3.12.** *Let* $(X_t) \in L^2([0,1], W^{1,2}_{C,loc}(TX)) \cap L^\infty([0,1], L^\infty(TX))$ *be such that* $X_t \in D(\mathrm{div}_{\mathrm{loc}})$ *for a.e.* $t \in [0,1]$*, with*

$$\int_0^1 \|\,|\nabla X_t|\,\|_{L^2(\tilde{X})} + \|\mathrm{div}(X_t)\|_{L^2(\tilde{X})} + \|(\mathrm{div}(X_t))^-\|_{L^\infty(\tilde{X})}\,\mathrm{d}t < \infty. \tag{3.6}$$

*Then a Regular Lagrangian flow* $F_s^{(X_t)}$ *for* $(X_t)$ *exists and is unique, in the sense that if* $\tilde{F}(X_t)$ *is another flow, then for* $m$*-a.e.* $x \in X$ *it holds that* $F_s(x) = \tilde{F}_s(x)$ *for every* $s \in [0,1]$*. Moreover, we have the quantitative bound for all* $s \in [0,1]$

$$(F_s^{(X_t)})_{\#} m \le \exp\left(\int_0^s \|(\mathrm{div}(X_t))^-\|_{L^\infty(X)\,\mathrm{d}t}\right) m.$$

We will apply the previous result in our setting. In the formulation of the previous definition, a regular Lagrangian flow is associated to a family of vector fields $X_t$. In our case, we are only dealing with a single vector field $\nabla u$. Hence, to fulfill the necessary integrability conditions, it is enough to prove that $\nabla u \in W^{1,2}_{C,loc}(T\tilde{X}) \cap L^\infty(T\tilde{X})$. From the proof of Lemma 3.4, we have that $u$ coincides locally with a test function on bounded sets and therefore these conditions are satisfied. On the other hand, $\nabla u \in D(\mathrm{div}_{loc})$ because $u \in D_{loc}(\Delta)$.

We now proceed to show the validity of (3.6) for $\nabla u$. Observing again the independence from the time variable of $\nabla u$, it is sufficient to show

$$\|\,|\nabla\nabla u|\,\|_{L^2(\tilde{X})} + \|\mathrm{div}(\nabla u)\|_{L^2(\tilde{X})} + \|(\mathrm{div}(\nabla u))^-\|_{L^\infty(\tilde{X})} < \infty.$$

The bounds concerning the divergence are obtained from the fact that $\Delta u = N-1$, so that $\mathrm{div}(\nabla u) = \mathrm{div}(\nabla u)^- = N-1$. Observe that the bound on the covariant derivative of $\nabla u$ is satisfied for every bounded set by using [Gig, Corollary 2.10] (coupled with the fact that $\nabla\nabla u = \mathrm{Hess}[u]^{\#}$) and that $|\nabla u| = 1$ $\tilde{m}$-a.e., which gives a bound for $\|\,|\nabla\nabla u|\,\|_{L^2(\tilde{X})}$. Therefore, by Theorem 3.12, a Regular Lagrangian Flow $F : [0,1] \times \tilde{X} \to \tilde{X}$ for $\nabla u$ exists and is unique in the sense of Definition 3.11. Moreover, as we are dealing with a single vector field $\nabla u$ (i.e. $X_t$ is independent of the time variable $t$), $F$ can be extended uniquely to a regular Lagrangian flow $F : [0,\infty) \times \tilde{X} \to \tilde{X}$. In addition, Theorem 3.12 yields that $F$ satisfies the following bound:

$$(F_t)_{\#}\tilde{m} \le \exp\left(\int_0^t \|(\mathrm{div}(\nabla u))^-\|_{L^\infty(\tilde{X})}\,\mathrm{d}s\right)\tilde{m} = \exp(-(N-1)t)\tilde{m}. \quad \forall t \in [0,\infty) \tag{3.7}$$

The uniqueness of a Regular Lagrangian Flow is tied to the uniqueness of solutions of the so-called *continuity equation* (see for example [GR, Definition 2.9]). Recall that two Borel maps $t \mapsto \mathcal{P}(X)$

and $t \mapsto X_t \in L^0(TX)$ are said to solve the continuity equation

$$\frac{d}{dt}\mu_t + \operatorname{div}(X_t\mu_t) = 0 \tag{3.8}$$

provided that the following conditions are satisfied:

(i) $\mu_t \leq Cm$ for every $t \in [0,1]$ and some $C > 0$,

(ii) $\int_0^1 \int |X_t|^2 \,\mathrm{d}\mu_t \,\mathrm{d}t < \infty$,

(iii) for any $f \in W^{1,2}(X)$ the map $t \mapsto \int f \,\mathrm{d}\mu_t$ is absolutely continuous and

$$\frac{d}{dt}\int f \,\mathrm{d}\mu_t = \int \mathrm{d}f(X_t)\,\mathrm{d}\mu_t \quad \text{a.e. } t.$$

The following result concerning the uniqueness of solutions of the continuity equation in connection with the uniqueness of Regular Lagrangian Flows was obtained in [AT]. We recall the formulation of [GR, Theorem 2.10].

**Theorem 3.13.** *Let $(X_t)$ be as in Theorem 3.12 and $\overline{\mu} \in \mathcal{P}(X)$ be such that $\mu_0 \leq Cm$ for some $C > 0$. Then there exists a unique $(\mu_t)$ such that the pair $(\mu_t, X_t)$ solves the continuity equation (3.8) and for which $\mu_0 = \overline{\mu}$. Moreover, such $(\mu_t)$ is given by $\mu_s = (F_s^{(X_t)})_{\#}\overline{\mu}$ for all $s \in [0,1]$.*

Observe that $\mu_t = e^{-(N-1)t}\tilde{m}$ is a solution to the continuity equation for $X_t = \nabla u$ and $\overline{\mu} = \tilde{m}$. Hence, it follows from the previous Theorem that inequality (3.7) is an equality for every $t \in [0,\infty)$.

Now we observe that the proof of [GR, Lemma 3.18] can be applied verbatim to our case and therefore, $F$ can be extended uniquely (preserving the bound 3.7) to a regular Lagrangian flow $F : (-\infty,\infty) \times \tilde{X} \to \tilde{X}$. Recalling [GR, Equation 2.3.3], that is,

$$|\dot{F}_s^{(X_t)}(x)| = |X_s|(F_s^{(X_t)}(x)) \quad \text{a.e. } s \in [0,1]. \tag{3.9}$$

We also have that $|\dot{F}_s(x)| = |\nabla u|(F_s(x)) = 1$ for all $s \in \mathbb{R}$.

Notice that the uniqueness statement in [GR, Theorem 2.8] implies that for $X_t$ independent of $t$, $F$ satisfies the semigroup property $F_t \circ F_s = F_{t+s}$, $\tilde{m}$-a.e. and for all $t,s \in \mathbb{R}$ (cf. [GR, Equation 2.3.10]). We summarize the previous discussion in the following proposition.

**Proposition 3.14.** *Let $u : \tilde{X} \to \mathbb{R}$ be the function constructed in this section. Then, there exists an $\tilde{m}$-a.e. unique Regular Lagrangian flow (in the sense of definition 3.11) $F : \mathbb{R} \times \tilde{X} \to \tilde{X}$ for $\nabla u$. Moreover, $F$ satisfies the semigroup property $F_t \circ F_s = F_{t+s}$, $\tilde{m}$-a.e. for all $t,s \in \mathbb{R}$, and the following change of measure formula holds,*

$$(F_t)_{\#}\tilde{m} = e^{-(N-1)t}\tilde{m}.$$

We end this section by pointing out that the following Lemma holds in our setting (cf. [Gig4, Theorem 2.3 (iv)]).

**Lemma 3.15.** *Let $F : (-\infty,\infty) \times \tilde{X} \to \tilde{X}$ be the regular Lagrangian flow associated to $\nabla u$. Then, for every $t,s \in (-\infty,\infty)$ and $x \in \tilde{X}$,*

$$\tilde{d}(F_s(x), F_t(x)) = |s-t| = |u(F_s(x)) - u(F_t(x))|.$$

*In particular, $u(F_{-u(x)}(x)) = 0$ for all $x \in \tilde{X}$ and the trajectories of $F_t$ are geodesics.*

*Proof.* Following the approach of the proof of $(a) \Rightarrow (b)$ in [GR, Proposition 2.7], from (3.5) we obtain that, for all $t < s$,

$$u \circ F_s - u \circ F_t = \int_t^s du(\nabla u) \circ F_r \,\mathrm{d}r.$$

Inverting the roles of $t$ and $s$, and using that $|\nabla u| = 1$ $\tilde{m}$-a.e.,

$$|u \circ F_s - u \circ F_t| = |s-t|.$$

Furthermore, by Equation (3.9), we find $d(F_s(x), F_t(x)) \leq |s-t|$ for all $t < s$. Moreover,

$$|u \circ F_s(x) - u \circ F_t(x)| \leq \tilde{d}(F_s(x), F_t(x))$$

because $u$ is 1-Lipschitz. Therefore $\tilde{d}(F_s(x), F_t(x)) = |s-t|$. ☐

## 4. Cheeger Energy along the Flow

Consider the map $f_t = f \circ F_t$, where $F : (-\infty, \infty) \times \tilde{X} \to \tilde{X}$ is the Regular Lagrangian Flow of the Busemann-type function $u : \tilde{X} \to \mathbb{R}$ obtained in the previous section and $f \in W^{1,2}(\tilde{X})$. In this section we focus on computing the $W^{1,2}(\tilde{X})$ norm of $f_t$. In the first subsection we calculate its $L^2$ norm. To calculate the norm of $|\nabla f_t|$ we use the formula for the Hessian of $u$ obtained in section 3.3. Then, we compute the derivative of the Cheeger energy along $F_t$, and finally localize the result.

4.1. **$L^2$ norm along the flow.** Let us consider the map $f_t = f \circ F_t$ where $F_t$ is the Regular Lagrangian Flow of $u$ and $f \in W^{1,2}(\tilde{X})$. In this section we study the $L^2$ norm of $f_t$. For that reason we begin by proving a version of [Gig4, Equation 3.39] in our setting.

**Lemma 4.1.** *For any $f \in S^2(\tilde{X}, \tilde{d}, \tilde{m})$ and $t \geq 0$,*

$$|f(F_t(x)) - f(x)| \leq \int_0^t |\nabla f|(F_s(x))\,\mathrm{d}s$$

*for $\tilde{m}$-a.e. $x \in \tilde{X}$. Furthermore, the result also holds for $t \leq 0$ by taking the integral from $t$ to $0$.*

*Proof.* Let us consider a probability measure $\bar{m}$ on $\tilde{X}$ satisfying $\bar{m} \leq \tilde{m}$ and $\tilde{m} \ll \bar{m}$. We define the measure $\pi := T_{\#}\bar{m} \in \mathcal{P}(C([0,1];\tilde{X}))$ where $T : \tilde{X} \to C([0,1], \tilde{X})$ is given by $T(x)_t = F_t(x)$. Let $e_t : C([0,1];\tilde{X}) \to \tilde{X}$ be the evaluation map at $t$. Notice that for all $t \geq 0$,

$$(e_t)_{\#}\pi = (F_t)_{\#}\bar{m} \leq (F_t)_{\#}\tilde{m} = e^{-(N-1)t}\tilde{m} \leq \tilde{m}.$$

So $\pi$ is a test plan (with compression constant $\leq 1$). Denote the set of trajectories of $F$ by $\Gamma_F$. Observe that for any set of curves $\Gamma \subset AC^2([0,1];\tilde{X})$, the point $x$ lies in $T^{-1}(\Gamma)$ if and only if there exists $\gamma \in \Gamma$ such that $\gamma(t) = F_t(x)$ for any $t \in [0,1]$. Hence, such a $\gamma$ is an element of $\Gamma_F$. It follows that $T^{-1}(\Gamma) = T^{-1}(\Gamma \cap \Gamma_F)$ and we find that $\pi$ concentrates on trajectories of $F$. By [Gig4, Theorem 2.3, (iii)] the elements of $\Gamma_F$ are constant speed geodesics satisfying $\gamma(1) \in \partial^c(u)(\gamma(0))$, hence $d(\gamma(1), \gamma(0)) = 1$. Therefore $\pi$ concentrates on 1-Lipschitz curves.

On the other hand, for any $\Gamma \subset C([0,1];\tilde{X})$

$$(e_t)_{\#}\pi(\Gamma) = \bar{m}(T^{-1}(e_t^{-1}(\Gamma)) = (F_t)_{\#}\bar{m}(\Gamma).$$

By [Gig4, (3.7)], for $0 \leq t \leq 1$ and $f \in S^2(\tilde{X})$, for $\pi$-a.e. $\gamma$,

$$|f(\gamma(t)) - f(\gamma(0))| \leq \int_0^t |\nabla f|(\gamma(s))|\gamma'(s)|ds = \int_0^t |\nabla f|(\gamma(s))ds.$$

Therefore, using that for $\tilde{m}$-a.e. $x \in \tilde{X}$ the flow $F$ is defined, and therefore for almost every $x$ there is a trajectory of $F$ passing through it, for every $0 \leq t \leq 1$,

$$|f(F_t(x)) - f(x)| \leq \int_0^t |\nabla f|(F_s(x))\,\mathrm{d}s. \tag{4.1}$$

An iteration of this argument will yield the result for any $t \in \mathbb{R}$. Let $1 \leq t \leq 2$, then by (4.1),

$$\int_{\tilde{X}} |f(F_{t-1}(x)) - f(x)| \, \mathrm{d}\tilde{m} \leq \int_{\tilde{X}} \int_0^{t-1} |\nabla f|(F_s(x)) \, \mathrm{d}s \, \mathrm{d}\tilde{m}.$$

A direct computation yields that the left-hand side of the previous inequality is equal to

$$\int_{\tilde{X}} |f(F_{t-1}(x)) - f(x)| \, \mathrm{d}\tilde{m} = \int_{\tilde{X}} |f(F_t(x)) - f(F_1(x))| \, \mathrm{d}(F_{-1})_{\#}\tilde{m} = e^{(N-1)} \int_{\tilde{X}} |f(F_t(x)) - f(F_1(x))| \, \mathrm{d}\tilde{m}$$

On the other hand, by (4.1) the right hand side becomes

$$\int_{\tilde{X}} \int_0^{t} |\nabla f|(F_s(x)) \, \mathrm{d}s \, \mathrm{d}\tilde{m} - \int_{\tilde{X}} \int_0^{1} |\nabla f|(F_s(x)) \, \mathrm{d}s \, \mathrm{d}\tilde{m} \leq \int_{\tilde{X}} \int_0^{t} |\nabla f|(F_s(x)) \, \mathrm{d}s \, \mathrm{d}\tilde{m} - \int_{\tilde{X}} |f(F_1(x) - f(x)| \, \mathrm{d}\tilde{m}.$$

Combining the previous equations, using that $e^{-(N-1)} \leq 1$, and the triangle inequality we obtain:

$$\begin{aligned}
\int_{\tilde{X}} |f(F_t(x)) - f(x)| &\leq \int_{\tilde{X}} |f(F_t(x)) - f(F_1(x))| \, \mathrm{d}\tilde{m} + \int_{\tilde{X}} |f(F_1(x) - f(x)| \, \mathrm{d}\tilde{m} \\
&\leq e^{-(N-1)} \int_{\tilde{X}} |f(F_{t-1}(x)) - f(x)| \, \mathrm{d}\tilde{m} + \int_{\tilde{X}} |f(F_1(x) - f(x)| \, \mathrm{d}\tilde{m} \\
&\leq \int_{\tilde{X}} \int_0^{t-1} |\nabla f|(F_s(x)) \, \mathrm{d}s \, \mathrm{d}\tilde{m} + \int_{\tilde{X}} |f(F_1(x) - f(x)| \, \mathrm{d}\tilde{m} \\
&\leq \int_{\tilde{X}} \int_0^{t} |\nabla f|(F_s(x)) \, \mathrm{d}s \, \mathrm{d}\tilde{m}
\end{aligned}$$

This is precisely the result we claim in the case that $1 \leq t \leq 2$. Iterating this process the inequality follows for any $t \geq 0$, and similarly for any $t \leq 0$. □

This implies a version of [Gig4, (3.40)] with appropriate modifications, as will be shown in the next lemma.

**Lemma 4.2.** *For any $f \in S^2(\tilde{X}, \tilde{d}, \tilde{m})$ and $t \in \mathbb{R}$,*

$$\int_{\tilde{X}} |f(F_t(x)) - f(x)|^2 \, \mathrm{d}\tilde{m}(x) \leq t \left( \frac{1 - e^{-(N-1)t}}{N-1} \right) \int_{\tilde{X}} |\nabla f|^2(x) \, \mathrm{d}\tilde{m}(x).$$

*Proof.* Taking squares, integrating the inequality of Lemma 4.1, and using Hölder's inequality we obtain:

$$\begin{aligned}
\int_{\tilde X} |f(F_t(x)) - f(x)|^2 \, \mathrm{d}\tilde m(x) &\le \int_{\tilde X} \left( \int_0^t |\nabla f|(F_s(x)) ds \right)^2 \mathrm{d}\tilde m(x) \le t \int_{\tilde X}\int_0^t |\nabla f|^2(F_s(x)) \, \mathrm{d}s \, \mathrm{d}\tilde m(x) \\
&\le t \int_0^t \int_{\tilde X} |\nabla f|^2(F_s(x)) \, \mathrm{d}\tilde m(x) \, \mathrm{d}s = t \int_0^t \int_{\tilde X} |\nabla f|^2(x) \, \mathrm{d}(F_s)_{\#}\tilde m(x) \, \mathrm{d}s \\
&= t \int_0^t \int_{\tilde X} e^{-(N-1)s} |\nabla f|^2(x) \, \mathrm{d}\tilde m(x) \, \mathrm{d}s \\
&= t \left( \int_0^t e^{-(N-1)s} \, \mathrm{d}s \right) \left( \int_{\tilde X} |\nabla f|^2(x) \, \mathrm{d}\tilde m(x) \right) \\
&= t \left( \frac{1 - e^{-(N-1)t}}{N-1} \right) \int_{\tilde X} |\nabla f|^2(x) \, \mathrm{d}\tilde m(x)
\end{aligned}$$

□

In the following Lemma we compute the $L^2$ norm of $f \circ F_t$ and investigate its regularity.

**Lemma 4.3.** *Let $f \in W^{1,2}(\tilde X)$, and fix $t \in \mathbb{R}$. Then $f \circ F_t \in L^2(\tilde X, \tilde d, \tilde m)$ and the map $t \mapsto f \circ F_t$ is Lipschitz.*

*Proof.* First we compute the $L^2$ norm of $f \circ F_t$:

$$\|f \circ F_t\|^2_{L^2} = \int_{\tilde X} (f \circ F_t)^2 \, \mathrm{d}\tilde m = \int_{\tilde X} f^2 e^{-(N-1)t} \, \mathrm{d}\tilde m = e^{-(N-1)t} \|f\|^2_{L^2}$$

Therefore, as $f \in W^{1,2}(\tilde X)$ and in particular $f \in L^2(\tilde X)$ it follows that $f \circ F_t \in L^2(\tilde X)$. Now we proceed with the second part of the lemma. Let $t < s \in \mathbb{R}$, by the previous lemma,

$$\begin{aligned}
\int_{\tilde X} |f \circ F_s - f \circ F_t|^2 \, \mathrm{d}\tilde m &= \int_{\tilde X} e^{-(N-1)t} |f \circ F_{s-t} - f|^2 \, \mathrm{d}\tilde m \le e^{-(N-1)t}(s-t) \left( \tfrac{1 - e^{-(N-1)(s-t)}}{N-1} \right) \int_{\tilde X} |\nabla f|^2 \, \mathrm{d}\tilde m \\
&= (s-t) \left( \tfrac{e^{-(N-1)t} - e^{-(N-1)s}}{N-1} \right) \int_{\tilde X} |\nabla f|^2 \, \mathrm{d}\tilde m \le (s-t)^2 \|\nabla f\|^2_{L^2}
\end{aligned}$$

Hence, $t \mapsto f \circ F_t$ is Lipschitz with Lipschitz constant dominated by $\|\nabla f\|_{L^2}$ (which is well defined because $f \in W^{1,2}(X)$). □

We will use the following technical result to compute the derivative of the Cheeger energy of $f_t$, which in turn will aid in computing $|\nabla f_t|$. We make use of the *heat flow* $h_t : L^2(\tilde X) \to L^2(\tilde X)$. Recall that $h_t$ is is the unique family of maps such that for any $f \in L^2(\tilde X)$ the curve $[0, \infty) \ni t \mapsto h_t(f) \in L^2(\tilde X)$ is continuous, locally absolutely continuous on $(0, \infty)$, satisfies that $h_0(f) = f$, $h_t(f) \in D(\Delta)$ for $t > 0$ and solves

$$\frac{\mathrm{d}}{\mathrm{d}t} h_t(f) = \Delta h_t(f), \quad \mathcal{L}^1 - \text{a.e. } t > 0.$$

We refer the reader to [Gig4, Section 4.1.2] for a thorough exposition of the main properties of the heat flow on infinitesimally Hilbertian metric measure spaces.

**Lemma 4.4.** *For each $t \geq 0$, let $h_t : L^2(\tilde{X}) \to L^2(\tilde{X})$ be the heat flow on $\tilde{X}$ and $\varepsilon > 0$ be fixed. Then the map $t \mapsto h_\varepsilon(f \circ F_t)$ is Lipschitz and, in particular, the map*

$$t \mapsto \frac{1}{2} \int_{\tilde{X}} |\nabla h_\varepsilon(f \circ F_t)|^2 d\tilde{m}$$

*is Lipschitz.*

*Proof.* Using the equivalence of (i) and (v) in [EKS, Theorem 7] and the fact that $\mathsf{BL}(K,N)$ implies $\mathsf{BL}(K,\infty)$, [AT, Corollary 6.3] implies that the $L^2$–$\Gamma$ inequality holds true. Therefore,

$$\| |\nabla\left(h_\varepsilon(f \circ F_s) - h_\varepsilon(f \circ F_t)\right)| \|_{L^2} \leq C(\varepsilon) \| f \circ F_s - f \circ F_t \|_{L^2}.$$

(See [AT, Definition 5.1] for the precise value of $C(\varepsilon)$). Moreover, by [Gig1, (3.1.2)]

$$\| h_\varepsilon(f \circ F_s - f \circ F_t) \|_{L^2} \leq \| f \circ F_s - f \circ F_t \|_{L^2}$$

Combining the previous inequalities, we find:

$$\| h_\varepsilon(f \circ F_s) - h_\varepsilon(f \circ F_t) \|_{W^{1,2}} \leq C(\varepsilon) \| f \circ F_s - f \circ F_t \|_{W^{1,2}} \tag{4.2}$$

□

4.2. **Derivative of the Cheeger energy along the flow.** We are now ready to compute the derivative of the Cheeger energy along the Regular Lagrangian flow of $u$, that is, the derivative of the Cheeger energy of $f_t$. In the following proposition we make use of the technical results developed in the previous Subsection. For the reader's convenience we recall the notation being used: $f$ will denote a function in $W^{1,2}(\tilde{X})$, $u : \tilde{X} \to \mathbb{R}$ is the Busemann-type function constructed in Section 3.2, $F : (-\infty, \infty) \times \tilde{X} \to \tilde{X}$ is the Regular Lagrangian Flow of $\nabla u$, $f_t := f \circ F_t$ and $\mathcal{E}(t) := \frac{1}{2} \int |\nabla f_t|^2 \, \mathrm{d}\tilde{m}$.

**Theorem 4.5.** *Let $f \in W^{1,2}(\tilde{X}, \tilde{d}, \tilde{m})$ and $h_t : L^2(\tilde{X}) \to L^2(\tilde{X})$ the heat flow. Then, for every $\varepsilon > 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_{\tilde{X}} |\nabla h_\varepsilon(f_t)|^2 \mathrm{d}\tilde{m} = -\int_{\tilde{X}} \Delta h_{2\varepsilon}(f_t) \langle \nabla f_t, \nabla u \rangle \, \mathrm{d}\tilde{m}.$$

*In particular, the derivative of the Cheeger energy of $f_t$ is given by*

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(t) = \int_{\tilde{X}} \mathrm{Hess}[u](\nabla f_t, \nabla f_t) - \frac{(N-1)}{2} \int_{\tilde{X}} \langle \nabla f_t, \nabla f_t \rangle \, \tilde{m}.$$

*Proof.* Using Equation (4.2) we get that

$$\begin{aligned}
\int_{\tilde{X}} |\nabla h_\varepsilon(f_{t+h})|^2 - |\nabla h_\varepsilon(f_t)|^2 \mathrm{d}\tilde{m} &= \int_{\tilde{X}} 2 \langle \nabla h_\varepsilon(f_t), \nabla h_\varepsilon(f_{t+h} - f_t) \rangle + |\nabla(h_\varepsilon(f_{t+h} - f_t))|^2 \mathrm{d}\tilde{m} \\
&\leq \int_{\tilde{X}} 2 \langle \nabla h_\varepsilon(f_t), \nabla h_\varepsilon(f_{t+h} - f_t) \rangle \, \mathrm{d}\tilde{m} + \left(C(\varepsilon) \| f_{t+h} - f_t \|_{L^2}\right)^2 \\
&\leq \int_{\tilde{X}} 2 \langle \nabla h_\varepsilon(f_t), \nabla h_\varepsilon(f_{t+h} - f_t) \rangle \, \mathrm{d}\tilde{m} + |h|^2 C(\varepsilon)^2 \| \nabla f \|_{L^2}^2.
\end{aligned}$$

Therefore, by the Dominated Convergence Theorem,

$$\lim_{h \to 0} \frac{1}{2} \int_{\tilde{X}} \frac{|\nabla h_\varepsilon(f_{t+h})|^2 - |\nabla h_\varepsilon(f_t)|^2}{h} \mathrm{d}\tilde{m} = \lim_{h \to 0} \int_{\tilde{X}} \left\langle \nabla h_\varepsilon(f_t), \nabla \frac{h_\varepsilon(f_{t+h} - f_t)}{h} \right\rangle \mathrm{d}\tilde{m}.$$

Since $h_\varepsilon$ is in the domain of the Laplacian, using [Gig4, 4.34] we have that the right hand side is

$$\begin{aligned}
\lim_{h\to 0}\int_{\tilde X}\left\langle \nabla h_\varepsilon(f_t), \nabla\frac{h_\varepsilon(f_{t+h}-f_t)}{h}\right\rangle \mathrm{d}\tilde m &= -\lim_{h\to 0}\int_{\tilde X}\left(\frac{h_\varepsilon(f_{t+h}-f_t)}{h}\right)\Delta h_\varepsilon f_t \mathrm{d}\tilde m\\
&= -\lim_{h\to 0}\int_{\tilde X}\Delta h_{2\varepsilon}(f_t)\frac{f_{t+h}-f_t}{h}\mathrm{d}\tilde m\\
&= -\lim_{h\to 0}\int_{\tilde X}\frac{\Delta h_{2\varepsilon}(f_t)f_{t+h}-\Delta h_{2\varepsilon}(f_t)f_t}{h}\mathrm{d}\tilde m\\
&= -\lim_{h\to 0}\left(\int_{\tilde X}\frac{\Delta h_{2\varepsilon}(f_t)\circ F_{-h}}{h}f_t\mathrm{d}(F_h)_\# m-\int_{\tilde X}\frac{\Delta h_{2\varepsilon}(f_t)}{h}f_t\mathrm{d}\tilde m\right)\\
&= -\lim_{h\to 0}\left(e^{-(N-1)h}\int_{\tilde X}\frac{\Delta h_{2\varepsilon}(f_t)\circ F_{-h}-\Delta h_{2\varepsilon}(f_t)}{h}f_t\mathrm{d}\tilde m\right.\\
&\qquad\left.+\frac{e^{-(N-1)h}-1}{h}\int_{\tilde X}\Delta h_{2\varepsilon}(f_t)f_t\mathrm{d}\tilde m\right).
\end{aligned}$$

Notice that [Gig4, 4.34] holds without modification in our setting. Hence, the previous expression equals

$$\int_{\tilde X}\langle\nabla\Delta h_{2\varepsilon}(f_t),\nabla u\rangle f_t\mathrm{d}\tilde m+(N-1)\int_{\tilde X}\Delta h_{2\varepsilon}(f_t)f_t\mathrm{d}\tilde m,$$

from which our first equality follows by using $\Delta u = N-1$.

Observe that for every $f,g\in W^{1,2}(\tilde X)$ the following holds, again by using $\Delta u = N-1$, (cf. [Gig4, 4.35])

$$\int f\langle\nabla g,\nabla u\rangle\mathrm{d}\tilde m = -(N-1)\int fg\mathrm{d}m-\int g\langle\nabla f,\nabla u\rangle\mathrm{d}\tilde m. \tag{4.3}$$

Therefore, we obtain

$$\lim_{h\to 0}\frac{1}{2}\int_{\tilde X}\frac{|\nabla h_\varepsilon(f_{t+h})|^2-|\nabla h_\varepsilon(f_t)|^2}{h}\mathrm{d}\tilde m = -\int_{\tilde X}\Delta h_{2\varepsilon}(f_t)\langle\nabla f_t,\nabla u\rangle\mathrm{d}\tilde m = \int_{\tilde X}\langle\nabla\langle\nabla f_t,\nabla u\rangle,\nabla h_{2\varepsilon}(f_t)\rangle\mathrm{d}\tilde m.$$

By the Cauchy-Schwarz inequality, the right hand side of the last equality is bounded by $\int_X|\nabla f_t||\nabla h_{2\varepsilon}(f_t)|\mathrm{d}\tilde m$. Therefore, by the Dominated Convergence Theorem and the continuity in $\varepsilon$ of the Laplacian of the heat flow taking limits when $\varepsilon\to 0$ yields

$$\lim_{h\to 0}\frac{1}{2}\int_{\tilde X}\frac{|\nabla(f_{t+h})|^2-|\nabla(f_t)|^2}{h}\mathrm{d}\tilde m = \int_{\tilde X}\langle\nabla\langle\nabla f_t,\nabla u\rangle,\nabla f_t\rangle\mathrm{d}\tilde m.$$

Using the definition for the Hessian of $u$, found in (2.5), and (2.6), and finally substituting (3.3), we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(t) = \int_{\tilde X}\mathrm{Hess}[u](\nabla f_t,\nabla f_t)-\frac{(N-1)}{2}\int_{\tilde X}\langle\nabla f_t,\nabla f_t\rangle$$

which is our second equality. □

In the following theorem we see how the Cheeger energy of $f_t$ decomposes along each of the summands of $\langle\nabla f_t,\nabla f_t\rangle = \mathrm{Hess}[u](\nabla f_t,\nabla f_t)+\langle\nabla f_t,\nabla u\rangle^2$.

**Theorem 4.6.** *Let $u : \tilde{X} \to \mathbb{R}$ the function built in Section 3.2. The following identities hold for any $f \in W^{1,2}(\tilde{X})$:*

$$\int_{\tilde{X}} \mathrm{Hess}[u](\nabla f_t, \nabla f_t)\,\mathrm{d}\tilde{m} = e^{-(N+1)t} \int_{\tilde{X}} \mathrm{Hess}[u](\nabla f, \nabla f)\,\mathrm{d}\tilde{m}$$

$$\int_{\tilde{X}} \langle \nabla f_t, \nabla u\rangle^2\,\mathrm{d}\tilde{m} = e^{-(N-1)t} \int_{\tilde{X}} \langle \nabla f, \nabla u\rangle^2\,\mathrm{d}\tilde{m}$$

*Proof.* We will first prove the second equality. We compute

$$\begin{aligned}\frac{\langle \nabla f_{t+h}, \nabla u\rangle^2 - \langle \nabla f_t, \nabla u\rangle^2}{h} &= \frac{\langle \nabla f_{t+h}, \nabla u\rangle - \langle \nabla f_t, \nabla u\rangle}{h}(\langle \nabla f_{t+h}, \nabla u\rangle + \langle \nabla f_t, \nabla u\rangle) \\ &= (\langle \nabla \frac{(f_{t+h} - f_t)}{h}, \nabla u)(\langle \nabla f_{t+h}, \nabla u\rangle + \langle \nabla f_t, \nabla u\rangle).\end{aligned}$$

Observe that

$$\begin{aligned}\lim_{h\to 0} \int_{\tilde{X}} \langle \nabla \frac{(f_{t+h} - f_t)}{h}, \nabla u\rangle\langle f_{t+h}, \nabla u\rangle \mathrm{d}\tilde{m} &= \lim_{h\to 0} -(N-1) \int_{\tilde{X}} \left(\frac{f_{t+h} - f_t}{h}\right)\langle \nabla f_{t+h}, \nabla u\rangle \mathrm{d}\tilde{m} \\ &\quad - \int_{\tilde{X}} \left(\frac{f_{t+h} - f_t}{h}\right)\langle \nabla\langle \nabla f_{t+h}, \nabla u\rangle, \nabla u\rangle \mathrm{d}\tilde{m}.\end{aligned}$$

Let us denote the first and second summands of the left hand side of the previous equation by $A_1$ and $A_2$ respectively. We claim that

$$A_1 = -(N-1) \int_{\tilde{X}} \langle \nabla f_t, \nabla u\rangle^2 \mathrm{d}\tilde{m}.$$

To prove this claim, notice that

$$\begin{aligned}\int_{\tilde{X}} \left(\frac{f_{t+h} - f_t}{h}\right)\langle \nabla f_{t+h}, \nabla u\rangle - \langle \nabla f_t, \nabla u\rangle^2 \mathrm{d}\tilde{m} &= \int_{\tilde{X}} \left(\frac{f_{t+h} - f_t}{h}\right)(\langle \nabla f_{t+h}, \nabla u\rangle - \langle \nabla f_t, \nabla u\rangle)\,\mathrm{d}\tilde{m} \\ &\quad + \int_{\tilde{X}} \langle \nabla f_t, \nabla u\rangle \left(\left(\frac{f_{t+h} - f_t}{h}\right) - \langle \nabla f_t, \nabla u\rangle\right)\mathrm{d}\tilde{m}.\end{aligned}$$

Holder's inequality implies that

$$\left|\int_{\tilde{X}} \left(\frac{f_{t+h} - f_t}{h}\right)(\langle \nabla f_{t+h}, \nabla u\rangle - \langle \nabla f_t, \nabla u\rangle)\,\mathrm{d}\tilde{m}\right| \le \|\frac{f_{t+h} - f_t}{h}\|_{L^2}\|\langle \nabla f_{t+h}, \nabla u\rangle - \langle \nabla f_t, \nabla u\rangle\|_{L^2}.$$

This last expression converges to 0 as $h \to 0$, since $\|\frac{f_{t+h}-f_t}{h}\|_{L^2}$ is bounded because $\frac{f_{t+h}-f_t}{h}$ is weakly convergent in $L^2$ and

$$\|\langle \nabla f_{t+h}, \nabla u\rangle - \langle \nabla f_t, \nabla u\rangle\|_{L^2} \to 0.$$

Moreover, by [Gig4, 4.34],

$$\int_{\tilde{X}} \langle \nabla f_t, \nabla u\rangle \left(\left(\frac{f_{t+h} - f_t}{h}\right) - \langle \nabla f_t, \nabla u\rangle\right)\mathrm{d}\tilde{m} \to 0,$$

as $h \to 0$, and therefore the claim is proved.

A similar procedure to the computation of $A_1$ yields

$$A_2 = -\int_{\tilde{X}} \langle \nabla f_t, \nabla u\rangle \langle \nabla \langle \nabla f_t, \nabla u\rangle, \nabla u\rangle \mathrm{d}\tilde{m}.$$

Now, we move on to compute

$$\begin{aligned} \lim_{h\to 0} \int_{\tilde{X}} \langle \nabla \left(\frac{f_{t+h} - f_t}{h}\right), \nabla u\rangle \langle \nabla f_{t+h}, \nabla u\rangle &= \lim_{h\to 0} -(N-1) \int_{\tilde{X}} \left(\frac{f_{t+h} - f_t}{h}\right) \langle \nabla f_{t+h}, \nabla u\rangle \mathrm{d}\tilde{m} \\ &\quad - \int_{\tilde{X}} \left(\frac{f_{t+h} - f_t}{h}\right) \langle \nabla \langle \nabla f_{t+h}, \nabla u\rangle, \nabla u\rangle \mathrm{d}\tilde{m}. \end{aligned}$$

As we have seen above, by [Gig4, 4.34] this last expression equals

$$-(N-1) \int_{\tilde{X}} \langle \nabla f_t, \nabla u\rangle^2 \mathrm{d}\tilde{m} - \int_{\tilde{X}} \langle \nabla f_t, \nabla u\rangle \langle \nabla \langle \nabla f_t, \nabla u\rangle, \nabla u\rangle \mathrm{d}\tilde{m}.$$

Therefore, combining our observations, and using [Gig4, 4.35] by taking $f = g = \langle \nabla (f \circ F_t), \nabla u\rangle)$, we obtain

$$\lim_{h\to 0} \int_{\tilde{X}} \frac{\langle \nabla f_{t+h}, \nabla u\rangle^2 - \langle \nabla f_t, \nabla u\rangle^2}{h} = \int_{\tilde{X}} (\langle \nabla \langle \nabla f_t, \nabla u\rangle, \nabla u\rangle) 2(\langle \nabla f_t, \nabla u\rangle) = -(N-1) \int_{\tilde{X}} \langle \nabla f_t, \nabla u\rangle^2.$$

In conclusion, we have found that $\frac{d}{dt} \int_{\tilde{X}} \langle \nabla f_t, \nabla u\rangle^2 = -(N-1) \int_{\tilde{X}} \langle \nabla f_t, \nabla u\rangle^2$. Hence,

$$\int_{\tilde{X}} \langle \nabla f_t, \nabla u\rangle^2 = e^{-(N-1)t} \int_{\tilde{X}} \langle \nabla f, \nabla u\rangle^2.$$

Now we will obtain the second equality. Observe that $\langle \nabla f, \nabla f\rangle = \mathrm{Hess}[u](\nabla f, \nabla f) + \langle \nabla f, \nabla u\rangle^2$ implies

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(t) = \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int \mathrm{Hess}[u](\nabla f_t, \nabla f_t) + \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int \langle \nabla f_t, \nabla u\rangle^2.$$

By the our arguments above,

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(t) = \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_{\tilde{X}} \mathrm{Hess}[u](\nabla f_t, \nabla f_t) - \frac{(N-1)}{2} \int_{\tilde{X}} \langle \nabla f_t, \nabla u\rangle^2.$$

From the previous theorem

$$\frac{d}{dt} \mathcal{E}(t) = \int_{\tilde{X}} \mathrm{Hess}[u](\nabla f_t, \nabla f_t) - \frac{(N-1)}{2} \int_{\tilde{X}} \langle \nabla f_t, \nabla f_t\rangle.$$

Using both expressions for $\frac{d}{dt} \mathcal{E}(t)$ and solving for $\frac{d}{dt} \int_{\tilde{X}} \mathrm{Hess}[u](\nabla f_t, \nabla f_t)$, we get

$$\frac{d}{dt} \int \mathrm{Hess}[u](\nabla f_t, \nabla f_t) = -(N-3) \int \mathrm{Hess}[u](\nabla f_t, \nabla f_t).$$

We conclude that

$$\int_{\tilde{X}} \mathrm{Hess}[u](\nabla f_t, \nabla f_t) = e^{-(N-3)t} \int_{\tilde{X}} \mathrm{Hess}[u](\nabla f, \nabla f).$$

□

**Remark 4.7.** *As $F_{t\sharp}\tilde{m} = e^{-(N-1)t}\tilde{m}$, we can rewrite the equalities in the previous theorem in the following way:*

$$\int_{\tilde{X}} \mathrm{Hess}[u](\nabla(f\circ F_t),\nabla(f\circ F_t))\,\mathrm{d}\tilde{m} = e^{2t}\int_{\tilde{X}} \mathrm{Hess}[u](\nabla f,\nabla f)\,\mathrm{d}F_{t\sharp}\tilde{m}$$

*and*

$$\int_{\tilde{X}} \langle\nabla(f\circ F_t),\nabla u\rangle^2\,\mathrm{d}\tilde{m} = \int_{\tilde{X}} \langle\nabla f,\nabla u\rangle^2\,\mathrm{d}F_{t\sharp}\tilde{m}$$

4.3. **Localization of the Cheeger energy along the flow.** Theorem 4.6 provides the behavior of $\langle\nabla(f\circ F_t),\nabla(f\circ F_t)\rangle$ in an integral form, i.e., at the level of the Cheeger energy. In this subsection we *localize* that result, that is, we obtain a pointwise expression for $\langle\nabla(f\circ F_t),\nabla(f\circ F_t)\rangle$.

**Theorem 4.8.** *Let $u:\tilde{X}\to\mathbb{R}$ be the function constructed in Section 3.2, $F:(-\infty,\infty)\times\tilde{X}\to\tilde{X}$ our Regular Lagrangian Flow. Then for every $f\in W^{1,2}(\tilde{X})$ the following identity holds*

$$\langle\nabla(f\circ F_t),\nabla(f\circ F_t)\rangle = e^{2t}\mathrm{Hess}[u](\nabla f,\nabla f)\circ F_t + \langle\nabla f,\nabla u\rangle^2\circ F_t$$

The proof of this theorem requires the following lemma.

**Lemma 4.9.** *Let $f,g\in W^{1,2}(\tilde{X},\tilde{d},F_{t\sharp}\tilde{m})$ then*

$$\int_{\tilde{X}} \langle\nabla(f\circ F_t),\nabla(g\circ F_t)\rangle\,d\tilde{m} = e^{2t}\int_{\tilde{X}}\langle\nabla f,\nabla g\rangle\,dF_{t\sharp}\tilde{m} + (1-e^{2t})\int_{\tilde{X}}\langle\nabla f,\nabla u\rangle\langle\nabla g,\nabla u\rangle\,dF_{t\sharp}\tilde{m} \tag{4.4}$$

*Proof.* By equation (3.3) and Remark 4.7 we write

$$\int_{\tilde{X}} \langle\nabla(f\circ F_t),\nabla(f\circ F_t)\rangle\,\mathrm{d}\tilde{m} = e^{2t}\int_{\tilde{X}}\langle\nabla f,\nabla f\rangle\,\mathrm{d}F_{t\sharp}\tilde{m} + \left(1-e^{2t}\right)\int_{\tilde{X}}\langle\nabla f,\nabla u\rangle^2\,\mathrm{d}F_{t\sharp}\tilde{m}.$$

Now, by the definition of $\langle\nabla\cdot,\nabla\cdot\rangle$,

$$\langle\nabla(f\circ F_t),\nabla(g\circ F_t)\rangle = \lim_{\varepsilon>0}\frac{|\nabla(g\circ F_t+\varepsilon f\circ F_t)|^2 - |\nabla(g\circ F_t)|^2}{2\varepsilon}.$$

Putting together both equations we find,

$$\begin{aligned}\int_{\tilde{X}} \langle\nabla(f\circ F_t),\nabla(g\circ F_t)\rangle\,\mathrm{d}\tilde{m} = \lim_{\varepsilon>0}\frac{1}{2\varepsilon}\Bigg[&e^{2t}\int_{\tilde{X}}\langle\nabla(g+\varepsilon f),\nabla(g+\varepsilon f)\rangle - \langle\nabla g,\nabla g\rangle\,\mathrm{d}F_{t\sharp}\tilde{m}\\ &+(1-e^{2t})\int_{\tilde{X}}\langle\nabla(g+\varepsilon f),\nabla u\rangle^2 - \langle\nabla g,\nabla u\rangle^2\,\mathrm{d}F_{t\sharp}\tilde{m}\Bigg].\end{aligned}$$

The result follows. □

*Proof of Theorem 4.8.* Let $f\in W^{1,2}(\tilde{X},\tilde{d},F_{t\sharp}\tilde{m})$ be non-negative. Assume that $f\in L^\infty(\tilde{X},\tilde{d},F_{t\sharp}\tilde{m})$ and let $g:\tilde{X}\to[0,\infty)$ be bounded and Lipschitz with $F_{t\sharp}\tilde{m}(\mathrm{supp}(g))<\infty$. For $\varepsilon>0$ let $f_\varepsilon = f+\varepsilon g$. Notice that $f_\varepsilon^2, f_\varepsilon g\in W^{1,2}(\tilde{X},\tilde{d},F_{t\sharp}\tilde{m})$. Let $\tilde{f}_\varepsilon = f_\varepsilon\circ F_t$ and $\tilde{g} = g\circ F_t$. Applying equation (4.4) to each term on the right hand side of the first equality below we obtain:

$$
\begin{aligned}
\int_{\tilde X} g|\nabla f_\varepsilon|^2 \, \mathrm{d}F_{t\sharp}\tilde m &= \int_{\tilde X} \langle \nabla(f_\varepsilon g), \nabla f_\varepsilon\rangle - \left\langle \nabla g, \nabla(\tfrac{f_\varepsilon^2}{2})\right\rangle \mathrm{d}F_{t\sharp}\tilde m \\
&= e^{-2t} \int_{\tilde X} \left\langle \nabla(\tilde f_\varepsilon \tilde g), \nabla \tilde f_\varepsilon\right\rangle \mathrm{d}\tilde m - e^{-2t} \int_{\tilde X} \left\langle \nabla \tilde g, \nabla(\tfrac{\tilde f_\varepsilon^2}{2})\right\rangle \mathrm{d}\tilde m \\
&\quad - (e^{-2t} - 1) \int_{\tilde X} \langle \nabla f_\varepsilon g, \nabla u\rangle \langle \nabla f_\varepsilon, \nabla u\rangle - \langle \nabla g, \nabla u\rangle \left\langle \nabla(\tfrac{f_\varepsilon^2}{2}), \nabla u\right\rangle \mathrm{d}F_{t\sharp}\tilde m
\end{aligned}
\tag{4.5}
$$

We now observe that $\langle \nabla(f_\varepsilon g), \nabla u\rangle \langle \nabla f_\varepsilon, \nabla u\rangle = (f_\varepsilon \langle \nabla g, \nabla u\rangle + g \langle \nabla f_\varepsilon, \nabla u\rangle) \langle \nabla f_\varepsilon, \nabla u\rangle$ and also that $\langle \nabla g, \nabla u\rangle \left\langle \nabla \frac{f_\varepsilon^2}{2}, \nabla u\right\rangle = \langle \nabla g, \nabla u\rangle f_\varepsilon \langle \nabla f_\varepsilon, \nabla u\rangle$. Thus, equation (4.5) may be rewritten as,

$$
\begin{aligned}
\int_{\tilde X} g|\nabla f_\varepsilon|^2 \, \mathrm{d}F_{t\sharp}\tilde m = e^{-2t} \int_{\tilde X} \left\langle \nabla(\tilde f_\varepsilon \tilde g), \nabla \tilde f_\varepsilon\right\rangle \, \mathrm{d}\tilde m - e^{-2t} \int_{\tilde X} \left\langle \nabla \tilde g, \nabla(\tfrac{\tilde f_\varepsilon^2}{2})\right\rangle \, \mathrm{d}\tilde m \\
-(e^{-2t} - 1) \int_{\tilde X} g \langle \nabla f_\varepsilon, \nabla u\rangle \langle \nabla f_\varepsilon, \nabla u\rangle \, \mathrm{d}F_{t\sharp}\tilde m.
\end{aligned}
\tag{4.6}
$$

Now, $|\nabla(f - f_\varepsilon)| \to 0$ in $L^2(\tilde X, \tilde d, F_{t\sharp}\tilde m)$ as $\varepsilon \downarrow 0$ and $|\nabla(\tilde f - \tilde f_\varepsilon)| \to 0$ in $L^2(\tilde X, d, \tilde m)$ as $\varepsilon \downarrow 0$, where $\tilde f := f \circ F_t$. In particular $\int_{\tilde X} g|\nabla f_\varepsilon|^2 \, \mathrm{d}F_{t\sharp}\tilde m \to \int_{\tilde X} g|\nabla f|^2 \, \mathrm{d}F_{t\sharp}\tilde m$ and $\int_{\tilde X} \tilde g|\nabla \tilde f_\varepsilon|^2 \, \mathrm{d}\tilde m \to \int_{\tilde X} \tilde g|\nabla \tilde f|^2 \, \mathrm{d}\tilde m$ as $\varepsilon \downarrow 0$. Therefore, passing to the limit in equation (4.6) as $\varepsilon \downarrow 0$ we obtain,

$$
\int_{\tilde X} g|\nabla f|^2 \, \mathrm{d}F_{t\sharp}\tilde m = e^{-2t} \int_{\tilde X} \tilde g|\nabla \tilde f|^2 \, \mathrm{d}\tilde m - (e^{-2t} - 1) \int g \langle \nabla f, \nabla u\rangle^2 \, \mathrm{d}F_{t\sharp}\tilde m.
$$

Rearranging terms,

$$
\begin{aligned}
e^{-2t} \int_{\tilde X} \tilde g|\nabla \tilde f|^2 \, \mathrm{d}\tilde m &= \int_{\tilde X} g|\nabla f|^2 \, \mathrm{d}F_{t\sharp}\tilde m - (1 - e^{-2t}) \int_{\tilde X} g \langle \nabla f, \nabla u\rangle^2 \, \mathrm{d}F_{t\sharp}\tilde m \\
&= \int_{\tilde X} g \, \mathrm{Hess}[u](\nabla f, \nabla f) \, \mathrm{d}F_{t\sharp}\tilde m + e^{-2t} \int_{\tilde X} g \langle \nabla f, \nabla u\rangle^2 \, \mathrm{d}F_{t\sharp}\tilde m.
\end{aligned}
\tag{4.7}
$$

As $g$ was arbitrarily chosen, we conclude that Theorem (4.8) holds for all non-negative $f \in W^{1,2} \cap L^\infty(\tilde X, \tilde d, F_{t\sharp}\tilde m)$. Finally, by a truncation argument the restriction to non-negative functions can be dropped, and then the general case follows. $\square$

## 5. The quotient metric measure space $(X', d', m')$

5.1. **Continuous representative of $F$.** Using our knowledge of $|\nabla f_t|$ we can now improve the regularity of the flow and show that for fixed $t$, the function $F_t$ is Lipschitz.

**Theorem 5.1.** *The map $F : \mathbb{R} \times \tilde X \to \tilde X$ admits a continuous representative with respect to the measure $\mathcal{L}^1 \times \tilde m$. Still denoting such representative by $F$, we have:*

i) *The semigroup property holds, i.e., for every $t, s \in \mathbb{R}$ and $x \in \tilde X$ we have $F_t(F_s(x)) = F_{t+s}(x)$. Moreover,*
$$
\tilde d(F_t(x), F_{t+s}(x)) = |s|.
$$
ii) *For every $t \in \mathbb{R}$, $F_t$ is a bi-Lipschitz map with $\mathrm{Lip}(F_t) \le \max\{e^t, 1\}$.*

iii) *Given a curve $\gamma$ let $\bar\gamma := F_t \circ \gamma$. Then one of the curves is absolutely continuous if and only if the other is and their metric speeds are related by the following inequality*

$$
\min\{1, e^t\}|\dot\gamma_s| \le |\dot{\bar\gamma}_s| \le \max\{1, e^t\}|\dot\gamma_s| \qquad \textit{for a.e. } s \in [0, 1].
\tag{5.1}
$$

*Proof.* For each $t \in \mathbb{R}$ we will first obtain a $\max\{1, e^t\}$-Lipschitz representative of $F_t$. Let $\mathcal{D} \subset W^{1,2}(\tilde X, \tilde d, \tilde m)$ be a countable set of 1-Lipschitz functions with compact support such that $\mathcal{D}$ is dense in the space of 1-Lipschitz functions with compact support with respect to uniform convergence. As in [Gig4, Lemma 4.19], let $f_{n,k} = \max\{0, \min\{d(\cdot, x_n), k - \tilde d(\cdot, x_n)\}\}$, is a dense subset of $\tilde X$. These functions are 1-Lipschitz with bounded support thus belong to $W^{1,2}(\tilde{X}, d, \tilde{m})$ with $|\nabla f_{k,n}| \leq 1$ $\tilde{m}$-a.e. Then, for all $y_0, y_1 \in \tilde{X}$,

$$\tilde{d}(y_0, y_1) = \sup_{f \in \mathcal{D}} |f(y_0) - f(y_1)|. \tag{5.2}$$

By Theorem (4.8) we know that

$$\langle \nabla(f \circ F_t), \nabla(f \circ F_t) \rangle = e^{2t} \mathrm{Hess}[u](\nabla f, \nabla f) \circ F_t + \langle \nabla f, \nabla u \rangle^2 \circ F_t.$$

Therefore,

$$\begin{aligned} \langle \nabla(f \circ F_t), \nabla(f \circ F_t) \rangle &\leq \max\{e^{2t}, 1\} \left( \mathrm{Hess}[u](\nabla f, \nabla f) \circ F_t + \langle \nabla f, \nabla u \rangle^2 \circ F_t \right) \\ &= \max\{e^{2t}, 1\} \langle \nabla f, \nabla f \rangle \circ F_t. \end{aligned}$$

Thus, $|\nabla(f \circ F_t)| \leq \max\{1, e^t\}$. Because $X$ has the Sobolev to Lipschitz property, $f \circ F_t$ has a $\max\{1, e^t\}$-Lipschitz representative. Given that $\mathcal{D}$ is countable, then there is an $\tilde{m}$-negligible Borel set $\mathcal{N}'$ such that the restrictions $f \circ F_t : \tilde{X} \setminus \mathcal{N}' \to \mathbb{R}$ are $\max\{1, e^t\}$-Lipschitz for every $f \in \mathcal{D}$. Therefore, by (5.2) for $x_0, x_1 \in F_t^{-1}(\tilde{X} \setminus \mathcal{N}')$ we have

$$\tilde{d}(F_t(x_0), F_t(x_1)) = \sup_{f \in \mathcal{D}} |f(F_t(x_0)) - f(F_t(x_1))| \leq \max\{1, e^t\} \tilde{d}(x_0, x_1).$$

Now, for each $(t, x), (s, y) \in \mathbb{R} \times \tilde{X}$ we obtain

$$\tilde{d}(F_t(x), F_s(y)) \leq \tilde{d}(F_t(x), F_t(y)) + \tilde{d}(F_t(y), F_s(y)) \leq \max\{1, e^t\} \tilde{d}(x, y) + |s - t|. \tag{5.3}$$

This proves that $F$ admits a continuous representative. From this, and Lemma (3.15), the statements in $i)$ and $ii)$ follow.

For $iii)$, let us assume that $\gamma$ is absolutely continuous. Then

$$\tilde{d}(\bar{\gamma}_h, \bar{\gamma}_s) = \tilde{d}(F_t(\gamma_h), F_t(\gamma_s)) \leq \max\{1, e^t\} \tilde{d}(\gamma_h, \gamma_s) \leq \max\{1, e^t\} \int_h^s |\dot{\gamma}_r| \, \mathrm{d}r.$$

Therefore, $|\dot{\bar{\gamma}}_s| \leq \max\{1, e^t\} |\dot{\gamma}_s|$ for a.e.-$s \in [0, 1]$. The other inequality is proven in a similar way. $\square$

We continue this section by defining a quotient metric measure space $(X', d', m')$ induced by the flow $F$. We will show that it is an infinitesimally Hilbertian space, and that it satisfies the Sobolev to Lipschitz property. We now provide the definition of $X'$.

**Definition 5.2.** *Let $X' = u^{-1}(0)$ and define $d' : X' \times X' \to \mathbb{R}$ by*

$$d'(z, y) = \inf\{L(\gamma) | \gamma \in AC([0, 1], \tilde{X}),\ u \circ \gamma = 0,\ \gamma_0 = z,\ \gamma_1 = y\}.$$

*Here $L(\gamma) = \int_0^1 |\dot{\gamma}_r| dr$.*

**Lemma 5.3.** *Let $X'$ be as in Definition 5.2, then $d'$ is a well defined function and $(X', d')$ is a metric space. The inclusion map $\iota : (X', d') \to (\tilde{X}, \tilde{d})$ is 1-Lipschitz.*

*Proof.* First we will show that the set

$$\{\gamma \in AC([0, 1], \tilde{X}),\ u \circ \gamma = 0,\ \gamma_0 = z,\ \gamma_1 = y\}$$

is nonempty for any $z, y \in X'$. As $\tilde{X}$ is a geodesic space there exists an absolutely continuous $\gamma : [0, 1] \to \tilde{X}$ such that $\gamma_0 = z$ and $\gamma_1 = y$. By Theorem (5.1), the curve $t \mapsto F_{-u(\gamma_t)}(\gamma_t)$ is contained in $u^{-1}(0)$. We only have to prove that it is absolutely continuous. To that end, let $M = \max\{\mathrm{Lip}(F_{-u(\gamma_s)}) | 0 \leq s \leq 1\}$. This maximum $M$ is achieved because $u$, $F$, and $\gamma$ are continuous.

Using the triangle inequality, together with $F_{-u(\gamma_s)}$ Lipschitz for all $s$ and that $u$ 1-Lipschitz, gives, for all $0 \le s \le t \le 1$,

$$
\begin{aligned}
\tilde{d}(F_{-u(\gamma_s)}(\gamma_s), F_{-u(\gamma_t)}(\gamma_t)) &\le \tilde{d}(F_{-u(\gamma_s)}(\gamma_s), F_{-u(\gamma_s)}(\gamma_t)) + \tilde{d}(F_{-u(\gamma_s)}(\gamma_t), F_{-u(\gamma_t)}(\gamma_t)) \\
&\le \mathrm{Lip}(F_{-u(\gamma_s)})\tilde{d}(\gamma_s, \gamma_t) + |u(\gamma_t) - u(\gamma_s)| \\
&\le (\mathrm{Lip}(F_{-u(\gamma_s)}) + 1)\tilde{d}(\gamma_s, \gamma_t) \\
&\le (M+1) \int_s^t |\dot{\gamma}_r| \, \mathrm{d}r.
\end{aligned} \tag{5.4}
$$

Hence, $F_{-u(\gamma_t)}(\gamma_t)$ is absolutely continuous in $(\tilde{X}, d)$ and $d'$ is well defined.

If $z, y \in u^{-1}(0)$ then,

$$
\begin{aligned}
\tilde{d}(z, y) &\le \inf\{L(\gamma) | \gamma \in AC([0,1], \tilde{X}),\ u \circ \gamma = 0,\ \gamma_0 = z,\ \gamma_1 = y\} \\
&= d'(z, y).
\end{aligned} \tag{5.5}
$$

This shows that $\iota$ is a 1-Lipschitz map and that $d'$ is positive definite. Symmetry and the triangle inequality follow from the definition of $d'$. □

5.2. **Metric speed of curves in the quotient space.** Let $\pi : \tilde{X} \to X'$ be given by $\pi(x) = F_{-u(x)}(x)$. By Lemma 3.15, $\pi$ is well defined and from now on we call it the projection map. The aim of this subsection is to study $\pi$ and its effect on the metric speed of curves in $\tilde{X}$. The main results of this subsection are collected in the following proposition, which will be used in the next subsection to relate a subspace of $W^{1,2}(\tilde{X}, \tilde{d}, \tilde{m})$ with $W^{1,2}(X', d', m')$.

**Proposition 5.4.** *Let $\boldsymbol{\pi}$ be a test plan on $\tilde{X}$. Then, for $\boldsymbol{\pi}$-a.e. $\gamma$, the curve $\tilde{\gamma} = \pi \circ \gamma$ in $(X', d')$ is absolutely continuous and for a.e. $t \in [0,1]$,*

*(1)* $|\dot{\tilde{\gamma}}_t| \le e^{-u(\gamma_t)} \, |\dot{\gamma}_t|$.

*(2) The projection map $\tilde{\pi} : \tilde{X} \to X'$ is locally Lipschitz, .i.e. for all $x_0 \in \tilde{X}$ and all $x, y \in B_r(x_0)$,*

$$d'(\pi(x), \pi(y)) \le e^{-u(x_0)+3r} \tilde{d}(x, y).$$

To prove (1) we will follow the strategy developed by De Philippis-Gigli (Section 3.6.2 [DePG]) and define a "truncated" and reparametrized flow $\hat{F}$ with the property that for large $s$ the maps $\hat{F}_s$ approximate the projection map $\pi : u^{-1}([-R, R]) \to u^{-1}(0)$, for $0 < R < 1$.

We proceed with the details in the following way. Let $0 < \overline{R} < R < 1$ and $\psi \in C^\infty(\mathbb{R})$ with support in $(-R, R)$ such that $\psi(z) = -\frac{1}{2}z^2$ for all $z \in [-\overline{R}, \overline{R}]$. Define the function $\hat{u} = \psi \circ u : \tilde{X} \to \mathbb{R}$ and consider a reparametrization function $\mathrm{rep}_s(r)$ defined by the property that $\partial_s \mathrm{rep}_s(r) = \psi'(\mathrm{rep}_s(r) + r)$. We now define the flow $\hat{F} : \mathbb{R} \times \tilde{X} \to \tilde{X}$ by $\hat{F}_s(x) \coloneqq F_{\mathrm{rep}_s(u(x))}(x)$ and note that $\hat{F}_s(x) = F_{(e^{-s}-1)u(x)}(x)$ on $u^{-1}([-\overline{R}, \overline{R}])$. It follows from these definitions that $\hat{F}$ is the regular Lagrangian flow associated to $\hat{u}$. Moreover, the following formulæ hold for all $x \in u^{-1}([-\overline{R}, \overline{R}])$,

$$\hat{u} = -\frac{1}{2}u^2, \tag{5.6}$$

$$\nabla \hat{u} = -u \nabla u, \tag{5.7}$$

$$\Delta \hat{u} = -u(N-1) - 1, \tag{5.8}$$

$$\mathrm{Hess}(\hat{u}) = -u\mathrm{Id} + (u-1)(\nabla u \otimes \nabla u). \tag{5.9}$$

The previous formulæ imply $\hat{u} \in \mathrm{Test}(\tilde{X})$, in particular it has bounded gradient, Laplacian and Hessian. When $s \to \infty$ then $\mathrm{rep}_s(u(x)) \to -u(x)$ for every $x \in u^{-1}([-\overline{R}, \overline{R}])$, that is $\hat{F}_s$

converges uniformly to $\pi := F_{-u(\cdot)}(\cdot)$, the projection map. We observe that $\hat{F}_s$ is the identity on $\tilde{X} \setminus u^{-1}([-R,R])$ and it sends $u^{-1}([-\overline{R},\overline{R}])$ to itself.

In the following, for each $s \in \mathbb{R}$, we only concern ourselves with $\hat{F}_s|_{u^{-1}([-\overline{R},\overline{R}])}$, because this will be sufficient for our purposes. Observe that [DePG, Lemma 3.30], [DePG, Proposition 3.31] hold in this setting because, as we will now see, $\hat{F}_s$ is of bounded deformation for any $s \in \mathbb{R}$. We begin by showing that $\hat{F}_s$ is Lipschitz on $u^{-1}([-\overline{R},\overline{R}])$:

$$\begin{aligned}
\tilde{d}(\hat{F}_s(x),\hat{F}_s(y)) &= \tilde{d}(F_{(e^{-s}-1)u(x)}(x), F_{(e^{-s}-1)u(y)}(y)) \\
&\le \tilde{d}(F_{(e^{-s}-1)u(x)}(x), F_{(e^{-s}-1)u(x)}(y)) + \tilde{d}(F_{(e^{-s}-1)u(x)}(y), F_{(e^{-s}-1)u(y)}(y)) \\
&\le \max\{1, e^{(e^{-s}-1)(u(x))}\}\tilde{d}(x,y) + |(e^{-s}-1)||(u(x)) - (u(y))| \\
&\le \max\{1, e^{(e^{-s}-1)(u(x))}\}\tilde{d}(x,y) + |(e^{-s}-1)|\tilde{d}(x,y) \\
&\le \left(\max\{1, e^{|(e^{-s}-1)|\overline{R})}\} + |(e^{-s}-1)|\right)\tilde{d}(x,y)
\end{aligned}$$

This proves $\hat{F}_s|_{u^{-1}([-\overline{R},\overline{R}])}$ is Lipschitz with Lipschitz constant

$$\max\{1, e^{|(e^{-s}-1)|\overline{R})}\} + |(e^{-s}-1)|$$

for any $s \in \mathbb{R}$.

Let us proceed by showing that $\hat{F}_s|_{u^{-1}([-\overline{R},\overline{R}])}$ is of bounded compression, as:

$$(\hat{F}_s|_{u^{-1}(-\overline{R},\overline{R}])})_{\#}\tilde{m} = (F_{(e^{-s}-1)u(\cdot)})_{\#}\tilde{m} = e^{-(N-1)(e^{-s}-1)u(\cdot)}\tilde{m} \le e^{(N-1)|(e^{-s}-1)|\overline{R}}\tilde{m}$$

It follows from these observations that [DePG, Lemma 3.30] and [DePG, Proposition 3.31] (which we recall below, note the different sign convention here) hold in our setting.

**Lemma 5.5** (De Philippis-Gigli, Lemma 3.30)**.** *Let $\varphi \in W^{1,2}(\tilde{X})$. Then the map $s \mapsto \varphi \circ \hat{F}_s \in L^2(X)$ is $C^1$ and its derivative is given by*

$$\frac{d}{ds}\varphi \circ \hat{F}_s = \langle \nabla\varphi, \nabla\hat{u}\rangle \circ \hat{F}_s. \tag{5.10}$$

*If $\varphi$ is further assumed to be in* $\mathrm{Test}(\tilde{X})$*, then the map $s \mapsto \mathrm{d}(\varphi \circ \hat{F}_s) \in L^2(T\tilde{X})$ is also $C^1$ and its derivative is given by*

$$\frac{d}{ds}\left(\mathrm{d}(\varphi \circ \hat{F}_s)\right) = \mathrm{d}(\langle \nabla\varphi, \nabla\hat{u}\rangle \circ \hat{F}_s). \tag{5.11}$$

**Proposition 5.6** (De Philippis-Gigli, Proposition 3.31)**.** *Let $v \in L^2(T\tilde{X})$ and put $v_s := \mathrm{d}\hat{F}_s(v)$. Then the map $s \mapsto \frac{1}{2}|v_s|^2 \circ \hat{F} \in L^1(\tilde{X})$ is $C^1$ on $\mathbb{R}$ and its derivative is given by the formula*

$$\frac{d}{ds}\frac{1}{2}|v_s|^2 \circ \hat{F} = \mathrm{Hess}[\hat{u}](v_s, v_s) \circ \hat{F}_s \tag{5.12}$$

*the incremental ratios being convergent both in $L^1(\tilde{X})$ and $\tilde{m}$-a.e. If $v$ is also bounded, then the curve $s \mapsto \frac{1}{2}|v_s|^2 \circ \hat{F}$ is $C^1$ also when seen with values in $L^2(\tilde{X})$, and in this case the incremental ratios in 5.12 also converge in $L^2(\tilde{X})$ to the right hand side.*

We will use the previous results to prove the following monotonicity formula. The proof is similar to that of [DePG, Corollary 3.32].

**Corollary 5.7.** *Let $v \in L^2(T\tilde{X})$ be concentrated on $B := u^{-1}([-\overline{R},\overline{R}])$ and set $v_s := \mathrm{d}\hat{F}_s(v)$. Then for every $s_1, s_2 \in \mathbb{R}$ such that $s_1 \le s_2$,*

$$\left(e^{-2u}|v_{s_2}|^2\right) \circ \hat{F}_{s_2} \le \left(e^{-2u}|v_{s_1}|^2\right) \circ \hat{F}_{s_1}, \qquad \tilde{m} - a.e.. \tag{5.13}$$

*Proof.* We may assume that $v$ is bounded up to replacing it with $v_n := \chi_{\{|v|\le n\}}v$, using the fact that $|\mathrm{d}\hat{F}_s(v_n)| \circ \hat{F}_s = |\mathrm{d}\hat{F}_s(v)| \circ \hat{F}_s$ on $\{|v| \le n\}$ and letting $n \to \infty$.

Now we observe that on the complement of $B$ both sides of 5.13 are 0 $\tilde{m}$-a.e. (as a consequence that $v$ is concentrated on $B$). So that we only need to prove

$$\left(e^{-2u}|v_s|^2 \circ \hat{F}_s\right)\chi_B \le \left(e^{-2u}|v|^2\right)\chi_B, \qquad \tilde{m} - a.e..$$

Observe that by Lemma 5.5 the derivative of $s \mapsto u \circ \hat{F}_s$ is

$$\frac{d}{ds}u \circ \hat{F}_s = \langle \nabla u, \nabla \hat{u}\rangle \circ \hat{F}_s = -u \circ \hat{F}_s. \tag{5.14}$$

Therefore, integrating with respect to $s$ we obtain $u \circ \hat{F}_s = e^{-s}u$. As we are assuming $v$ is bounded, by Proposition 5.6, the map $s \mapsto \frac{|v_s|^2}{2} \circ \hat{F}_s\chi_B \in L^1(\tilde{X})$ is $C^1$ and then

$$\begin{aligned}
\frac{d}{ds}\left(e^{-2u\circ\hat{F}_s}\frac{|v_s|^2}{2} \circ \hat{F}_s\chi_B\right) &= \left(\frac{d}{ds}\left(e^{-2u\circ\hat{F}_s}\right)\frac{|v_s|^2}{2} \circ \hat{F}_s + \frac{d}{ds}\left(\frac{|v_s|^2}{2} \circ \hat{F}_s\right)e^{-2u\circ\hat{F}_s}\right)\chi_B \\
&= \left(\frac{|v_s|^2}{2} \circ \hat{F}_s\left\langle \nabla e^{-2u}, \nabla\hat{u}\right\rangle \circ \hat{F}_s + e^{-2u\circ\hat{F}_s}\mathrm{Hess}[\hat{u}]\left(v_s, v_s\right) \circ \hat{F}_s\right)\chi_B \\
&= \left((u \circ \hat{F}_s)e^{-2u\circ\hat{F}_s}|v_s|^2 \circ \hat{F}_s + e^{-2u\circ\hat{F}_s}\mathrm{Hess}[\hat{u}]\left(v_s, v_s\right) \circ \hat{F}_s\right)\chi_B \\
&= e^{-2u\circ\hat{F}_s}\left((u \circ \hat{F}_s - 1)\left\langle \nabla u, v_s\right\rangle^2 \circ \hat{F}_s\right)\chi_B \\
&\le 0.
\end{aligned}$$

Recall that $\bar{R} < R \le 1$, from which follows that $u(x) \le 1$ for all $x \in B$. This concludes the proof. ☐

**Proposition 5.8.** *Let $\boldsymbol{\pi}$ be a test plan and $\gamma : [0,1] \to u^{-1}([-\overline{R}, \overline{R}])$. Then* $\mathrm{ms}_t(\tilde{\gamma}) \le e^{-u(\gamma_t)}\mathrm{ms}_t(\gamma)$ *for a.e. $t \in [0,1]$, $\boldsymbol{\pi}$-a.e. $\gamma$, where $\tilde{\gamma} := \pi \circ \gamma$.*

The proof of the proposition follows along the lines of [Gig4, Proposition 3.33], as follows.

*Proof.* Abusing the notation we will still denote by $\hat{F}_s$ the map $C([0,1], \tilde{X}) \to C([0,1], \tilde{X})$ taking $\gamma \mapsto \hat{F}_s \circ \gamma$. Recall that for every $t \in [0,1]$ the differential of $\hat{F}_s$ induces a map, still denoted by $d\hat{F}_s$, from $L^2(T\tilde{X}, e_t, \boldsymbol{\pi})$ to $L^2(T\tilde{X}, e_t, \boldsymbol{\pi}_s)$. We claim that for any $s_1 \le s_2$ and any $V \in L^2(T\tilde{X}, e_t, \boldsymbol{\pi})$,

$$\left(e^{-2u\circ e_t}|d\hat{F}_{s_2}(V)|^2\right) \circ \hat{F}_{s_2} \le \left(e^{-2u\circ e_t}|d\hat{F}_{s_1}(V)|^2\right) \circ \hat{F}_{s_1} \qquad \boldsymbol{\pi} - \text{a.e.} \tag{5.15}$$

To prove the claim we first consider $V$ to be of the form $e_t^*v$ for some $v \in L^2(T\tilde{X})$. By Proposition 5.7, for $s_1 \le s_2$, $\boldsymbol{\pi}$-a.e. ,

$$\begin{aligned}
\left(e^{-2u\circ e_t}|d\hat{F}_{s_2}(e_t^*v)|^2\right) \circ \hat{F}_{s_2} &= \left(e^{-2u\circ e_t}|e_t^*d\hat{F}_{s_2}(v)|^2\right) \circ \hat{F}_{s_2} \\
&= \left(e^{2u}|d\hat{F}_{s_2}(v)|^2\right) \circ e_t \circ \hat{F}_{s_2} \\
&= \left(e^{2u}|d\hat{F}_{s_2}(v)|^2\right) \circ \hat{F}_{s_2} \circ e_t \\
&\le \left(e^{2u}|d\hat{F}_{s_1}(v)|^2\right) \circ \hat{F}_{s_1} \circ e_t \\
&= \left(e^{-2u\circ e_t}|d\hat{F}_{s_1}(e_t^*v)|^2\right) \circ \hat{F}_{s_1}
\end{aligned}$$

Let $(A_i)_{i\in\mathbb{N}}$ be a Borel partition of $C([0,1], \tilde{X})$. The locality property of $d\hat{F}_s : L^2(T\tilde{X}, e_t, \boldsymbol{\pi}) \to L^2(T\tilde{X}, e_t, \boldsymbol{\pi}_s)$ implies that any combination of the form $\sum \chi_{A_i}e_t^*v_i$, with $v_i \in L^2(T\tilde{X})$, satisfies

$$\left(e^{-2u\circ e_t}|d\hat{F}_{s_2}(\textstyle\sum \chi_{A_i}e_t^*v_i)|^2\right) \circ \hat{F}_{s_2} \le \left(e^{-2u\circ e_t}|d\hat{F}_{s_1}(\textstyle\sum \chi_{A_i}e_t^*v_i)|^2\right) \circ \hat{F}_{s_1} \qquad \boldsymbol{\pi} - \text{a.e.}$$

As the elements of the form $\sum \chi_{A_i}e_t^*v_i$ are dense in $L^2(T\tilde{X}, e_t, \boldsymbol{\pi})$ and $d\hat{F}_s$ is continuous when considered as a map $L^2(T\tilde{X}, e_t, \boldsymbol{\pi}) \to L^2(T\tilde{X}, e_t, \boldsymbol{\pi}_s)$ the claim follows.

Let $(\boldsymbol{\pi}_s)'_t \in L^2(T\tilde{X}, e_t, \boldsymbol{\pi}_s)$ be the speed at time $t$ of the test plan $\boldsymbol{\pi}_s$. Applying (5.15) to $\boldsymbol{\pi}'_t$ and using the Chain Rule for Speeds [DePG, Proposition 3.28] we obtain that for $s_1 \leq s_2$ and a.e. $t \in [0,1]$,

$$\left(e^{-2u\circ e_t}|(\boldsymbol{\pi}_{s_2})'_t|^2\right) \circ \hat{F}_{s_2} \leq \left(e^{-2u\circ e_t}|(\boldsymbol{\pi}_{s_1})'_t|^2\right) \circ \hat{F}_{s_1}, \qquad \boldsymbol{\pi} - a.e..$$

Now we integrate with respect to $t$ and recall the link between point-wise norm and metric speed given in [DePG, (3.58)] to obtain,

$$\int\int_0^1 e^{-2u(\gamma_t)}|\dot{\gamma}_t|^2 \mathrm{d}t \; \mathrm{d}\boldsymbol{\pi}_{s_2}(\gamma) \leq \int\int_0^1 e^{-2u(\gamma_t)}|\dot{\gamma}_t|^2 \mathrm{d}t \; \mathrm{d}\boldsymbol{\pi}_{s_1}(\gamma). \tag{5.16}$$

The lower semicontinuity of the corresponding functional follows analogously as in [DePG, Proposition 3.33]. Now let us consider the functions $\hat{F}_s$ as functions from $\overline{B} \to \overline{B}$ and recall that they converge uniformly to the projection map $\pi : \tilde{X} \to u^{-1}(0)$ as $s \to \infty$. Then the test plans $\boldsymbol{\pi}_s$ weakly converge to $\pi_*\boldsymbol{\pi}$ as $s \to \infty$ and therefore,

$$\int\int_0^1 |\dot{\gamma}_t|^2 \mathrm{d}\pi_*\boldsymbol{\pi} \leq \liminf_{s\to\infty} \int\int_0^1 e^{-2u(\gamma_t)}|\dot{\gamma}_t|^2 \mathrm{d}\boldsymbol{\pi}_s.$$

From the last expression it follows that

$$\int\int_0^1 \mathrm{ms}_t^2(\pi\circ\gamma)\mathrm{d}t \; \mathrm{d}\boldsymbol{\pi} \leq \int\int_0^1 e^{-2u(\gamma_t)}\mathrm{ms}_t^2(\gamma)\mathrm{d}t \; \mathrm{d}\boldsymbol{\pi}.$$

Now, the argument to conclude the proof from this integral formulation follows exactly as the corresponding part of [DePG, Proposition 3.33]. □

*Proof of Proposition 5.4.* We start by proving (1). By Proposition 5.8, (1) holds for $\gamma \in u^{-1}[-\overline{R}, \overline{R}]$. Proceeding as in the proof of Proposition 5.8 it is possible to show that if $\boldsymbol{\pi}$ is a test plan and $\gamma : [0,1] \to u^{-1}([c-\overline{R}, c+\overline{R}])$. Then $\mathrm{ms}_t(pr_c\gamma) \leq e^{-u(\gamma_t)+c}\mathrm{ms}_t(\gamma)$ for a.e. $t \in [0,1]$, $\boldsymbol{\pi}$-a.e. $\gamma$, where $pr_c\gamma := F_{-u(\gamma)+c} \circ \gamma$. Take $c = \overline{R}$ and

$$\gamma : [0,1] \to u^{-1}([c-\overline{R}, c+\overline{R}]) = u^{-1}([0, 2\overline{R}]).$$

It follows by iii) in Theorem 5.1 and Proposition 5.8 that for almost every $t \in [0,1]$,

$$e^{-\overline{R}}\mathrm{ms}_t(pr_{\overline{R}}(\gamma)) \leq \mathrm{ms}_t(pr_0(pr_{\overline{R}}\gamma)) \leq e^{-\overline{R}}\mathrm{ms}_t(pr_{\overline{R}}(\gamma)).$$

Note that $pr_0(\gamma) = pr_0(pr_{\overline{R}}\gamma)$. Thus, for almost every $t \in [0,1]$,

$$\mathrm{ms}_t(pr_0\gamma) = e^{-\overline{R}}\mathrm{ms}_t(pr_{\overline{R}}(\gamma)) \leq e^{-\overline{R}}e^{-u(\gamma_t)+\overline{R}}\mathrm{ms}_t(\gamma).$$

This shows that (1) is satisfied for curves on $u^{-1}(([0, 2\overline{R}])$. Proceeding in the same way, (1) follows.

Now we prove part (2). Let $x, y \in B_r(x_0)$ and $\gamma : [0,1] \to \tilde{X}$ be a minimal geodesic joining them. As $u$ is 1-Lipschitz :

$$u(\gamma_t) \geq \max\{u(\gamma_0), u(\gamma_1)\} - \tilde{d}(\gamma_0, \gamma_1),$$
$$u(\gamma_0) \geq -r + u(x_0),$$
$$u(\gamma_1) \geq -r + u(x_0).$$

Thus, $u(\gamma_t) \geq -r + u(x_0) - 2r = u(x_0) - 3r$. From the previous paragraph $|\dot{\tilde{\gamma}}_t| \leq e^{-u(\gamma_t)}\,|\dot{\gamma}_t|$. Therefore, $d'(\pi(x), \pi(y)) \leq L(\tilde{\gamma}) \leq e^{-u(x_0)+3r}\tilde{d}(x,y)$. □

5.3. **Properties of the quotient metric measure space.** Here we show that $(X', d')$ is a complete, separable and geodesic metric space. Then we define a measure $m'$ on $X'$ and study the relationship between the spaces $W^{1,2}(X', d', m')$ and $W^{1,2}(\tilde{X}, d, \tilde{m})$. At the end of the subsection we show that $(X', d', m')$ is an infinitesimally Hilbertian space that satisfies the Sobolev to Lipschitz property.

**Theorem 5.9.** *With the same notation and assumptions of Definition 5.2, $(X', d')$ is a complete, separable and geodesic metric space.*

*Proof.* By Proposition 5.4 the map $\pi$ is continuous, we will show that $X'$ is separable. Since $\tilde{X}$ is separable there exist a countable dense subset $\{x_j\} \in \tilde{X}$. Consider an open set $U \subset X'$, then $\pi^{-1}(U)$ is open in $\tilde{X}$. As $\tilde{X}$ is separable there exists $x_j \in \pi^{-1}(U)$, and then $\pi(x_j) \in U$. Thus, $\{\pi(x_j)\}$ is a dense subset of $X'$.

To prove that $(X', d')$ is complete let $\{x_j\} \in X'$ be a Cauchy sequence. Then, because $\iota : X' \to \tilde{X}$ is 1-Lipschitz, $\{\iota(x_j)\}$ is a Cauchy sequence in $\tilde{X}$, and hence it has a convergent subsequence $\iota(x_{j_k}) \to x$. Given that $\pi$ is continuous, $x_{j_k} = \pi(\iota(x_{j_k})) \to \pi(x)$.

To prove that $(X', d')$ is a geodesic space recall that a complete, locally compact length space is geodesic. So it is enough to prove that $(X', d')$ is locally compact. This is very similar to the previous paragraph. Let $x \in X'$ and $r > 0$. If $\{x_j\} \subset \overline{B_r^{d'}(x)}$, then $\{\iota(x_j)\} \subset \overline{B_r^{\tilde{d}}(\iota(x))}$. Now, since $(\tilde{X}, \tilde{d})$ is locally compact, there exists a convergent subsequence $\iota(x_{j_k}) \to y$. Because $\pi$ is continuous, $x_{j_k} = \pi(\iota(x_{j_k})) \to \pi(y)$ and $d'(\pi(y), x) = \lim_{k\to\infty} d'(x_{j_k}, x) \leq r$. This concludes the proof. □

Given that $u : \tilde{X} \to \mathbb{R}$ and $\pi : \tilde{X} \to X'$ are continuous (see (2) in Proposition 5.4 where it is shown that $\pi$ is locally Lipschitz and recall that $u$ is Lipschitz), we define a Borel measure on $X'$.

**Definition 5.10.** *We define the measure $m'$ on $(X', d')$ by*

$$m'(A) = (\int_0^1 e^{(N-1)s} ds)^{-1} \tilde{m}(\pi^{-1}(A) \cap u^{-1}[0,1])$$

*for any Borel set $A \subset X'$.*

**Lemma 5.11.** *Given $A \subset X'$ Borel, let $A_a^b = \{x \in \tilde{X} | u(x) \in [a,b],\, \pi(x) \in A\}$. Then,*

$$\tilde{m}(A_a^b) = m'(A) \int_a^b e^{(N-1)s}\, \mathrm{d}s. \tag{5.17}$$

*Proof.* The proof follows that of Proposition 5.28 [Gig4]. For completeness we give some details. Note that by the definition of $m'$, equation (5.17) holds for $a = 0$ and $b = 1$. By Proposition 3.14 and Theorem 5.1, we know that $F_{a\sharp}\tilde{m} = e^{-(N-1)a}\tilde{m}$ and $F_a^{-1} = F_{-a}$. Thus,

$$\begin{aligned}\tilde{m}(A_a^{a+1}) &= e^{(N-1)a}\tilde{m}({F_a}^{-1}(A_a^{a+1})) = e^{(N-1)a}\tilde{m}(A_0^1)\\ &= m'(A)\int_0^1 e^{(N-1)a} e^{(N-1)s} ds = m'(A)\int_a^{a+1} e^{(N-1)s}\, \mathrm{d}s.\end{aligned}$$

To prove that equation (5.17) holds for $a = 0$ and $b = 1/2$, we use again Proposition 3.14 and Theorem 5.1. Thus,

$$\tilde{m}(A_0^1) = \tilde{m}(A_0^{1/2}) + \tilde{m}(A_{1/2}^1) = (1 + e^{\frac{1}{2}(N-1)})\tilde{m}(A_0^{1/2}).$$

With some algebra we conclude

$$\tilde{m}(A_0^{1/2}) = (1 + e^{\frac{1}{2}(N-1)})^{-1}\tilde{m}(A_0^1) = m'(A)\int_0^{1/2} e^{(N-1)s}\, \mathrm{d}s.$$

Continuing in this way, equation (5.17) holds for $a \in \mathbb{R}$ and $b = a + k/2^n$ with $k, n \in \mathbb{N}$. Then an approximation argument concludes the proof. □

**Proposition 5.12.** *Let $h \in \mathrm{Lip}(\mathbb{R})$ with compact support and identically 1 on $[a,b]$. Let $f \in L^2(\tilde{X})$ be of the form $f(x) = g(\pi(x))h(u(x))$ for some $g \in L^2(m')$. If $f \in W^{1,2}(\tilde{X})$ then $g \in W^{1,2}(X')$ and for $\tilde{m}$-ae. $x \in u^{-1}[a,b]$ we have*

$$|\nabla g|_{X'}(\pi(x)) \le e^{u(x)} |\nabla f|_{\tilde{X}}(x). \tag{5.18}$$

*Proof.* Let $\boldsymbol{\pi}'$ be a test plan on $X'$. Define

$$T : X' \times [a',b'] \to \tilde{X}, \quad \hat{T} : C([0,1],X') \times [a',b'] \to C([0,1],\tilde{X}),$$

and $\boldsymbol{\pi} \in \mathcal{P}(C([0,1],\tilde{X}))$ given by $T(x,s) = F_s(\iota(x))$, $\hat{T}(\gamma,s)_t = T(\gamma_t,s)$ and

$$\boldsymbol{\pi} = \hat{T}_\sharp(\boldsymbol{\pi}' \times (b'-a')^{-1}\mathcal{L}^1_{[a',b']}),$$

with $[a',b'] \subset [a,b]$.

We claim that $\boldsymbol{\pi}$ is a test plan on $\tilde{X}$. That is, $\boldsymbol{\pi}$ has finite kinetic energy and bounded compression. Finite kinetic energy for $\boldsymbol{\pi}$ follows from the fact that $\boldsymbol{\pi}'$ is a test plan and so it has finite kinetic energy, and that $\mathrm{ms}_t(\hat{T}(\gamma,s)) \le \mathrm{Lip}(F_s)|\dot{\gamma}_t|$ (where $\mathrm{ms}_t(\hat{T}(\gamma,s))$ denotes the metric speed of $\hat{T}(\gamma,s)$), by Theorem 5.1 and Theorem 5.3. Set $M = \max\{\mathrm{Lip}(F_s) \,|\, s \in [a',b']\}$, then,

$$\begin{aligned}\frac{1}{2}\int\int_0^1 |\dot{\gamma}_t|^2 \,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma) =& \frac{1}{2}\int\int_0^1\int_{a'}^{b'} (b'-a')^{-1}\mathrm{ms}_t(\hat{T}(\gamma,s))^2\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}'(\gamma)\\ \le& M\frac{1}{2}\int\int_0^1 |\dot{\gamma}_t|^2\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}'(\gamma) < \infty.\end{aligned}$$

To show that $\boldsymbol{\pi}$ has bounded compression it is enough to consider sets of the form

$$A_c^d = \{x \in \tilde{X} | u(x) \in [c,d],\, \pi(x) \in A\},$$

for some Borel set $A \subset X'$. Thus, using that $\boldsymbol{\pi}'$ has bounded compression, and equation (5.17),

$$\begin{aligned}e_{t\sharp}\boldsymbol{\pi}(A_c^d) =& \boldsymbol{\pi}' \times (b'-a')^{-1}\mathcal{L}^1_{[a',b']}((e_t \circ \hat{T})^{-1}(A_c^d))\\ =& \boldsymbol{\pi}'(e_t^{-1}(A))(b'-a')^{-1}\mathcal{L}^1_{[a',b]}([c,d])\\ \le& Cm'(A).\end{aligned}$$

The definition of $\boldsymbol{\pi}$ and $f$ yield,

$$\int |f(\gamma_1) - f(\gamma_0)|\,\mathrm{d}\boldsymbol{\pi}(\gamma) = \int |g(\gamma_1) - g(\gamma_0)|\,\mathrm{d}\boldsymbol{\pi}'(\gamma). \tag{5.19}$$

Now, the definition of $|\nabla f|_{\tilde{X}}$, and $\boldsymbol{\pi}$, imply the following estimates:

$$\begin{aligned}\int |f(\gamma_1) - f(\gamma_0)|\,\mathrm{d}\boldsymbol{\pi} \le& \int\int_0^1 |\nabla f|_{\tilde{X}}(\gamma_t)|\dot{\gamma}|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}\\ \le& \int\int_0^1\int_a^{b'} (b'-a')^{-1}|\nabla f|_{\tilde{X}}(\hat{T}(\gamma,s)_t)\mathrm{ms}_t(\hat{T}(\gamma,s))\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}'\\ \le& \int\int_0^1 (b'-a')^{-1}\int_a^b \mathrm{Lip}(F_s)|\nabla f|_{\tilde{X}}(\hat{T}(\gamma,s)_t)|\dot{\gamma}_t|\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}'.\end{aligned} \tag{5.20}$$

In the previous inequalities we used Theorem 5.1 to bound $\mathrm{ms}_t(\hat{T}(\gamma,s)) \le \mathrm{Lip}(F_s)|\dot{\gamma}_t|$, and $\mathrm{Lip}(F_s) \le \max\{e^s, 1\}$.

Combining equality (5.19), inequality (5.20), and that $\boldsymbol{\pi}'$ was chosen arbitrarily, we conclude that $g \in W^{1,2}(X')$, and that for $m'$-a.e. $x'$,

$$|\nabla g|_{X'}(x') \le (b'-a')^{-1}\int_{a'}^{b'} \mathrm{Lip}(F_s)|\nabla f|_{\tilde{X}}(T(x',s))\,\mathrm{d}s.$$

This proves $g \in W^{1,2}(X')$, and gives the right estimate for $|\nabla g|_{X'}$ if $0 \le a < b$, as for $s \in [a,b]$ the inequality $\mathrm{Lip}(F_s) \le e^s$ holds.

If $a < b \le 0$ write $f = \tilde{f} \circ F_t$, here $t \ge -a$. Then, $\tilde{f}(x) = g(\pi(x))$ for $x \in u^{-1}[a+t, b+t]$ and $0 \le a+t \le b+t$. We note that $\langle \nabla f, \nabla u \rangle = 0$. Then by the definition of Regular Lagrangian Flow, Definition 3.11 (iii), and Corollary 3.10, the equality $|\nabla f|^2_{\tilde{X}} = \mathrm{Hess}[u](\nabla f, \nabla f)$ holds $\tilde{m}$ a.e. in $u^{-1}[a,b]$. In combination with Theorem 4.8 we have thus found,

$$\langle \nabla f, \nabla f \rangle = e^{2t}\mathrm{Hess}[u](\nabla \tilde{f}, \nabla \tilde{f}) \circ F_t + \left\langle \nabla \tilde{f}, \nabla u \right\rangle^2 \circ F_t = e^{2t} \left\langle \nabla \tilde{f}, \nabla \tilde{f} \right\rangle \circ F_t.$$

The previous equality holds for $0 \le a \le b$, and so we conclude that $m'$-a.e. $x'$,

$$|\nabla f|(x) = e^t |\nabla \tilde{f}|(F_t(x)) \ge e^t e^{-(u(x)+t)} |\nabla g|_{X'}(\pi(x)).$$

□

**Theorem 5.13.** *Assume $h \in \mathrm{Lip}(\mathbb{R})$ has compact support and is identically 1 on $[a,b]$. Let $f \in L^2(\tilde{X})$ be of the form $f(x) = g(\pi(x))h(u(x))$, for some $g \in L^2(X', d', m')$. Then $g \in W^{1,2}(X', d', m')$ if and only if $f \in W^{1,2}(\tilde{X}, d, \tilde{m})$, and for $\tilde{m}$-a.e. $x \in u^{-1}[a,b]$ we have*

$$|\nabla f|_{\tilde{X}}(x) = e^{-u(x)} |\nabla g|_{X'}(\pi(x)). \tag{5.21}$$

*Proof.* By Proposition 5.12 it is enough to prove that if $g \in W^{1,2}(X', d', m')$ then $f \in W^{1,2}(\tilde{X}, d, \tilde{m})$ and $|\nabla f|_{\tilde{X}}(x) \le e^{-u(x)} |\nabla g|_{X'}(\pi(x))$ holds for $\tilde{m}$-a.e. $x \in u^{-1}[a,b]$. Let $G : \tilde{X} \to \mathbb{R}$ be given by

$$G(x) = e^{-u(x)} |Dg|_{X'}(\pi(x)) h(u(x)) + g(\pi(x)) |h'|(u(x)). \tag{5.22}$$

We will show that $G$ is a weak upper gradient of $f$. Notice that $G$ is in $L^2(\tilde{m})$ and that $G(x) = e^{-u(x)} |\nabla g|_{X'}(\pi(x))$ for $x \in u^{-1}[a,b]$.

For $x \in \mathsf{supp} f$ following the same arguments of the proof of Theorem 4.19 in [AGS114] (this is the property of weak gradient being a local object) it is sufficient to check the definition of weak upper gradients for $f$ using test plans $\boldsymbol{\pi}$ such that for each $t \in [0,1]$,

$$\gamma_t \subset A(x,r) = \{ y \in \tilde{X} | u(y) \in [u(x) - r, u(x) + r],\ d'(\pi(x), \pi(y)) \le r \}$$

and $\gamma \in \mathsf{supp}(\boldsymbol{\pi})$. Fix such $\boldsymbol{\pi}$. By (2) in Proposition 5.4, the map $\hat{\pi} : C([0,1], \overline{A(x,r)}) \to C([0,1], X')$ given by $\hat{\pi}(\gamma) = \pi \circ \gamma$ is Lipschitz. Arguing as in the proof of Proposition 5.12, we conclude that $\boldsymbol{\pi}' = \hat{\pi}_\sharp \boldsymbol{\pi}$ is a test plan on $X'$.

Since $g \in W^{1,2}(X')$ and the way $\boldsymbol{\pi}'$ was defined, by Proposition 2.1, for $\boldsymbol{\pi}$-a.e. $\gamma$ the map $t \mapsto g(\hat{\pi}(\gamma)_t)$ is equal a.e. on $[0,1]$ and $\{0,1\}$ to an absolutely continuous map $g_{\hat{\pi}(\gamma)}$ such that for a.e. $t \in [0,1]$

$$|g'_{\hat{\pi}(\gamma)}|(t) \le |\nabla g|_{X'}(\hat{\pi}(\gamma)_t) |\dot{\hat{\pi}(\gamma)}_t| \le e^{-u(\gamma_t)} |\nabla g|_{X'}(\pi(\gamma_t) |\dot{\gamma}_t|. \tag{5.23}$$

In the last inequality we used (1) from Proposition 5.4.

For any absolutely continuous curve $\gamma$ in $\tilde{X}$, $h \circ u \circ \gamma$ is absolutely continuous with derivative $|(h \circ u \circ \gamma)'| \le |h'|(u \circ \gamma)|\dot{\gamma}|$. Hence, for $\boldsymbol{\pi}$-a.e. $\gamma$ the map $t \mapsto f(\gamma_t) = g(\pi(\gamma_t))h(u(\gamma_t))$ is equal a.e. on $[0,1]$ and $\{0,1\}$ to the absolutely continuous map $f_\gamma(t) = g_{\hat{\pi}(\gamma)}(t) h(u(\gamma_t))$ such that for a.e. $t \in [0,1]$ satisfies

$$|f'_\gamma|(t) \le \left( e^{-u(\gamma_t)} |\nabla g|_{X'}(\pi(\gamma_t)) h(u(\gamma_t)) + g(\pi(\gamma_t)) |h'|(u \circ \gamma_t) \right) |\dot{\gamma}_t|. \tag{5.24}$$

This proves that for $\tilde{m}$-a.e. $x \in u^{-1}[a,b]$

$$|\nabla f|_{\tilde{X}}(x) \le e^{-u(x)} |\nabla g|_{X'}(\pi(x)).$$

□

**Proposition 5.14.** *Under the assumptions of Definition 5.2 and Definition 5.10, the space $(X', d', m')$ is infinitesimally Hilbertian, almost everywhere locally doubling and a measured-length space. Hence, it satisfies the Sobolev to Lipschitz property.*

For the definition of locally doubling and measured-length space see Definition 2.13 and Definition 2.12 in Subsection 2.6.

*Proof.* By Theorem 5.13 and the infinitesimally Hilbertianity of $(\tilde{X}, d, \tilde{m})$ it is easy to see that $(X', d', m')$ is infinitesimally Hilbertian. We now prove that $(X', d', m')$ is almost everywhere locally doubling.

We will show that there exists a Borel set B with $m'$-negligible complement such that for every $x' \in B$ there exist an open set $U$ containing $x'$ and constants $C, R > 0$ such that if $r \in (0, R)$ and $y \in U$, then $m'(B'_{2r}(y)) \leq Cm'(B_r(y))$.

Given $x' \in X'$ and $R > 0$, for $r < R/2$ define

$$A(x', r) = \{x \in \tilde{X} | u(x) \in [-r, r], \, d'(x', \pi(x)) < r\} \subset B(\iota(x'), 2r). \tag{5.25}$$

By (2) in Proposition 5.28, there exists a Lipschitz constant $L > 1$ for $\pi : \overline{B_R(\iota(x'))} \to X'$. Notice that $B_{r/L}(\iota(x')) \subset B_{2r}(\iota(x'))$ because $L > 1$. Since $u$ is 1-Lipschitz and by the triangle inequality, if $y \in B_{2r}(\iota(x'))$ then $|u(y)| \leq u(\iota(x'))) + \tilde{d}(y, \iota(x'))) \leq 2r$. Thus, $B_{2r}(\iota(x')) \subset \overline{B_R(\iota(x'))}$. This shows $d'(\pi(y), x') \leq r$ for any $y \in B_{r/L}(\iota(x'))$. Since $u$ is 1-Lipschitz it follows that $u(y) \leq r/L < r$ for any $y \in B_{r/L}(\iota(x'))$. Thus,

$$B_{r/L}(\iota(x')) \subset A(x', r). \tag{5.26}$$

Equation (5.17) gives

$$\tilde{m}(A(x', r)) = m'(B'_r(x')) \int_{-r}^{r} e^{(N-1)s} \, \mathrm{d}s. \tag{5.27}$$

Let $c(r) = \int_{-r}^{r} e^{(N-1)s} ds$. Starting with equation (5.27), then using equation (5.26), that $(\tilde{X}, \tilde{d}, \tilde{m})$ is locally doubling with constant $C_{\tilde{X}}$ [Vil], equation (5.25) and equation (5.27) once more, we estimate

$$\begin{aligned} m'(B'_r(x')) &= c^{-1}(r)\tilde{m}(A(x, r)) \geq c^{-1}(r)\tilde{m}(B_{r/L}(\iota(x'))) \\ &\geq C_{\tilde{X}} c^{-1}(r)\tilde{m}(B_{r/2L}(\iota(x'))) \geq C_{\tilde{X}} c^{-1}(r)\tilde{m}(A(x, r/4L)) \\ &= C_{\tilde{X}} c^{-1}(r)c(r/4L)m'(B'_{r/4L}(x')). \end{aligned}$$

That is, $m'(B'_r(x')) \geq Cm'(B'_{r/4L}(x'))$, for $C = C_{\tilde{X}} c^{-1}(r)c(r/4L)$. Therefore $(X', d', m')$ is almost everywhere locally doubling.

Now we show that $(X', d', m')$ is a measured-length space. Let $x_0, x_1 \in X'$, define $\varepsilon = 1$ and take $\varepsilon_0, \varepsilon_1 \in (0, \varepsilon]$. Let $\tilde{\gamma}$ be a geodesic in $X'$ from $x_0$ to $x_1$, and $x_i = \tilde{\gamma}_{i/n}$ for $i = 0, 1, ..., n$, $n = \lfloor 1 + 1/\sqrt{\varepsilon'} \rfloor$ and $\varepsilon' = \max\{\varepsilon_0, \varepsilon_1\}$.

Let $\varepsilon_i = \varepsilon_0 + \frac{i}{n}(\varepsilon_1 - \varepsilon_0)$, and define $\mu_i^{\varepsilon_0,\varepsilon_1} = (\tilde{m}(A(x_i, \varepsilon_i))^{-1} \tilde{m}|_{A(x_i, \varepsilon_i)}$. Here $A(x_i, \varepsilon_i)$ is defined by equation (5.25). From equation (5.17),

$$\pi_\sharp \mu_i^{\varepsilon_0,\varepsilon_1} = (m'(B'_{\varepsilon_i}(x_i))^{-1} m'|_{B'_{\varepsilon_i}(x_i)}. \tag{5.28}$$

Let $\boldsymbol{\pi}_i^{\varepsilon_0,\varepsilon_1}$ be the only optimal geodesic plan from $\mu_i^{\varepsilon_0,\varepsilon_1}$ to $\mu_{i+1}^{\varepsilon_0,\varepsilon_1}$ ([GRS16]). By the triangle inequality and our choices of $x_i$ and $\varepsilon_i$, for $y_i \in A(x_i, \varepsilon_i)$ we have

$$\tilde{d}(y_i, y_{i+1}) \leq 2\varepsilon_i + d'(x_i, x_{i+1}) + 2\varepsilon_{i+1} \leq 4\varepsilon' + \frac{1}{n} d'(x_i, x_{i+1}).$$

It follows that

$$\int \int_0^1 |\dot{\gamma}_t| \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}_i^{\varepsilon_0,\varepsilon_1}(\gamma) = W_2^2(\mu_i^{\varepsilon_0,\varepsilon_1}, \mu_{i+1}^{\varepsilon_0,\varepsilon_1}) \leq (4\varepsilon' + \frac{1}{n} d'(x_i, x_{i+1}))^2. \tag{5.29}$$

From the definition of $\varepsilon$ and $\varepsilon'$ for $\boldsymbol{\pi}_i^{\varepsilon_0,\varepsilon_1}$ a.e. $\gamma$, $u(\gamma_t) \subset [-\varepsilon', \varepsilon'] \subset [-1, 1]$.

Gluing the plans $\boldsymbol{\pi}_i^{\varepsilon_0,\varepsilon_1}$ we construct a plan $\boldsymbol{\pi}^{\varepsilon_0,\varepsilon_1}$ that satisfies

(i)

$$(Restr_{\frac{i}{n}}^{\frac{i+1}{n}})_\sharp \boldsymbol{\pi}^{\varepsilon_0,\varepsilon_1} = \boldsymbol{\pi}_i^{\varepsilon_0,\varepsilon_1}, \quad i = 0,1,...,n.$$

(ii)

$$\int\int_0^1 |\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}^{\varepsilon_0,\varepsilon_1}(\gamma) = n\sum_{i=0}^{n-1}\int\int_0^1 |\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}_i^{\varepsilon_0,\varepsilon_1}(\gamma) \tag{5.30}$$

$$\leq n^2(4\varepsilon' + \frac{1}{n}d'(x_i,x_{i+1}))^2 \leq (8\sqrt{\varepsilon'} + d'(x_i,x_{i+1}))^2. \tag{5.31}$$

Note that $n = \lfloor 1 + 1/\sqrt{\varepsilon'}\rfloor$ and $\varepsilon' < 1$ implies $4n\varepsilon' \leq 8\sqrt{\varepsilon'}$. Then using (5.29) and taking into account the rescaling factor we get the previous inequality.

(iii) $\boldsymbol{\pi}^{\varepsilon_0,\varepsilon_1}$ a.e. $\gamma$,

$$u(\gamma_t) \subset [-\varepsilon',\varepsilon'] \subset [-1,1]. \tag{5.32}$$

Define:

$$\bar{\boldsymbol{\pi}}^{\varepsilon_0,\varepsilon_1} := \pi_\sharp \boldsymbol{\pi}^{\varepsilon_0,\varepsilon_1} \tag{5.33}$$

From (5.28) we get

$$e_{i\sharp}\bar{\boldsymbol{\pi}}^{\varepsilon_0,\varepsilon_1} = \frac{1}{m'(B'_{\varepsilon_i}(x_i))} m'|_{B'_{\varepsilon_i}(x_i)} \quad i = 1,2.$$

By (1) in Proposition (5.28) we know that

$$\int\int_0^1 |\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\bar{\boldsymbol{\pi}}^{\varepsilon_0,\varepsilon_1}(\gamma) \leq \int\int_0^1 e^{u(\gamma_t)}|\dot\gamma_t|\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}^{\varepsilon_0,\varepsilon_1}(\gamma).$$

From equations (5.30), (5.32), and $\varepsilon' = \max\{\varepsilon_0,\varepsilon_1\}$ it follows that

$$\limsup_{\varepsilon_0,\varepsilon_1\downarrow 0}\int\int_0^1 |\dot\gamma_t| dt d\bar{\boldsymbol{\pi}}^{\varepsilon_0,\varepsilon_1}(\gamma) \leq \limsup_{\varepsilon_0,\varepsilon_1\downarrow 0} e^{\varepsilon'}(8\sqrt{\varepsilon'} + d'(x_i,x_{i+1}))^2 = d'(x_i,x_{i+1})^2.$$

□

## 6. $(\tilde X,\tilde d,\tilde m)$ is isomorphic to $(X'_\omega,d'_\omega,m'_\omega)$

Let $X'_w$ denote the warped product of $(X',d',m')$ with warping functions $w_{d'},w_{m'}:\mathbb{R}\to\mathbb{R}$ given by $w_{m'}(t) = e^{(N-1)t}$ and $w_{d'}(t) = e^t$. In subsection 6.1 we prove that there is a locally biLipschitz map from $(\tilde X,\tilde d,\tilde m)$ to $(X'_\omega,d'_\omega,m'_\omega)$ that preserves the measures. Then we show that the spaces are isomorphic by showing that their $W^{1,2}$ spaces are isomorphic.

### 6.1. $\tilde X$ is measure preserving homeomorphic to a warping product.

Here we prove that there is a locally biLipschitz map from $(\tilde X,\tilde d,\tilde m)$ to $(X'_\omega,d'_\omega,m'_\omega)$ that preserves the measures.

Proceeding as in Proposition 5.4 we obtain the following.

**Proposition 6.1.** *For all $(x'_0,t_0)\in X'_w$ and $r>0$,*

$$d'(x',y') \leq e^{-t_0+3r} d'_w((x',t),(y',s)),$$

*for all $(x',t),(y',s)\in B_r(x_0,t_0)$.*

**Proposition 6.2.** *Let $T : X'_w \to \mathsf{supp}(\tilde{m})$ and $S : \mathsf{supp}(\tilde{m}) \to X'_w$ be defined by*

$$T(x', t) = F_t(\iota(x'))$$

*and*

$$S(x) = (\pi(x), u(x)).$$

*Then $T$ and $S$ are inverse of each other, $S$ is 2-Lipschitz and $T$ is locally Lipschitz.*

*Proof.* It is clear that $T \circ S = \mathrm{Id}_{\mathsf{supp}(\tilde{m})}$ and $S \circ T = \mathrm{Id}_{X'_\omega}$. Let us prove that $T$ is locally Lipschitz. Let $(x'_0, t_0) \in X'_w$ and $r > 0$. Consider $(x'_1, t_1), (x'_2, t_2) \in B_r(x_0, t_0)$. By the triangle inequality, Theorem 5.1, and Proposition 6.1, we obtain

$$\begin{aligned}\tilde{d}(T(x'_1, t_1), T(x'_2, t_2)) &= \tilde{d}(F_{t_1}(\iota(x'_1)), F_{t_2}(\iota(x'_2)))\\ &\leq \tilde{d}(F_{t_1}(\iota(x'_1)), F_{t_1}(\iota(x'_2)) + \tilde{d}(F_{t_1}(\iota(x'_2)), F_{t_2}(\iota(x'_2))\\ &\leq \mathrm{Lip}(F_{t_1}) d'(x'_1, x'_2) + |t_1 - t_2|\\ &\leq \mathrm{Lip}(F_{t_1}) e^{-t_0+3r} d'_w((x', t_1), (y', t_2)) + d'_w((x', t_1), (y', t_2)).\end{aligned}$$

It follows that $T$ is locally Lipschitz.

Now we prove that $S$ is Lipschitz. Let $\gamma : [0,1] \to \tilde{X}$ be a geodesic from $T(x'_1, t_1)$ to $T(x'_2, t_2)$. As $u : X \to \mathbb{R}$ is 1-Lipschitz, the curve $u \circ \gamma$ is absolutely continuous and $|\dot{u}(\gamma_t)| \leq |\dot{\gamma}_t|$. From Proposition 5.4 (1), we know that $e^{u(\gamma_t)} |\dot{\tilde{\gamma}}_t| \leq |\dot{\gamma}_t|$, here $\tilde{\gamma} = \pi \circ \gamma$. Thus,

$$\begin{aligned}2\tilde{d}(T(x'_1, t_1), T(x'_2, t_2)) =& 2\int |\dot{\gamma}_t| dt\\ \geq& \int e^{u(\gamma_t)} |\dot{\tilde{\gamma}}_t| + |\dot{u}(\gamma_t)|\\ \geq& \int \sqrt{e^{2u(\gamma_t)} |\dot{\tilde{\gamma}}_t|^2 + |\dot{u}(\gamma_t)|^2}\\ \geq& d'_w\big((x'_1, t_1), (x'_2, t_2)\big).\end{aligned}$$

□

Applying Lemma 5.11 we see that $T$ and $S$ are measure preserving:

**Proposition 6.3** (T and S are measure preserving)**.** *Let $T : X'_w \to \mathsf{supp}(\tilde{m})$ and $S : (\mathsf{supp}(\tilde{m})) \to X'_w$ be given by $T(x', t) = F_t(\iota(x'))$ and $S(x) = (\pi(x), u(x))$. Then $T_\sharp(m'_\omega) = \tilde{m}$ and $S_\sharp \tilde{m} = m'_\omega$.*

*Proof.* As $S$ and $T$ are inverses of each other, it is sufficient to prove that $S_\sharp \tilde{m} = m'_w$. Given that both $m'_w$ and $S_\sharp \tilde{m}$ are Borel measures defined on $X'_w$ which has positive warping functions, it is enough to prove that for any Borel set $E \subset X'$ and any interval $I = [a, b] \subset \mathbb{R}$ the following equality holds

$$S_\sharp \tilde{m}(E \times I) = m'_w(E \times I).$$

Equation (5.17) implies,

$$S_\sharp \tilde{m}(E \times I) = \tilde{m}(S^{-1}(E \times I)) = \tilde{m}(E^b_a) = m'(E) \int_a^b e^{(N-1)s} ds.$$

By the definition of $m'_w$,

$$m'_w(E \times I) = \int_I \left( \int_{X'} \chi_E(x) w_{m'}(t)\, dm'(x) \right) dt = m'(E) \int_a^b w_{m'}(t)\, dt = m'(E) \int_a^b e^{(N-1)t} dt.$$

□

The following proposition will be helpful in the next subsection.

**Proposition 6.4.** *Let $h \in S^2_{\mathrm{loc}}(w_{m'}\mathbb{R})$ and define $f : \tilde{X} \to \mathbb{R}$ by $f := h \circ u$. Then $f \in S^2_{\mathrm{loc}}(\tilde{X})$ and*

$$|\nabla f|_{\tilde{X}}(x) = |\nabla h|_{w_{m'}\mathbb{R}}(u(x)), \qquad \tilde{m} - a.e.\, x \in \tilde{X}.$$

*Proof.* The proof follows the same strategy as that of [Gig4, Proposition 5.29].

Let $R > 0$ and $\chi : \mathbb{R} \to [0,1]$ be a Lipschitz function which is compactly supported and identically 1 on $[-R, R]$. Firstly we observe that, since the claim is a local statement, to provide a proof it is enough to show that, if $h \in W^{1,2}(w_{m'}\mathbb{R})$ then $f(\chi \circ u) \in W^{1,2}(\tilde{X})$ and that

$$|\nabla f|_{\tilde{X}}(x) = |\nabla h|_{w_{m'}\mathbb{R}}(u(x)) \tag{6.1}$$

is valid for $\tilde{m}$-a.e. $x \in u^{-1}([-R, R])$.

Let $h_n$ be a sequence of Lipschitz functions on $w_{m'}\mathbb{R}$ such that $h_n \to h$ and $\mathrm{lip}_{w_{m'}\mathbb{R}} h_n \to |\nabla h|_{w_{m'}\mathbb{R}}$ in $L^2(w_{m'}\mathbb{R})$. Such a sequence exists by [Gig4, Theorem 4.3]. Now, we consider the functions $f_n := (h_n \circ u)(\chi \circ u)$. Proposition 6.3 implies that $f_n \to f(\chi \circ u)$ in $L^2(\tilde{X})$. Moreover, since $u$ is 1-Lipschitz, for $x \in u^{-1}([-R, R])$ and $n \in \mathbb{N}$,

$$\mathrm{lip}_{\tilde{X}}(f_n)(x) = \limsup_{y \to x} \frac{|f_n(y) - f_n(x)|}{\tilde{d}(x,y)} \leq \limsup_{y \to x} \frac{|h_n \circ u(y) - h_n \circ u(x)|}{|u(y) - u(x)|} = \mathrm{lip}_{w_{m'}\mathbb{R}} h_n \circ u(x). \tag{6.2}$$

From the previous inequality, the Leibniz rule [Gig4, (3.9)] and the convergence of $h_n$ we conclude that $\mathrm{lip}_{\tilde{X}}(f_n)$ is bounded in $L^2(\tilde{X})$. Therefore, passing to a (non-relabeled) subsequence if necessary, we can assume that there exists a Borel function $G : \tilde{X} \to \mathbb{R}$ such that $\mathrm{lip}_{\tilde{X}}(f_n) \to G$ weakly in $L^2(X)$.

The lower semicontinuity of minimal weak upper gradients (see the paragraph after [Gig4, Definition 3.8]) and the convergence of $f_n$ to $f(\chi \circ u)$ in $L^2(\tilde{X})$ imply that $|\nabla f(\chi \circ u)|_{\tilde{X}} \leq G$ $\tilde{m}$-a.e.. Moreover by the locality of minimal weak upper gradients [Gig4, (3.6)], $|\nabla f|_{\tilde{X}} = |\nabla f(\chi \circ u)|_{\tilde{X}}$, $\tilde{m}$-a.e. on $u^{-1}([-R, R])$. Now, passing to the limit in 6.2, we obtain the $\leq$ inequality in 6.1.

We now proceed to prove the other inequality in 6.1 by showing the following result, and applying it to $t = u(x')$: Let $f \in W^{1,2}(\tilde{X})$ and for $x' \in X'$ let $f^{(x')} : w_{m'}\mathbb{R} \to \mathbb{R}$ be given by $f^{(x')}(t) := f(T(x', t))$. Then for $m'$-a.e. $x'$, $f^{(x')} \in S^2_{loc}(w_{m'}\mathbb{R})$ and

$$|\nabla f^{(x')}|_{w_{m'}\mathbb{R}}(t) \leq |\nabla f|_{\tilde{X}}(T(x', t)), \quad m'_w - \text{a.e. } (x', t) \in X'_w.$$

Using that for any $x, y \in \mathsf{supp}(\tilde{m})$ with $\pi(x) = \pi(y)$ we have $|u(x) - u(y)| = \tilde{d}(x, y)$, we observe the following inequality

$$\begin{aligned} \mathrm{lip}_{\tilde{X}} f(x) &= \limsup_{y \to x} \frac{|f(x) - f(y)|}{\tilde{d}(x,y)} \geq \limsup_{\substack{y \to x \\ \pi(y) = \pi(x)}} \frac{|f(x) - f(y)|}{\tilde{d}(x,y)} \\ &= \limsup_{\substack{y \to x \\ \pi(y) = \pi(x)}} \frac{\left|f^{(\pi(x))}(u(x)) - f^{(\pi(x))}(u(y))\right|}{|u(x) - u(y)|} = \mathrm{lip}_{w_{m'}\mathbb{R}} f^{(\pi(x))}(u(x)). \end{aligned} \tag{6.3}$$

By [Gig4, Theorem 4.3], there exists a sequence $(f_n) \subset L^2(\tilde{X})$ of Lipschitz functions such that $f_n \to f$ and $\mathrm{lip}_{\tilde{X}}(f_n) \to |\nabla f|_{\tilde{X}}$ in $L^2(\tilde{X})$. Passing to a subsequence if necessary we can further assume that $\sum_n \|f_n - f_{n+1}\|_{L^2(\tilde{X})} < \infty$ and $\sum_n \|\mathrm{lip}_{\tilde{X}} f_n - |\nabla f|_{\tilde{X}}\|_{L^2(\tilde{X})} < \infty$. This, together with Proposition 6.3, implies that for $m'$-a.e. $x'$, $f_n(T(x', \cdot)) \to f(T(x', \cdot))$ and $\mathrm{lip}_{\tilde{X}}(f_n)(T(x', \cdot)) \to |\nabla f|_{\tilde{X}}(T(x', \cdot))$ in $L^2(w_{m'}\mathbb{R})$.

We now fix such an $x'$, apply inequality 6.3 to the function $f_n$ on $u^{-1}(t)$ and take the limit when $n \to \infty$. Finally, we use that $|\nabla f^{(\pi(x))}|_{w_{m'}\mathbb{R}} \leq \mathrm{lip}_{w_{m'}\mathbb{R}} f^{(\pi(x))}$ (by [Gig4, (3.8)]) and the lower semicontinuity of the minimal weak upper gradients to conclude. $\square$

6.2. $W^{1,2}(\tilde{X}, \tilde{d}, \tilde{m})$ **is isomorphic to** $W^{1,2}(X'_\omega, d'_\omega, m'_\omega)$**.** The aim of this section is to show that $(\tilde{X}, \tilde{d}, \tilde{m})$ and $(X'_\omega, d'_\omega, m'_\omega)$ are isomorphic. This will be achieved applying Proposition 2.11. Thus, we only need to show that right composition with $S$ provides an isometry from $W^{1,2}(X'_w)$ to $W^{1,2}(\tilde{X})$.

In Proposition 2.18 we showed that $\mathcal{A} \cap W^{1,2}(X'_w)$ is dense in $W^{1,2}(X'_w)$. Here

$$\mathcal{G} = \Big\{ g \in S^2_{\text{loc}}(X'_w) \mid g(x',t) = \tilde{g}(x') \text{ for some } \tilde{g} \in S^2(X') \cap L^\infty(X') \Big\},$$

$$\mathcal{H} = \Big\{ h \in S^2_{\text{loc}}(X'_w) \mid h(x',t) = \tilde{h}(t) \text{ for some } \tilde{h} \in S^2(w'_{m'}\mathbb{R}) \cap L^\infty(\mathbb{R}) \Big\}$$

$$\mathcal{A} = \text{algebra spanned by } \mathcal{G} \cup \mathcal{H} \subset S^2_{\text{loc}}(X'_w).$$

The proof that right composition with $S$ provides an isometry from $W^{1,2}(X'_w)$ to $W^{1,2}(\tilde{X})$ is divided in the following way.

0) Proposition 6.5: For every $f \in \mathcal{G}$ or $f \in \mathcal{H}$, we have that $f \circ S \in S^2_{\text{loc}}(\tilde{X})$ and $|\nabla(f \circ S)|_{\tilde{X}} = |\nabla f|_{X'_w} \circ S$ $\tilde{m}$-a.e.
1) Lemma 6.6: For every $g \in \mathcal{G}$ and $h \in \mathcal{H}$, $\langle \nabla g, \nabla h \rangle_{X'_w} = 0$ and $\langle \nabla(g \circ S), \nabla(h \circ S) \rangle_{\tilde{X}} = 0$ hold $\tilde{m} - a.e.$.
2) Proposition 6.7: Every $f \in \mathcal{A}$ satisfies $f \circ S \in S^2_{\text{loc}}(\tilde{X})$ and $|\nabla(f \circ S)|_{\tilde{X}} = |\nabla f|_{X'_w} \circ S$ $\tilde{m}$-a.e..
3) Proposition 6.8: Right composition with $S$ is a homeomorphism between $W^{1,2}(X'_w)$ and $W^{1,2}(\tilde{X})$

**Proposition 6.5.** *The maps*

$$\mathcal{G} \to S^2_{\text{loc}}(\tilde{X}), \quad g \mapsto g \circ S,$$
$$\mathcal{H} \to S^2_{\text{loc}}(\tilde{X}), \quad h \mapsto h \circ S,$$

*are well defined, and satisfy* $|\nabla(g \circ S)|_{\tilde{X}} = |\nabla g|_{X'_w} \circ S$ *and* $|\nabla(h \circ S)|_{\tilde{X}} = |\nabla h|_{X'_w} \circ S$ $\tilde{m} - a.e.$.

*Proof.* Combining Corollary 2.16 with a cut off such that $\mathsf{supp}(f) \subset u^{-1}[a,b]$ and Theorem 5.13 shows that $g \circ S \in S^2_{\text{loc}}(\tilde{X})$, and $|\nabla(g \circ S)|_{\tilde{X}} = |\nabla g|_{X'_w} \circ S$ $\tilde{m} - a.e.$.

Similarly, Corollary 2.16 and Proposition 6.4 give $h \circ S \in S^2_{\text{loc}}(\tilde{X})$ and $|\nabla(h \circ S)|_{\tilde{X}} = |\nabla h|_{X'_w} \circ S$ $\tilde{m} - a.e.$. □

**Lemma 6.6.** *(Orthogonality relations) With the same notation as above, let $g \in \mathcal{G}$ and $h \in \mathcal{H}$. Then,*

$$\langle \nabla g, \nabla h \rangle_{X'_w} = 0 \qquad m'_w - a.e., \tag{6.4}$$

*and*

$$\langle \nabla(g \circ S), \nabla(h \circ S) \rangle_{\tilde{X}} = 0 \qquad \tilde{m} - a.e.. \tag{6.5}$$

*Proof.* Let $\tilde{g} \in S^2(X') \cap L^\infty(X')$ and $\tilde{h} \in S^2(w_{m'}\mathbb{R}) \cap L^\infty(\mathbb{R})$ be such that $g(x',t) = \tilde{g}(x')$ and $h(x',t) = \tilde{h}(t)$. Corollary 2.16 implies

$$|\nabla(g+h)|^2_{X'_w}(x',t) = w_d^{-2}(t)|\nabla \tilde{g}|^2_{X'}(x') + |\nabla \tilde{h}|^2_{w_{m'}\mathbb{R}}(t) \qquad m'_w - a.e.\ (x',t).$$

Using equation (2.10) we acquire equation (6.4):

$$2\langle \nabla g, \nabla h \rangle_{X'_w} = |\nabla(g+h)|^2_{X'_w} - |\nabla g|^2_{X'_w} - |\nabla h|^2_{X'_w} = 0, \qquad m'_w - a.e..$$

To prove equation (6.5) holds, notice that the chain rule and the identity $h \circ S = \tilde{h} \circ u$ yield

$$\langle \nabla(g \circ S), \nabla(h \circ S) \rangle_{\tilde{X}} = \tilde{h}' \circ u \langle \nabla(g \circ S), \nabla u \rangle_{\tilde{X}}, \qquad \tilde{m} - a.e..$$

Then to conclude it is sufficient to show that

$$\langle \nabla(g \circ S), \nabla u \rangle_{\tilde{X}} = 0, \qquad \tilde{m} - a.e..$$

The previous equality holds because $\tilde g \circ \pi \circ F_t = \tilde g \circ \pi$, and with a truncation argument we can see that the following derivation rule is also valid for functions in $S^2_{\rm loc}(\tilde X)$:

$$\langle \nabla(g\circ S), \nabla u\rangle_{\tilde X} = \lim_{t\to 0} \frac{g\circ S\circ F_t - g\circ S}{t} = \lim_{t\to 0}\frac{\tilde g\circ \pi\circ F_t - \tilde g\circ\pi}{t} = 0, \qquad \tilde m - a.e.$$

□

**Proposition 6.7.** *With the same notation as above, every $f \in \mathcal{A}$ satisfies $f\circ S \in S^2_{\rm loc}(\tilde X, \tilde d, \tilde m)$, and*

$$|\nabla(f\circ S)|_{\tilde X} = |\nabla f|_{X'_w}\circ S, \qquad \tilde m - a.e..$$

*Proof.* Let $f \in \mathcal{A}$. Then $f$ can be written as $f = \sum_{i\in I} g_i h_i$ for some finite set $I$, $g_i \in \mathcal{G}$ and $h_i \in \mathcal{H}$, $i \in I$. By the infinitesimal Hilbertianity of $X'_w$, Proposition 5.14 and Corollary 2.17, we know that $m'_w$-a.e.

$$\begin{aligned} |\nabla f|^2_{X'_w} &= \sum_{i,j\in I} g_i g_j \langle\nabla h_i, \nabla h_j\rangle_{X'_w} + g_i h_j\langle \nabla h_i, \nabla g_j\rangle_{X'_w} \\ &\qquad\qquad + h_i g_j \langle \nabla g_i, \nabla h_j\rangle_{X'_w} + h_i h_j \langle\nabla g_i, \nabla g_j\rangle_{X'_w} \\ &= \sum_{i,j\in I} g_i g_j\langle \nabla h_i, \nabla h_j\rangle_{X'_w} + h_i h_j\langle\nabla g_i,\nabla g_j\rangle_{X'_w}, \end{aligned} \tag{6.6}$$

where we used (6.4) in the second step.

Corollary 6.5 grants

$$\begin{aligned} \langle \nabla h_i, \nabla h_j\rangle_{X'_x}\circ S &= \langle \nabla(h_i\circ S), \nabla(h_j\circ S)\rangle_{\tilde X}, \\ \langle \nabla g_i, \nabla g_j\rangle_{X'_w}\circ S &= \langle \nabla(g_i\circ S), \nabla(g_j\circ S)\rangle_{\tilde X}, \end{aligned}$$

$\tilde m$-a.e. for any $i,j \in I$. Thus writing—to shorten the notation—$\bar g_i, \bar h_i$ in place of $g_i\circ S, h_i\circ S$ respectively, from (6.6) we have

$$|\nabla f|^2_{X'_w}\circ S = \sum_{i,j\in I} \bar g_i \bar g_j \langle \nabla \bar h_i, \nabla\bar h_j\rangle_{X'_w} + \bar h_i \bar h_j\langle \nabla \bar g_i, \nabla \bar g_j\rangle_{X'_w}.$$

Using the orthogonality relation (6.5) and the fact that $\tilde X$ is infinitesimally Hilbertian we can do the same computations as in (6.6), in reverse order, to get

$$\begin{aligned} |\nabla f|^2_{X'_w}\circ S = \sum_{i,j\in I} &\bar g_i\bar g_j\langle\nabla\bar h_i,\nabla\bar h_j\rangle_{\tilde X} + \bar g_i \bar h_j\langle \nabla\bar h_i, \nabla\bar g_j\rangle_{\tilde X} \\ &+ \bar h_i \bar g_j\langle\nabla \bar g_i, \nabla\bar h_j\rangle_{\tilde X} + \bar h_i\bar h_j\langle\nabla\bar g_i,\nabla\bar g_j\rangle_{\tilde X} = |\nabla(f\circ S)|^2_{\tilde X}, \end{aligned}$$

$\tilde m$-a.e. □

Recall that in Proposition 6.2 we defined functions $S : \tilde X \to X'_w$ and $T : X'_w \to \tilde X$ inverses of each other, such that $S$ is 1 Lipschitz, and $T$ is locally Lipschitz.

**Proposition 6.8.** *With the same notation as above the following holds*

*(i) If $f \in W^{1,2}(X'_w)$ then $f\circ S \in W^{1,2}(\tilde X)$ and*

$$\| |\nabla(f\circ S)| \|_{L^2(\tilde X)} \le \| |\nabla f| \|_{L^2(X'_w)}. \tag{6.7}$$

*(ii) If $f\circ S \in W^{1,2}(\tilde X)$ then $f \in S^2_{loc}(X'_w)$ and each $x \in \tilde X$ has a neighbourhood $\Omega_x$ such that*

$$L^{-1}\| |\nabla f| \|_{L^2(S(\Omega_x))} \le \| |\nabla(f\circ S)| \|_{L^2(\Omega_x)}. \tag{6.8}$$

*Here $L = \mathrm{Lip}^{-1}(T^{-1}(x))$.*

*Proof.* Note that $(\tilde{X}, d, \tilde{m})$ and $X'_w = (X' \times_w \mathbb{R}, d'_w, m'_w)$ satisfy the hypotheses of Lemma 2.10. That is, they satisfy the Sobolev to Lipschitz property, see the paragraph after [Gig4, Definition 4.9] and Proposition 5.14. Moreover, $T_\sharp(m'_w) = \tilde{m}$ and $S_\sharp\tilde{m} = m'_w$ by Proposition 6.3. To prove the first inequality recall that by Proposition 6.2 the map $S$ is 1-Lipschitz. Then equation 6.7 follows by Lemma 2.10.

To prove the second inequality, choose $\Omega_x = T(B_r(T^{-1}(x)))$ and rescale $d'_w$ by $L$. Then we get $\mathrm{Lip}(T|_{B_r(T^{-1}(x))}) \leq 1$. With this rescaling the corresponding gradient part of the Sobolev norm is scaled by $\frac{1}{L}$. The result follows by Lemma 2.10. □

The main theorem of this section follows.

**Theorem 6.9** ($(\tilde{X}, \tilde{d}, \tilde{m})$ is isomorphic to $(X'_\omega, d'_\omega, m'_\omega)$)**.** *The maps $T$ and $S$ given in Proposition 6.2 are isomorphisms of metric measure spaces.*

*Proof.* By the paragraph after [Gig4, Definition 4.9] $\tilde{X}$ has the Sobolev to Lipschitz property and by Proposition 5.14 and Theorem 2.14 $X'_w$ also has the Sobolev to Lipschitz property. Hence, it is enough to apply Proposition 2.11. By Proposition 6.3 we know that $T$ and $S$ are measure preserving. It remains to prove that $f \in W^{1,2}(X'_w)$ if and only if $f \circ S \in W^{1,2}(\tilde{X})$ and that

$$\| |\nabla(f \circ S)|_{\tilde{X}} \|_{L^2(\tilde{X})} = \| |\nabla f|_{X'_w} \|_{L^2(X'_w)}. \tag{6.9}$$

Let $f \in W^{1,2}(X'_w)$. By Proposition 2.18 there exists a sequence $f_n \subset \mathcal{A} \cap W^{1,2}(X'_w)$ converging to $f$ in $W^{1,2}(X'_w)$. Then the the first inequality in Proposition 6.8 implies that both $f_n \circ S$, and $f \circ S$ are in $W^{1,2}(\tilde{X})$, with $f_n \circ S$ converging to $f \circ S$ in $W^{1,2}(\tilde{X})$. From Proposition 6.7 we get,

$$|\nabla f_n|_{X'_w} \circ S = |\nabla(f_n \circ S)|_{\tilde{X}}, \qquad \tilde{m} - a.e..$$

Taking the $L^2$ norm of the functions in the previous equality and taking the limit as $n \to \infty$ we get (6.9).

If $f : X'_w \to \mathbb{R}$ is such that $f \circ S \in W^{1,2}(\tilde{X}_{a,b})$, the second inequality in Proposition 6.8 implies that each $x \in \tilde{X}$ has neighbourhood $\Omega_x$ on which the above argument can be repeated. Thus

$$|\nabla f_n|_{S(\Omega_x)} \circ S = |\nabla(f_n \circ S)|_{\Omega_x}, \qquad \tilde{m} - ae.$$

By the locality of the weak upper gradient we have equality in the whole space and therefore $f \in W^{1,2}(X'_w)$. □

## 7. $\mathsf{RCD}^*$-condition for $X'$

Recall that $X'$ is an infinitesimally Hilbertian space satisfying the Sobolev to Lipschitz property. Under these conditions, [EKS, Theorem 7] implies that the validity of the Bochner inequality is equivalent to the $\mathsf{RCD}^*$ condition. Hence, to prove that $X'$ is an $\mathsf{RCD}^*(-(N-1), N)$-space, we will show that the weak Bochner inequality holds.

We begin with the following technical lemma. From this we will obtain that $(X', d', m')$ is an $\mathsf{RCD}^*(-(N-1), N-1)$ space. Denote the Laplacian operator of $X'$ by $\underline{\Delta}$.

**Lemma 7.1.** *Let $I = [0,1]$ and $f \in D(\underline{\Delta}) \cap L^\infty(X')$ be such that $\underline{\Delta} f \in W^{1,2}(X') \cap L^\infty(X')$. Let $\overline{f} : X'_w \to \mathbb{R}$ be defined as $\overline{f}(x,t) = f(x)$ and $\overline{\chi_I} : X'_w \to \mathbb{R}$ as $\overline{\chi_I}(x,t) = \chi_I(t)$. Then $\overline{f}\overline{\chi_I} \in D(\Delta) \cap L^\infty(X'_w)$ and $\Delta(\overline{f}\overline{\chi_I}) \in W^{1,2}(X'_w) \cap L^\infty(X'_w)$.*

*Proof.* It is immediate to check that $\|\overline{f}\chi_I\|_{L^\infty(X'_w)} \leq \|f\|_{L^\infty(X')}$, which implies that $f\chi_I \in L^\infty(X'_w, m'_w)$.

Observe that $\overline{f}\chi_I \in L^2(X'_w, m_w)$ because

$$\int_{X'_w} |\overline{f}\chi_I|^2 \,\mathrm{d}m'_w = \int_{X'_w} |\overline{f}|^2 |\chi_I|^2 \,\mathrm{d}m'_w = \int_{\mathbb{R}} \int_{X'} |f|^2 |\chi_I|^2 w_{m'}(s) \,\mathrm{d}m' \,\mathrm{d}s \leq \int_I w_{m'}(s) \,\mathrm{d}s \|f\|^2_{L^2(X')}.$$

We will now show that $\overline{f}\overline{\chi_I} \in S^2(X'_w, d'_w, m'_w)$. Let $\pi$ be a test plan in $X'_w$ and let $p_1 : X'_w \to X'$, $p_2 : X'_w \to \mathbb{R}$ be the usual projections. Then, using that the projection maps commute with the evaluation maps, that $(p_1)_\#\pi$ and $(p_2)_\#\pi$ are test plans on $X'$ and $\mathbb{R}$ respectively, and that $f \in S^2(X')$, $\chi_I \in S^2(\mathbb{R})$ with $|\nabla\chi_I| = 0$, the following holds (we omit the integration subscript for simplicity):

$$\begin{aligned}
\int \left|\overline{f}\overline{\chi_I}(\gamma_1) - \overline{f}\overline{\chi_I}(\gamma_0)\right| \mathrm{d}\pi(\gamma) &\leq \int \left|\overline{f}\overline{\chi_I}(\gamma_1) - \overline{f}(\gamma_0)\overline{\chi_I}(\gamma_1)\right| \mathrm{d}\pi(\gamma) + \int \left|\overline{f}(\gamma_0)\overline{\chi_I}(\gamma_1) - \overline{f}\overline{\chi_I}(\gamma_0)\right| \mathrm{d}\pi(\gamma) \\
&\leq \int |\overline{\chi_I}(\gamma_1)| \left|\overline{f}(\gamma_1) - \overline{f}(\gamma_0)\right| \mathrm{d}\pi(\gamma) + \int |f(\gamma_0)| |\chi_I(\gamma_1) - \chi_I(\gamma_0)| \mathrm{d}\pi(\gamma) \\
&\leq \int |f \circ p_1(\gamma_1) - f \circ p_1(\gamma_0)| \mathrm{d}\pi(\gamma) \\
&+ \|f\|_{L^\infty(X')} \int_{X'w} |\chi_I \circ p_2(\gamma_1) - \chi_I \circ p_2(\gamma_0)| \mathrm{d}\pi(\gamma) \\
&= \int |f(\alpha_1) - f(\alpha_0)| \mathrm{d}(p_1)_\#\pi \\
&+ \|f\|_{L^\infty(X')} \int |\chi_I(\beta_1) - \chi_I(\beta_0)| \mathrm{d}(p_2)_\#\pi \\
&\leq \int_0^1 \int |\nabla f|_{X'}(\alpha_t) |\dot{\alpha}_t| \mathrm{d}(p_1)_\#\pi \\
&\leq \int_0^1 \int |\nabla f|_{X'} \circ p_1(\gamma_t) |\dot{\gamma}_t| \mathrm{d}\pi
\end{aligned}$$

Therefore $|\nabla f|_{X'} \circ p_1 : X'_w \to \mathbb{R}$ is a weak upper gradient for $\overline{f}\overline{\chi_I}$. Putting our previous observations together, so far we have shown that $\overline{f}\overline{\chi_I} \in W^{1,2}(X'_w) \cap L^\infty(X'_w)$. We will now prove that $\overline{f}\overline{\chi_I} \in D(\Delta)$. It is clear that $\mathrm{Test}(X'_w) \cap \mathcal{A} \neq \varnothing$. Let $\varphi \in \mathrm{Test}(X'_w) \cap \mathcal{A}$ be given by $\varphi = \sum_i^n a_i h_i g_i$, where $a_i \in \mathbb{R}$, $h_i \in \mathcal{H}$ and $g_i \in \mathcal{G}$. Then,

$$\begin{aligned}
\int_{X'_w} \left\langle \nabla(\overline{f}\overline{\chi_I}), \nabla\varphi \right\rangle_{X'_w} \mathrm{d}m'_w &= \int_{X'_w} \left\langle \nabla(\overline{f}\overline{\chi_I}), \nabla \sum a_i h_i g_i \right\rangle_{X'_w} \mathrm{d}m'_w \\
&= \sum a_i \int_{X'_w} g_i \left\langle \nabla(\overline{f}\overline{\chi_I}), \nabla h_i \right\rangle_{X'_w} + h_i \left\langle \nabla(\overline{f}\overline{\chi_I}), \nabla g_i \right\rangle_{X'_w} \mathrm{d}m'_w \\
&= \sum a_i \int_{X'_w} \overline{\chi_I} g_i \left\langle \nabla\overline{f}, \nabla h_i \right\rangle_{X'_w} + \overline{f} g_i \left\langle \nabla\overline{\chi_I}, \nabla h_i \right\rangle_{X'_w} \\
&+ \overline{\chi_I} h_i \left\langle \nabla\overline{f}, \nabla g_i \right\rangle_{X'_w} + \overline{f} h_i \left\langle \nabla\overline{\chi_I}, \nabla g_i \right\rangle_{X'_w} \mathrm{d}m'_w \\
&= \sum a_i \int_{X'_w} \overline{\chi_I} h_i \left\langle \nabla\overline{f}, \nabla g_i \right\rangle_{X'_w} \mathrm{d}m'_w
\end{aligned}$$

Here we have used the validity of the Leibniz rule due to the regularity of the functions involved as well as the orthogonality relations. Now we note that $\left\langle \nabla\overline{f}, \nabla g_i \right\rangle_{X'_w} = \left\langle \nabla\overline{f}, \nabla g_i \right\rangle_{X'}$ as a consequence of Theorem 2.15 and polarization. Therefore we have obtained,

$$\begin{aligned}
\sum a_i \int_{X'_w} \overline{\chi_I} h_i \left\langle \nabla\overline{f}, \nabla g_i \right\rangle_{X'_w} \mathrm{d}m'_w &= \sum a_i \int_{\mathbb{R}} \overline{\chi_I} h_i w_{m'}(s) \int_{X'} \left\langle \nabla\overline{f}, \nabla g_i \right\rangle_{X'} \mathrm{d}m' \,\mathrm{d}s \\
&= -\sum a_i \int_{\mathbb{R}} \overline{\chi_I} h_i w_{m'}(s) \int_{X'} g_i \underline{\Delta} f \,\mathrm{d}m' \,\mathrm{d}s
\end{aligned}$$

Hence, for all $\varphi \in \mathrm{Test}(\mathrm{X'_w}) \cap \mathcal{A}$, we have that

$$\int_{X'_w} \left\langle \nabla(\overline{f}\overline{\chi_I}), \nabla\varphi \right\rangle_{X'_w} \mathrm{d}m'_w = -\int_{X'_w} \varphi\overline{\chi_I} \left(\underline{\Delta} f \circ p_1\right) \mathrm{d}m'_w$$

Now, notice that $\mathrm{Test}(\mathrm{X}'_{\mathrm{w}}) \cap \mathcal{A}$ is dense in $W^{1,2}(X'_w)$, so by an approximation argument the previous equality holds for all $\varphi \in W^{1,2}(X'_w)$. Hence $\overline{f}\overline{\chi_I} \in D(\Delta)$ and $\Delta(\overline{f}\overline{\chi_I}) = \overline{\chi_I}\,(\underline{\Delta} f \circ p_1)$. Furthermore, it immediatly follows that $\Delta(\overline{f}\overline{\chi_I}) \in W^{1,2}(X'_w) \cap L^\infty(X'_w)$. ◻

**Proposition 7.2.** *For all $f \in D(\underline{\Delta})$ such that $\underline{\Delta} f \in W^{1,2}(X', d', m')$ and all non-negative $g \in D(\underline{\Delta}) \cap L^\infty(X', m')$ such that $\underline{\Delta} \in L^\infty(X', m')$, the following is satisfied:*

$$\frac{1}{2}\int_{X'} \underline{\Delta} g |\nabla f|^2_{X'}\,\mathrm{d}m' - \int_{X'} g\,\langle \nabla(\underline{\Delta} f), \nabla f\rangle_{X'}\,\mathrm{d}m' \geq -(N-1)\int_{X'} g|\nabla f|^2_{X'}\,\mathrm{d}m' + \frac{1}{N}\int_{X'} g(\underline{\Delta} f)^2\,\mathrm{d}m'$$

*Proof.* We will show the inequality holds for functions $f, g \in \mathrm{Test}(X')$ since, as the general case follows by the density of $\mathrm{Test}(X')$ in $W^{1,2}(X')$. With this assumption, it follows from Lemma 7.1 that we can apply Bochner's inequality on $X'_w$ for the functions $\overline{f}\overline{\chi_I}$ and $\overline{g\chi_I}$, that is

$$\begin{aligned}
&\frac{1}{2}\int_{X'_w} \Delta(\overline{g\chi_I})|\nabla(\overline{f}\overline{\chi_I})|^2_{X'_w}\,\mathrm{d}m'_w - \int_{X'_w} \overline{g\chi_I}\left\langle \nabla(\Delta(\overline{f}\overline{\chi_I})), \nabla\overline{f}\overline{\chi_I}\right\rangle_{X'_w}\,\mathrm{d}m'_w \\
&\quad \geq -(N-1)\int_{X'_w} \overline{g\chi_I}|\nabla(\overline{f}\overline{\chi_I})|^2_{X'_w}\,\mathrm{d}m'_w + \frac{1}{N}\int_{X'_w} \overline{g\chi_I}(\Delta(\overline{f}\overline{\chi_I}))^2\,\mathrm{d}m'_w.
\end{aligned}$$

We now simplify each of the summands composing the inequality. In the following computations we use the fact that $|\nabla\overline{f}|_{X'_w} = |\nabla\overline{f}|_{X'}$

$$\int_{X'_w} \Delta(\overline{g\chi_I})|\nabla(\overline{f}\overline{\chi_I})|^2_{X'_w}\,\mathrm{d}m'_w = \int_{X'_w} \overline{\chi_I}(\underline{\Delta}\overline{g} \circ p_1)|\overline{\chi_I}|^2|\nabla\overline{f}|^2_{X'_w}\,\mathrm{d}m'_w = \int_{\mathbb{R}} \overline{\chi_I}^3 w_{m'}(s)\int_{X'}(\underline{\Delta}\overline{g})|\nabla\overline{f}|^2_{X'}\,\mathrm{d}m'\,\mathrm{d}s$$

$$\begin{aligned}
\int_{X'_w} \overline{g\chi_I}\left\langle \nabla(\Delta(\overline{f}\overline{\chi_I})), \nabla\overline{f}\overline{\chi_I}\right\rangle_{X'_w}\,\mathrm{d}m'_w &= \int_{X'_w} \overline{g\chi_I}\left\langle \nabla(\overline{\chi_I}\underline{\Delta} f \circ p_1), \nabla\overline{f}\overline{\chi_I}\right\rangle_{X'_w}\,\mathrm{d}m'_w \\
&= \int_{X'_w} \overline{g\chi_I}\left(\overline{\chi_I}^2\left\langle \nabla(\underline{\Delta} f \circ p_1), \nabla\overline{f}\right\rangle_{X'_w}\right)\mathrm{d}m'_w \\
&= \int_{\mathbb{R}} \overline{\chi_I}^3 w_{m'}(s)\int_{X'} \overline{g}\left\langle \nabla(\underline{\Delta} f), \nabla\overline{f}\right\rangle_{X'_w}\,\mathrm{d}m'_w
\end{aligned}$$

$$\int_{X'_w} \overline{g\chi_I}|\nabla(\overline{f}\overline{\chi_I})|^2_{X'_w}\,\mathrm{d}m'_w = \int_{X'_w} \overline{g\chi_I}|\overline{\chi_I}|^2|\nabla\overline{f}|^2_{X'_w}\,\mathrm{d}m'_w = \int_{\mathbb{R}} \overline{\chi_I}^3 w_{m'}(s)\int_{X'} \overline{g}|\nabla\overline{f}|^2_{X'}\,\mathrm{d}m'\,\mathrm{d}s$$

$$\int_{X'_w} \overline{g\chi_I}(\Delta(\overline{f}\overline{\chi_I}))^2\,\mathrm{d}m'_w = \int_{X'_w} \overline{g\chi_I}\left(\overline{\chi_I}(\underline{\Delta} f \circ p_1)^2\right)\mathrm{d}m'_w = \int_{\mathbb{R}} \overline{\chi_I}^3 w_{m'}(s)\int_{X'} \overline{g}(\underline{\Delta} f)^2\,\mathrm{d}m'\,\mathrm{d}s$$

Therefore, dividing every term by $\int_{\mathbb{R}} \overline{\chi_I}^3 w_{m'}(s)ds$ and replacing $\overline{f}$ and $\overline{g}$ by $f$ and $g$ respectively where necessary, we obtain the result. ◻

## 8. $(\tilde{X}, d, \tilde{m})$ is isometric to a hyperbolic space

Here we adapt the ideas of Chen-Rong-Xu [CRX] to conclude that $(\tilde{X}, \tilde{d}, \tilde{m})$ is isometric to the hyperbolic space.

*Proof of Theorem 1.1.* By Theorem 3.3 and Theorem 1.2, $(\tilde{X}, \tilde{d}, \tilde{m})$ is isomorphic to the warped product space $(X'_w, d'_w, m'_w)$, with $w_{d'}(t) = e^t$ and $w_{m'}(t) = e^{(N-1)t}$. From the previous section we know that $(X', d', m')$ is an $\mathsf{RCD}^*(-(N-1), N)$. Thus, by Mondino-Naber [MN] there exists a point $y \in X'$ such that every tangent space of $(X', d', m')$ at $y$ is isometric to $(\mathbb{R}^{k-1}, d_{Euc}, \mathcal{H}^{k-1}, 0)$ for some $k \leq N+1$.

Let $\tilde{p} = (0, y)$. From now on, we identify $(\tilde{X}, \tilde{d}, \tilde{m})$ with $(X'_w, d'_w, m'_w)$. For any $t \in \mathbb{R}$ there is a deck transform $\gamma_t \in \bar{\pi}_1(X)$ such that

$$\tilde{d}(\gamma_t(\tilde{p}), (t, y)) \leq \mathrm{diam}(X) < \infty.$$

Note that in the last inequality we used that $X$ is compact.

Then in the pointed measured Gromov-Hausdorff (pmGH) sense:

$$(\tilde{X},\tilde{d},\tilde{m},\tilde{p}) = (\tilde{X},\tilde{d},\tilde{m},\gamma_{t_i}(\tilde{p})) = \lim_{i\to\infty}(\tilde{X},\tilde{d},\tilde{m},(t_i,y)) \tag{8.1}$$

Now we calculate the previous limit. For $t_i \in \mathbb{R}$, define $(X'_i,d'_i,m'_i) = (X', e^{t_i}d', e^{(N-1)t_i}m')$. From the definition of tangent space, in the pmGH sense,

$$\lim_{i\to\infty}(X'_i,d'_i,m'_i,y) = (\mathbb{R}^{k-1}, d_{Euc}, \mathcal{H}^{k-1}, 0). \tag{8.2}$$

The function, $(t,x)\mapsto(t-t_i,x)$ is a pointed isometry and hence

$$(X'_w,d'_w,m'_w,(t_i,y)) \cong (X'_{iw},d'_{iw},m'_{iw},(0,y)).$$

In combination with equation (8.2) it implies that in the pmGH sense (8.1) can be written as,

$$\begin{aligned}(\tilde{X},\tilde{d},\tilde{m},\tilde{p}) = \lim_{i\to\infty}(\tilde{X},\tilde{d},\tilde{m},(t_i,y)) &= \lim_{i\to\infty}(X'_{iw},d'_{iw},m'_{iw},(0,y))\\ &= (\mathbb{R}\times_w\mathbb{R}^{k-1}, d_{Eucw}, \mathcal{H}^{k-1}{}_w, 0).\end{aligned} \tag{8.3}$$

If $k\neq N$ then the measure $\mathcal{H}^{k-1}{}_w$ is non locally finite. This contradicts the previous equation, that is, $(\tilde{X},\tilde{d},\tilde{m},\tilde{p})$ and $(\mathbb{R}\times_w\mathbb{R}^{k-1}, d_{Eucw}, \mathcal{H}^{k-1}{}_w, 0)$ are isomorphic and $\tilde{m}$ locally finite. Hence, $k=N$ and we conclude that $(\tilde{X},\tilde{d},\tilde{m},\tilde{p}) = (\mathbb{H}^N, d_{\mathbb{H}^N}, \mathcal{H}^N, 0)$. □

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