1 Toward Schemes

Exercise 3.1.A. Suppose that $\pi : X \to Y$ is a continuous map of differentiable manifolds (as topological spaces not a priori differentiable). Show that π is differentiable if differentiable functions pull back to differentiable functions, i.e., if pullback by π gives a map $\mathcal{O}_Y \to \pi_*\mathcal{O}_X$.

Proof: Assume $\pi(p) = q$ and the local charts at $p \in X$ and $q \in Y$ are $(U, \phi; x_1, ..., x_m)$ and $(V, \psi : y_1, ..., y_n)$. Further assume that $\pi(U) = V$. View y_j as functions on V (which are clearly differentiable), by assumption $\pi_* y_j$ must be differentiable. This implies the function $(\pi_* y_1, ..., \pi_* y_n)$ is differentiable, which is just the map $\psi \circ \pi \circ \phi^{-1}$. □

Exercise 3.1.B. Show a morphism of differential manifolds $\pi : X \to Y$ with $\pi(p) = q$ induces a morphism of stalks $\pi^\# : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p}$. Show that $\pi^\#(\mathfrak{m}_{Y,q}) \subset \mathfrak{m}_{X,p}$.

Proof: For $\overline{\{V, g\}} \in \mathcal{O}_{Y,q}$, define $\pi^\#(\overline{\{V, g\}}) = \overline{\{\pi^{-1}(V), \pi_* g\}}$ where $\pi_* g(x) := g(\pi(x))$ for all $x \in \pi^{-1}(V)$. Now we have to verify that this is well defined. In fact, if $\{V, g\} \sim \{V', g'\}$, then there exist $W \subset V \cap V'$ such that $g|_W = g'|_W$. Then $\pi_* g|_{\pi^{-1}(W)} = \pi_* g'|_{\pi^{-1}(W)}$, which means that $\{\pi^{-1}(V), \pi_* g\} \sim \{\pi^{-1}(V'), \pi_* g'\}$. Since $\mathfrak{m}_{Y,q} = \{\overline{\{V, g\}} : g(q) = 0\}$, the second result comes naturally by our construction. □

2 The Underlying Set of Affine Schemes

Exercise 3.2.A.

1. Describe the set $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$.
2. Describe the set $\operatorname{Spec} k[x]_{(x)}$.

Proof:

1. $\operatorname{Spec} k[\epsilon]/(\epsilon^2) = \{[(\epsilon)]\}$.
2. $\operatorname{Spec} k[x]_{(x)} = \{[(x)], [(0)]\}$.

□

Exercise 3.2.B. Show that $\mathbb{R}[x]/(x^2 + ax + b) \cong \mathbb{C}$ if $x^2 + ax + b$ is irreducible in $\mathbb{R}$.

Proof: Trivial. □

Exercise 3.2.C. Describe the set $\mathbb{A}^1_{\mathbb{Q}}$.

Proof: $\mathbb{A}^1_{\mathbb{Q}} = \{(f) : f \text{ irreducible in } \mathbb{Q}[x]\}$. □

Exercise 3.2.D. If k is a field, show that $\operatorname{Spec} k[x]$ has infinitely many points.

Proof: Assume $\mathfrak{p}_1, ..., \mathfrak{p}_n$ are all the nonzero prime ideals in $k[x]$, assume that there are finitely many of them. Assume that $f_i \in \mathfrak{p}_i$ for all $i = 1, ..., n$ with $deg\, f_i \geq 1$. Then $f := f_1 f_2 ... f_n + 1$ has degree greater than or equal to n, which means that it's not a unit. Since $k[x]$ is UFD, f can be divided by at least one irreducible element, p say, which can not belong to any of the $\mathfrak{p}_i$'s. Then (p) is a new prime ideal, which leads to contradiction. □

Exercise 3.2.E. Show that all of the prime ideals of $\mathbb{C}[x, y]$ has one of the following forms: (0), $(x - a, y - b)$ and (f) when f is irreducible.

Proof: Assume that $\mathfrak{p}$ is a prime ideal which is not principle. Then there must be at least two irreducibles $f(x, y), g(x, y) \in \mathfrak{p}$ which are not differ by a unit. By considering the degree of y as an Euclidean function, we can find $0 \neq h(x) \in (f(x, y), g(x, y)) \subset \mathfrak{p}$. Since $\mathfrak{p}$ is prime, we know that all the linear divisors of $h(x)$, $x - a$ say, is in $\mathfrak{p}$. Thus only one linear term is of this form. Similarly we can find $y - b \in \mathfrak{p}$ for some b. Now we claim that $\mathfrak{p} = (x - a, y - b)$, since if $f(x, y) \in \mathfrak{p} - (x - a, y - b)$, we have $0 \neq f(a, b) \in \mathfrak{p}$, which is a unit. □

Exercise 3.2.F. Show that the Nullstellensatz 3.2.5 implies the Weak Nullstellensatz. 3.2.4.
weak Nullstellensatz: If k is an algebraically closed field then the maximal ideals of $k[x_1, ..., x_n]$ are precisely $(x_1 - a_1, ..., x_n - a_n)$.
Nullstellensatz: If k is a field then the maximal ideals of $k[x_1, ..., x_n]$ have residue fields finite extension of k.

Proof: Assume k algebraically closed, and $\mathfrak{m}$ a maximal ideal of $A := k[x_1, ..., x_n]$. By Nullstellensatz we know $A/\mathfrak{m}$ is just k. Let $\phi : A \twoheadrightarrow A/\mathfrak{m}$ be the natural surjection. Then for any $f \in \mathfrak{m}$, $0 = \phi(f(x_1, ..., x_n)) = f(\phi(x_1), ..., \phi(x_n))$. Let $a_i = \phi(x_i)$, we conclude that $\mathfrak{m} \subset (x_1 - a_1, ..., x_n - a_n) \neq A$. Since $\mathfrak{m}$ is maximal, $\mathfrak{m} = (x_1 - a_1, ..., x_n - a_n)$, which completes the proof. □

Exercise 3.2.G. Any integral domain A which is a finite k-algebra (i.e., a k-algebra that is finite dimensional vector space over k) must be a field.

Proof: Assume that $0 \neq x \in A$. Consider the linear transformation $\times x : A \to A$ when viewing A as a vector space over k. It's kernel must be zero since A is an integral domain. Thus it must be an isomorphism since A is finite dimensional. Hence we can find the inverse of x, which completes the proof. □

Exercise 3.2.H. Describe the maximal ideal of $\mathbb{Q}[x, y]$ corresponding to $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. Describe the maximal ideal of $\mathbb{Q}[x, y]$ corresponding to $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$. What are the residue fields in each case?

Proof: The first maximal ideal should be $(x^2 - 2, x - y)$. The second is $(x^2 - 2, x + y)$. The residue fields are both isomorphic to $\mathbb{Q}[x]/(x^2 - 2)$, which is just $\mathbb{Q}[\sqrt{2}]$. □

Exercise 3.2.I. Consider the map of sets $\phi : \mathbb{C}^2 \to \mathbb{A}^2_{\mathbb{Q}}$ defined as follows. (z_1, z_2) is sent to the prime ideal of $\mathbb{Q}[x, y]$ consisting of polynomials vanishing at (z_1, z_2).
(a) What is the image of (π, π^2)?
*(b) Show that ϕ is surjective.

Proof: (a) Assume that $f(x, y) \in \mathbb{Q}[x, y]$ vanishes at (π, π^2). Then $f(x, x^2) \in \mathbb{Q}[x]$ vanishes at π, which makes it a zero polynomial. Hence $(x^2 - y) \supset \phi(\pi, \pi^2)$. The other inclusion is obvious.
(b) (c.f. Exercise 9.4.D) Since $\mathbb{C}[x, y] = \mathbb{Q}[x, y] \otimes_{\mathbb{Q}} \mathbb{C}$, and the fact that the map $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{Q}$ i surjective, by Exercise 9.4.D we know that the map $\psi : \operatorname{Spec} \mathbb{C}[x, y] \to \operatorname{Spec} \mathbb{Q}[x, y]$ is surjective, the restriction of which on the closed points is just ϕ. Now for any prime ideal $\mathfrak{p} \subset \mathbb{Q}[x, y]$, there is at least one preimage under ψ, which must be one of the following three cases by Exercise 3.2.E:

1. Of the form $(x-a, y-b)$, in which case we are done.
2. The zero ideal. Since the transcendental degree of $\mathbb{C}/\mathbb{Q}$ is of continuous cardinarity, we can find at least two of them, a, b, say, and thus $\phi(a,b) = (0)$.
3. Of the form (f) for some irreducible element $f \in \mathbb{C}[x,y]$. If $(f) \cap \mathbb{Q}[x,y] = (0)$, then we come back to the previous situation. Hence we can assume that there exist $g \in \mathbb{C}[x,y]$ such that $h := fg \in \mathbb{Q}[x,y]$. Without loss of generality we can assume $f(x,y) = \sum_{k=0}^{n} g_k(y)x^k$ with $g_k \in \mathbb{C}[y]$ for all $k = 0, ..., n$, with $n > 0$. Since $g_n(y) = 0$ has finitely many roots, we can pick $q \in \mathbb{Q}$ such that $g_n(\pi + q) \neq 0$. Thus $f(x, \pi + q)$ is not zero function and we can find at least one root α.

   Now I claim that $(f) \cap \mathbb{Q}[x,y] = (x-\alpha, y-\pi-q) \cap \mathbb{Q}[x,y]$.

   The left hand side is clearly a subset of the right hand side, since $(x-\alpha, y-\pi-q)$ is clearly a maximal ideal containing (f).

   Now assume the inclusion is strict, by the fact that $\mathbb{Q}[x,y]$ is of dimension 2, we must have $(x-\alpha, y-\pi-q)$ maximal. Then $\mathbb{Q}[x,y]/(x-\alpha, y-\pi-q)$ must be a finite extension of $\mathbb{Q}$ by Nullstellensatz. But π is transcendental over $\mathbb{Q}$, which leads to contradiction.

□

Exercise 3.2.J. Suppose A is a ring, and I an ideal of A. Let $\phi : A \to A/I$. Show that ϕ^{-1} gives an inclusion-preserving bijection between prime ideals of A/I and prime ideals in A containing I.

Proof: Direct verification. If $\mathfrak{p}$ is a prime ideal in A/I, consider $f, g \in A$ such that $fg \in \phi^{-1}\mathfrak{p}$. Thus $\overline{fg}$ is in $\mathfrak{p}$. Then $\bar{f} \in \mathfrak{p}$ or $\bar{g} \in \mathfrak{p}$. Let's assume that $\bar{f} \in \mathfrak{p}$. Then $f \in \phi^{-1}\mathfrak{p}$, which means that $\phi^{-1}\mathfrak{p}$ is a prime ideal in A. The other direction is similar. □

Exercise 3.2.K. Suppose S is a multiplicative subset of A. Describe an order-preserving bijection of the prime ideals of $S^{-1}A$ with the prime ideals of A that don't meet the multiplicative subset S.

Proof: To boring to write it down. □

Exercise 3.2.L. Give an isomorphism of rings $(\mathbb{C}[x,y]/(xy))_x \simeq \mathbb{C}[x]_x$.

Proof: Define $\phi : (\mathbb{C}[x,y]/(xy))_x \to \mathbb{C}[x]_x$ given by $f(x,y) \mapsto f(x,0)$. Both rings are localizing at x so this map is well defined. We also need to check that the map does not depend on the choice of $f(x,y)$, which is not hard. The map is obviously surjective. Now consider $f(x,y) \in (\mathbb{C}[x,y]/(xy))_x$ such that $f(x,0) = 0$. Then $f(x,y) = yg(y) = (xyg(y))/x = 0$, which proves injectivity. □

Exercise 3.2.M. If $\phi : B \to A$ is a map of rings, and $\mathfrak{p}$ is a prime ideal of A, show that $\phi^{-1}(\mathfrak{p})$ is a prime ideal of B.

Proof: Let $f, g \in B$ such that $fg \in \phi^{-1}(\mathfrak{p})$. Thus $\phi(f)\phi(g) = \phi(fg) \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, we have either $\phi(f) \in \mathfrak{p}$ or $\phi(g) \in \mathfrak{p}$, which means that $f \in \phi^{-1}(\mathfrak{p})$ or $g \in \phi^{-1}(\mathfrak{p})$. $\square$

Exercise 3.2.N. Let B be a ring.

1. Suppose $I \subset B$ is an ideal. Show that the map $\operatorname{Spec} B/I \to \operatorname{Spec} B$ is the inclusion of 3.2.7.

2. Suppose $S \subset B$ is a multiplicative set. Show that the map $\operatorname{Spec} S^{-1}B \to \operatorname{Spec} B$ is the inclusion of 3.2.8.

Proof: The corresponding ring map $B \to B/I$ (resp. $B \to S^{-1}B$) gives the same bijection in 3.2.7 (resp. 3.2.8). $\square$

Exercise 3.2.O. Consider the map of complex manifolds sending $\mathbb{C} \to \mathbb{C}$ via $x \mapsto y = x^2$. We interpret the "source" as the x-line, and the "target" y-line. Interpret the corresponding map of rings as given by $\mathbb{C}[y] \to \mathbb{C}[x]$ by $y = x^2$. Verify that the preimage above the point $a \in \mathbb{C}$ is the points $\pm\sqrt{a} \in \mathbb{C}$, using the definition given above.

Proof: $a \in \mathbb{C}$ is interpreted by a maximal ideal $(y-a) \subset \mathbb{C}[y]$. Assume that $(x-b) \subset \mathbb{C}[x]$ is a preimage of $(y-a)$. Then $y - a = x^2 - a \in (x-b)$, which means that $(x-b)(x-c) = x^2 - a$ for some $c \in \mathbb{C}$. Hence $c = -b = \pm\sqrt{a}$. $\square$

Exercise 3.2.P. Suppose k is a field, and $f_1, ..., f_n \in k[x_1, ..., x_m]$ are given. Let $\phi : k[y_1, ..., y_n] \to k[x_1, ..., x_m]$ be the ring homomorphism defined by $y_i \mapsto f_i$.

1. Show that ϕ induces a map of sets $\operatorname{Spec} k[x_1, ..., x_m]/I \to \operatorname{Spec} k[y_1, ..., y_n]/J$ for any ideals $I \subset k[x_1, ..., x_m]$ and $J \subset k[y_1, ..., y_n]$ such that $\phi(J) \subset I$.

2. Show that the map of part (1) sends the point $(a_1, ..., a_m) \in k^m$ to $(f_1(a_1, ..., a_m), ..., f_n(a_1, ..., a_m)) \in k^n$.

Proof:

1. $\phi(J) \subset I$ provides that we can induce a ring map $\bar{\phi} : k[y_1, ..., y_n]/J \to k[x_1, ..., x_m]/I$, thus induces the map described.

2. Let $I = (x_1 - a_1, ..., x_m - a_m)$. We want to find $\phi^{-1}(I)$. Assume $\phi^{-1}(I) := J := (y_1 - b_1, ..., y_n - b_n)$. Then we induce a ring map $\bar{\phi} : k[y_1, ..., y_n]/J \to k[x_1, ..., x_m]/I$. But both the source and target are isomorphic to k, which makes $\bar{\phi}$ be the identity map on k, since it clearly can not be the zero map. So consider $y_i \in k[y_1, ..., y_n]/J$. It is mapped to $f_i(x_1, ..., x_m) \in k[x_1, ..., x_m]/I$, which is $f_i(a_1, ..., a_m) \in k$. On the other hand, it is directly mapped to b_i. Thus $b_i = f_i(a_1, ..., a_m)$, which proves the claim.

$\square$

Exercise 3.2.Q. Consider the map of sets $\pi : \mathbb{A}^n_{\mathbb{Z}} \to \operatorname{Spec} \mathbb{Z}$, given by the ring map $\mathbb{Z} \to \mathbb{Z}[x_1, ..., x_n]$. If p is prime, describe a bijection between the fiber $\pi^{-1}([(p)])$ and $\mathbb{A}^n_{\mathbb{F}_p}$.

Proof: The fiber $\pi^{-1}([(p)])$, corresponds to all prime ideals of $\mathbb{Z}[x_1, ..., x_n]$, which pull backs to (p) under π, thus corresponds to all prime ideals of $\mathbb{Z}[x_1, ..., x_n]$ that contains the ideal (p), which corresponds to all prime ideals of $\mathbb{Z}[x_1, ..., x_n]/(p)$ according to Exercise 3.2.J, which is just $\mathbb{A}^n_{\mathbb{F}_p}$. □

Exercise 3.2.R. Ring elements that have a power that is 0 are called nilpotents.

1. Show that if I is an ideal of nilpotents, then the inclusion $\operatorname{Spec} B/I \to \operatorname{Spec} B$ of Exercise 3.2.J is a bijection. Thus nilpotents dont affect the underlying set.

2. Show that the nilpotents of a ring B form an ideal. This ideal is called the nilradical, and is denoted $\mathfrak{N} = \mathfrak{N}(B)$.

Proof:

1. We just need to show that all prime ideals contains nilpotents, and the result will follow from Exercise 3.2.J. In fact, since 0 is in any ideal, so any nilpotents should be contained in any prime ideals since its power is contained.

2. Step-by-step verification, which can easily be found in any textbook.

□

Exercise 3.2.S. The nilradical $\mathfrak{N}(A)$ is the intersection of all the prime ideals of A. Geometrically: a function on $\operatorname{Spec} A$ vanishes at every point if and only if it is nilpotent.

Proof: Can be easily found in other textbooks. □

Exercise 3.2.T. Suppose we have a polynomial $f(x) \in k[x]$. Instead, we work in $k[x, \epsilon]/(\epsilon^2)$. What then is $f(x + \epsilon)$?

Proof: $f(x + \epsilon) = f(x) + \epsilon f'(x)$. In fact, since $f(x)$ is polynomial, we have the Taylor expansion $f(x) = \sum_{i=0}^{n} f^{(i)}(x)\frac{\epsilon^i}{i!}$, where n is the degree of $f(x)$. Hence the result holds since $\epsilon^2 = 0$. □

3 Visualizing schemes I: generic points

4 The underlying topological space of an affine scheme

Exercise 3.4.A. Check that the x-axis is contained in $V(xy, yz)$. (The x-axis is defined by $y = z = 0$, and the y-axis and z-axis are defined analogously.)

Proof: $xy \in (y, z) \subset (x - a, y, z)$ for every $a \in \mathbb{C}$. Same thing with yz □

Exercise 3.4.B. Show that if (S) is the ideal generated by S, then $V(S) = V((S))$.

Proof: By definition, clearly $V((S)) \subset V(S)$. Now assume $[\mathfrak{p}] \in V(S)$. Then for any $f \in (S)$, $f = \sum s_i a_i$ for some $s_i \in S$, $a_i \in A$. By assumption $s_i \in \mathfrak{p}$. Thus $f \in \mathfrak{p}$ since $\mathfrak{p}$ is an ideal. □

Exercise 3.4.C. 1. Show that $\emptyset$ and $\operatorname{Spec} A$ are both open in $\operatorname{Spec} A$.

2. If I_i is a collection of ideals, show that $\cap_i V(I_i) = V(\sum_i I_i)$. Hence the union of any collection of open sets is open.

3. Show that $V(I_1) \cup V(I_2) = V(I_1 I_2)$. Hence the intersection of any finite number of open sets is open.

Proof:

1. $V(0) = \operatorname{Spec} A$ and $V(1) = \emptyset$.

2. If $[\mathfrak{p}] \in \cap_i V(I_i)$, then $\forall i \in \Lambda, f_i \in I_i$, $f \in \mathfrak{p}$, thus for a typical element $\sum_i f_i \in \sum_i I_i$, $\sum_i f_i \in \mathfrak{p}$. On the other hand, if $[\mathfrak{p}] \in V(\sum_i I_i)$, then $\sum_i f_i \in \sum_i I_i$ for all $f_i \in I_i$. In typical let $f_j = 0$ when $j \neq i$ we get what we want.

3. If $[\mathfrak{p}] \in V(I_1 I_2) - V(I_1)$, which means that there exist $u \in I_1$ such that $u \notin \mathfrak{p}$. Then since $\forall v \in I_2$, $uv \in \mathfrak{p}$, and the fact that $\mathfrak{p}$ is a prime ideal, we have $v \in \mathfrak{p}$, and so $[\mathfrak{p}] \in V(I_2)$. The other inclusion is obvious since $I_1 I_2 \subset I_1$ and $I_1 I_2 \subset I_2$.

□

Exercise 3.4.D. Define radical of an ideal I to be

$$\sqrt{I} := \{r \in A : r^m \in I \text{ for some } n \in \mathbb{Z}^+\}.$$

Show that $\sqrt{I}$ is an ideal. Show that $V(\sqrt{I}) = V(I)$. We say an ideal is radical if it equals its own radical. Show that $\sqrt{\sqrt{I}} = \sqrt{I}$, and prime ideals are radical.

Proof:

1. If $r \in \sqrt{I}$, then there exist $m > 0$ such that $r^m \in I$, then for any $a \in A$ we have $(ra)^m = r^m a^m \in I$, which indicates that $ra \in \sqrt{I}$. If $r_1, r_2 \in \sqrt{I}$ such that $r_1^{m_1}, r_2^{m_2} \in I$ for some $m_1, m_2 > 0$, then $(r_1 + r_2)^{m_1+m_2-1} \in I$ by binomial expansion, which indicates that $r_1 + r_2 \in \sqrt{I}$.

2. If $[\mathfrak{p}] \in V(I)$, then for any $f \in \sqrt{I}$, $f^m \in \mathfrak{p}$ for some $m > 0$. Since $\mathfrak{p}$ is prime, $f \in \mathfrak{p}$. Thus $V(I) \subset V(\sqrt{I})$. The other inclusion is trivial since $\sqrt{I} \supset I$.

3. Clearly $\sqrt{\sqrt{I}} \supset \sqrt{I}$. Now if $r \in \sqrt{\sqrt{I}}$. Then $r^m \in \sqrt{I}$ for some $m > 0$, which means $r^{mn} = (r^m)^n \in I$ for some $n > 0$.

4. For any prime ideal $\mathfrak{p}$, and $p \in \sqrt{\mathfrak{p}}$, then $p^m \in \mathfrak{p}$ for some $m > 0$. But since it is a prime ideal, we must have $p \in \mathfrak{p}$.

□

Exercise 3.4.E. If $I_1, ..., I_n$ are ideals of a ring A, show that $\sqrt{\cap_{i=1}^n I_i} = \cap_{i=1}^n \sqrt{I_i}$.

Proof: For $f \in \sqrt{\cap_{i=1}^n I_i}$, there exist $m > 0$ such that $f^m \in I_i$ $(\forall i)$, which indicates that $f \in \cap_{i=1}^n \sqrt{I_i}$.

For $f \in \cap_{i=1}^n \sqrt{I_i}$, there exist $m_i > 0$ such that $f^{m_i} \in I_i$. Let $m = \max m_i$ and we have $f^m \in \cap_{i=1}^n I_i$, and so $f \in \sqrt{\cap_{i=1}^n I_i}$. □

Exercise 3.4.F. Show that $\sqrt{I}$ is the intersection of all the prime ideals containing I.

Proof: By Exercise 3.2.J, all the prime ideals containing I corresponds to the prime ideals of A/I with preserve of inclusion, the intersection of which is the nilradical of A/I, which corresponds to $\sqrt{I}$ by definition. □

Exercise 3.4.G. Describe the topological space $\mathbb{A}_k^1$.

Proof: Since $k[x]$ is PID, then all the ideals have the form (f). Clearly $[(0)]$ is in all $V(f)$'s. All the other points in $\mathbb{A}_k^1$ is are maximal ideals of the form $[(x-a)]$ for some $a \in k$, or a in short. $a \in V((f))$ if and only if $f(a) = 0$. We know that a function can only have finitely many roots, and we can always find a function with exactly finitely many preassigned roots. Hence we conclude that all the closed sets in $\mathbb{A}_k^1$ are the whole set, empty set, or any finitely many points union $[(0)]$. □

Exercise 3.4.H. By showing that closed sets pull back to closed sets, show that $\pi : \operatorname{Spec} A \to \operatorname{Spec} B$ is a continuous map. Interpret Spec as a contravariant functor from Rings to Tops.

Proof: $\pi^{-1}(V(I)) = \{[\mathfrak{p}] \in \operatorname{Spec} A : I \subset \pi(\mathfrak{p})\} = \{[\mathfrak{p}] \in \operatorname{Spec} A : \pi^{-1}(I) \subset \mathfrak{p}\} = V(\pi^{-1}(I))$. Hence π is continuous.

Clearly the Spec functor preserves identity and composition. □

Exercise 3.4.I. Suppose that $I, S \subset B$ are an ideal and multiplicative subset respectively.

1. Show that $\operatorname{Spec} B/I$ is naturally a closed subset of $\operatorname{Spec} B$. If $S = \{1, f, f^2, ...\}$ $(f \in B)$, show that $\operatorname{Spec} S^{-1}B$ is naturally an open set of $\operatorname{Spec} B$. Show that for arbitrary S, $\operatorname{Spec} S^{-1}B$ need not be open or closed.

2. Show that the Zariski topology on $\operatorname{Spec} B/I$ (resp. $\operatorname{Spec} S^{-1}B$) is the subspace topology induced by inclusion in $\operatorname{Spec} B$.

Proof:

1. $\operatorname{Spec} B/I$ is just $V(I)$ of $\operatorname{Spec} B$. $\operatorname{Spec} S^{-1}B = \operatorname{Spec} B - V(f)$.

   Consider the localization $\mathbb{Z} \to \mathbb{Q}$ and the corresponding map $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$. $[(0)] \in \operatorname{Spec} \mathbb{Z}$ is not open or closed.

2. Every closed set $V(J)$ of $\operatorname{Spec} B/I$ (resp. $\operatorname{Spec} S^{-1}B$) is the intersection of the vanishing set of pullback of J and $\operatorname{Spec} B/I$ (resp. $\operatorname{Spec} S^{-1}B$).

□

Exercise 3.4.J. Suppose $I \subset B$ is an ideal. Show that f vanishes on $V(I)$ if and only if $f \in \sqrt{I}$.

Proof: $\Leftarrow$: If $f \in \sqrt{I}$, then $f^n \in I$ for some $n > 0$. For any $[\mathfrak{p}] \in V(I)$, we have $f^n \in \mathfrak{p}$, and so $f \in \mathfrak{p}$ since $\mathfrak{p}$ is prime.

$\Rightarrow$: If f vanishes on $V(I)$, then $f \in \mathfrak{p}$ for all $[\mathfrak{p}] \in V(I)$, hence $f \in \cap_{[\mathfrak{p}]\in V(I)}\mathfrak{p} = \sqrt{I}$ by Exercise 3.4.F. □

Exercise 3.4.K. Describe the topological space $\operatorname{Spec} k[x]_{(x)}$.

Proof: $\operatorname{Spec} k[x]_{(x)} = \{[(x)], [(0)]\}$. Clearly the whole set and the empty set is in the topology. Now since $[(x)] \in V(x)$ while $[(0)] \notin V(x)$, we know $\{[(0)]\}$ is in the topology. Since $(0) \subset (x)$, $\{[(x)]\}$ can not be in the topology. □

5 A base of the Zariski topology on $\operatorname{Spec} A$: Distinguished open sets

Exercise 3.5.A. Show that the distinguished open sets form a base for the (Zariski) topology.

Proof: Given $S \subset A$, claim that $\operatorname{Spec} A - V(S) = \cup_{f\in S} D(f)$. In fact, if $[\mathfrak{p}] \in D(f)$ for some $f \in S$, then $f \notin \mathfrak{p}$, then $[\mathfrak{p}] \notin V(S)$. On the other hand, if $[\mathfrak{p}] \notin V(S)$, then $f \notin \mathfrak{p}$ for some $f \in S$, which means that $[\mathfrak{p}] \in D(f)$. □

Exercise 3.5.B. Suppose $f_i \in A$ as i runs over some index set J. Show that $\cup_{i\in J} D(f_i) = \operatorname{Spec} A$ if and only if $(\{f_i\}_{i\in J}) = A$, or equivalently, if there are a_i, all but finitely many nonzero, such that $\sum a_i f_i = 1$.

Proof: By proof of 3.5.A, $\cup_{i\in J} D(f_i) = \operatorname{Spec} A - V(\{f_i : i \in J\}) = \operatorname{Spec} A - V((\{f_i\}_{i\in J}))$. The result hold by the fact that if I is an ideal of A, $V(I) = \emptyset$ if and only if $I = A$. □

Exercise 3.5.C. Show that if $\operatorname{Spec} A$ is an infinite union of distinguished open sets $\cup_{i\in J} D(f_i)$, then in fact is a union of finitely number of these.

Proof: By Exercise 3.5.B, $\cup_{i\in J} D(f_i) = \operatorname{Spec} A$ if and only if there are a_i, all but finitely many nonzero, such that $\sum a_i f_i = 1$. Hence $(f_i)_{a_i\neq 0} = A$, and by using Exercise 3.5.B again we know that $\cup_{a_i\neq 0} D(f_i) = \operatorname{Spec} A$. □

Exercise 3.5.D. Proof that $D(f) \cap D(g) = D(fg)$.

Proof: If $[\mathfrak{p}] \in D(fg)$, then $fg \notin \mathfrak{p}$, and thus $f, g \notin \mathfrak{p}$, hence $[\mathfrak{p}] \in D(f) \cap D(g)$.

On the other hand, if $[\mathfrak{p}] \in D(f) \cap D(g)$, then $f, g \notin \mathfrak{p}$, thus $fg \notin \mathfrak{p}$ since $\mathfrak{p}$ is prime, and so $[\mathfrak{p}] \in D(fg)$. □

Exercise 3.5.E. Show that $D(f) \subset D(g)$ if and only if $f^n \in (g)$ for some $n \geq 1$, if and only if g is an invertible element of A_f.

Proof: $\Rightarrow$: If $D(f) \subset D(g)$, then $V(g) \subset V(f)$. Since f vanishes on $V(f)$, thus it vanishes on $V(g)$, and by Exercise 3.4.J, $f \in \sqrt{(g)}$, which means that $f^n \in (g)$ for some $n \geq 1$.

$\Leftarrow$: If $f^n \in (g)$ for some $n \geq 1$, then $f \in \sqrt{(g)}$, and by Exercise 3.4.J, f vanishes on $V(g)$, thus $V(g) \subset V(f)$, and so $D(f) \subset D(g)$.

The second if and only if is obvious. □

Exercise 3.5.F. Show that $D(f) = \emptyset$ if and only if $f \in \mathfrak{N}$.

Proof: $D(f) = \emptyset$ if and only if $D(f) \subset D(0)$, if and only if $f \in \mathfrak{N}$ by Exercise 3.5.E. □

6 Topological (and Noetherian) properties

Exercise 3.6.A. If $A = A_1 \times ... \times A_n$, describes a homomorphism $\operatorname{Spec} A_1 \coprod ... \coprod \operatorname{Spec} A_n \to \operatorname{Spec} A$ for which each $\operatorname{Spec} A_i$ is mapped to a distinguished open set $D(f_i)$ of $\operatorname{Spec} A$. Thus $\operatorname{Spec} \prod_{i=1}^n A_i = \coprod_{i=1}^n \operatorname{Spec} A_i$ as topological spaces.

Proof: WLOG, we can assume that $n = 2$. We claim that all prime ideals $\mathfrak{p}$ of A has the form $\mathfrak{p}_1 \times A_2$ or $A_1 \times \mathfrak{p}_2$ for some prime ideals $\mathfrak{p}_i \subset A_i$. Now, let $f_1 = (1, 0)$ and $f_2 = (0, 1)$, we know that $f_1 f_2 = 0 \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, we know one of f_i's is in $\mathfrak{p}$. WLOG we assume $f_1 \in \mathfrak{p}$. Then $f_1 A = A_1 \times 0 \subset \mathfrak{p}$. Consider the natural projection $\pi : A \to A_2$. By Exercise 3.2.J, $\pi(\mathfrak{p})$ is a prime ideal of A_2, which completes the claim. Besides, $D(f_1)$ contains prime ideals of A that contains A_1 but not A_2, which corresponds to $\operatorname{Spec} A_2$ under our map.

On the other hand (in remark), if $\operatorname{Spec} A$ is not connected, then it can written as a disjoint union of two closed subsets, namely $V(I_1)$ and $V(I_2)$. Thus, $V(I_1 I_2) = V(I_1) \cup V(I_2) = \operatorname{Spec} A$ meaning $I_1 I_2 \subset \mathfrak{N}(A)$, while $V(I_1 + I_2) = V(I_1) \cap V(I_2) = \emptyset$ meaning $1 \in I_1 + I_2$. Now we can choose $a_i \in I_i$ such that $a_1 + a_2 = 1$. Then $a_1 a_2 \subset \mathfrak{N}(A)$ meaning $(a_1 a_2)^n = 0$ for some n. Let $b_1 = \sum_{k=0}^{n-1} C(2n, k) a_1^{2n-k} a_2^k \in I_1$ and $b_2 = \sum_{k=0}^{n-1} C(2n, k) a_2^{2n-k} a_1^k \in I_2$, where $C(-, -)$ is the combination number. Then $b_1 + b_2 = \sum_{k=0}^{2n} C(2n, k) a_1^{2n-k} a_2^k = (a_1 + a_2)^{2n} = 1$, and $b_1 b_2 = 0$. Thus $b_1 + b_2 = 1 = (b_1 + b_2)^2 = b_1^2 + 2b_1 b_2 + b_2^2 = b_1^2 + b_2^2$. Let $k = b_1 - b_1^2 = b_2^2 - b_2$. Then $k \in I_1 \cap I_2$. Let $c_1 = b_1 - k \in I_1$ and $c_2 = b_2 + k \in I_2$. Thus $c_1 c_2 = 0$. Then $c_1 + c_2 = b_1 + b_2 = 1$, and $c_1^2 = (b_1 - k)^2 = b_1^2 - 2b_1 k + k^2 = b_1^2 = b_1 - k = c_1$. Similarly $c_2^2 = c_2$. Now consider the ring map $A \to (c_1) \times (c_2)$ via $a \mapsto (c_1 a, c_2 a)$. This map is injective, since if $(c_1 a, c_2 a) = 0$, we have $a = (c_1 + c_2) a = 0$. It is also surjective, since for any $(c_1 u, c_2 v)$ in range, we have $(c_1 u + c_2 v) \mapsto (c_1(c_1 u + c_2 v), c_2(c_1 u + c_2 v)) = (c_1 u, c_2 v)$. Hence the map is ring isomorphism, which proves the claim. □

Exercise 3.6.B.

1. Show that in an irreducible topological space, any nonempty open set is dense.

2. If X is a topological space, and Z (with the subspace topology) is an irreducible subset, then the closure $\bar{Z}$ in X is irreducible as well.

Proof:

1. Assume U is a nonempty open set of an irreducible topological space X, and p is a point outside of U. For any open neighborhood V of p such that $U \cap V = \emptyset$, we have $U^c \cup V^c = X$, thus by irreducibility one of them must be the whole space, which means that one of U and V must be empty, which leads to contradiction.

2. Let U and V be two closed subsets in $\bar{Z}$ such that $U \cup V = \bar{Z}$. Then $U \cap Z$ and $V \cap Z$ are closed in Z, and their union is Z. Since Z is irreducible, $U \cap Z = Z$ or $V \cap Z = Z$. WLOG let's assume it's the first case. Then $U = \bar{Z}$ since U is closed, which proofs the claim.

□

Exercise 3.6.C. If A is an integral domain, show that $\operatorname{Spec} A$ is irreducible.

Proof: Assume that $V(I)$ and $V(J)$ are two closed subsets, the union of which is the whole set $\operatorname{Spec} A$. Then we have $\operatorname{Spec} A = V(IJ)$, which means $IJ = (0)$, and hence one of them must be 0. □

Exercise 3.6.D. Show that an irreducible topological space is connected.

Proof: If not, then the two connected components are closed and their union is the whole space. However, non of them is the whole space, which contradicts to the irreducible hypothesis. □

Exercise 3.6.E. Give (with proof!) an example of a ring A where $\operatorname{Spec} A$ is connected but reducible.

Proof: Let $A = \mathbb{C}[x, y]/(xy)$. Consider $U_1 = V(x)$ and $U_2 = V(y)$. they are closed subsets, the union of which is $V((xy)) = V(0) = \operatorname{Spec} A$. However, neither of the U_i's the whole space, thus proving that $\operatorname{Spec} A$ not irreducible. However, $V(x) = (\mathbb{C}[x, y]/(xy))/(x) = \mathbb{C}[y]$, which is connected, and so is $V(y)$. So $\operatorname{Spec} A$ is connected, based on the fact that $[(0)] \in V(x) \cap V(y)$. □

Exercise 3.6.F.

1. Suppose $I = (wz - xy, wy - x^2, xz - y^2) \subset k[w, x, y, z]$. Show that $\operatorname{Spec} k[w, x, y, z]/I$ is irreducible, by showing that $k[w, x, y, z]/I$ is an integral domain.

2. Note that the generators of the ideal of part (1) may be rewritten as the equations ensuring that
$$\operatorname{rank} \begin{pmatrix} w & x & y \\ x & y & z \end{pmatrix} \leq 1,$$
Generalize part (1) to the ideal of rank one $2 \times n$ matrices.

Proof:

1. Consider the ring map $f : k[w, x, y, z] \to k[a, b]$ via $w \mapsto a^3$, $x \mapsto a^2b$, $y \mapsto ab^2$, $z \mapsto b^3$. We claim that the kernel is I. In fact, it is easy to verify that I is in the kernel. Now consider u in the kernel. We can find $v \in k[w, x, y, z]$ such that $u - v \in I$, and v does not contain any terms having wz, wy or xz as factor. That leaves with the following options: $w^i x^j$, $x^i y^j$, $y^i z^j$, z^i ($i > 0, j \geq 0$). They will be mapped to $a^{3i+2j}b^j$, $a^{2i+j}b^{2j+i}$, $a^i b^{2i+3j}$, b^{3i}, respectively. They are linearly independent: all $a^m b^n$ are linearly independent for different $m.n$, and different i, j's in different options give different m, n's, since the options coincides with $m > 2n$, $n/2 < m \leq 2n$, $0 < 2m \leq n$, $m = 0$, respectively. Thus we can conclude that $v = 0$, which completes our claim.

   Now that we have $k[w, x, y, z]/I$ isomorphic to a subring of the integral domain $k[a, b]$, which means that itself must also be an integral domain, which proves irreducibility by Exercise 3.6.C.

2. cf Exercise 8.2.J

□

Exercise 3.6.G. 1. Show that Spec A is quasicompact.

2. Show that in general Spec A can have nonquasicompact open sets.

Proof:

1. Given an open cover of Spec A, we can always find an open finer cover consisting of distinguished open sets $D(f_i)$. By Exercise 3.5.D, there is a finite subcover, which gives a way to find a finite subcover of the original cover.

2. Let $A = k[x_1, x_2, ...]$ and $\mathfrak{m} = (x_1, ...)$. Then Spec $A - V(\mathfrak{m}) = \cup_{i=1}^{\infty} D(x_i)$, which does not have finite subcover.

□

Exercise 3.6.H. 1. If X is a topological space that is a finite union of quasicompact spaces, show that X is quasicompact.

2. Show that every closed subset of a quasicompact topological space is quasicompact.

Proof: Could be found in any topology textbook. □

Exercise 3.6.I. Show that the closed points of Spec A correspond to the maximal ideals.

Proof: If $[\mathfrak{p}]$ is a closed point, then it is $V(I)$ for some ideal I, then we have that the only prime ideal containing I is $\mathfrak{p}$. However, there must be a maximal ideal containing I, and hence it must be $\mathfrak{p}$, which means that $\mathfrak{p}$ must be maximal.

On the other hand, if $\mathfrak{m}$ is maximal, $\{[\mathfrak{m}]\} = V(\mathfrak{m})$. □

Exercise 3.6.J. 1. Suppose that k is a field, and A is a finitely generated k-algebra. Show that closed points of Spec A are dense, by showing that if $f \in A$, and $D(f)$ is a nonempty (distinguished) open subset of Spec A, then $D(f)$ contains a closed point of Spec A.

2. Show that if A is a k-algebra that is not finitely generated the closed points need not be dense.

Proof:

1. If f is a zero divisor, then $D(f) = \operatorname{Spec} A$, which makes the conclusion trivial. Now assume that f is not a zerodivisor.

   By Nullstellensatz, closed points of Spec of a finitely generated k-algebra B corresponds to maximal ideals of B (Exercise 3.6.I), as those for which the residue field is a finite extension of k. Now, $\operatorname{Spec} A_f$ must has a closed point $[\mathfrak{m}]$, which corresponds to a maximal ideal $\mathfrak{m}$ of A_f whose residue field is a finite extension of k, provided that A_f is also finitely generated as a k-algebra. Now since the natural map $A \to A_f$ gives an injective map $A/(\mathfrak{m}|_A) \to A_f/\mathfrak{m}$. Hence $A/(\mathfrak{m}|_A)$ is a subring of a finite extension of k, thus itself is a finite extension of k, and by Nullstellensatz again, we know that $\mathfrak{m}|_A$ is maximal, and by Exercise 3.6.J we get our conclusion.

2. Consider $\operatorname{Spec} k[x]_{(x)}$. The only closed point is $[(x)]$. It is not dense, since the closure is still itself, not the whole set.

□

Exercise 3.6.K. Suppose that k is an algebraically closed field, and $A = k[x_1, ..., x_n]/I$ is a finitely generated k-algebra with $\mathfrak{N}(A) = \{0\}$. Consider the set $X = \operatorname{Spec} A$ as a subset of $\mathbb{A}^n_k$. The space $\mathbb{A}^n_k$ contains the "classical" points k^n. Show that functions on X are determined by their values on the closed points (by the weak Nullstellensatz 3.2.4, the classical points $k^n \cap \operatorname{Spec} A$ of $\operatorname{Spec} A$).

Proof: If f and g are two different functions on X, then $f - g \neq 0$. Now since $\mathfrak{N}(A) = \{0\}$, we know that $f - g$ does not vanish on some point of X by Remark 3.2.13. That means $D(f-g)$ is not empty. Now apply Exercise 3.6.J we know that $D(f-g)$ contains some closed points of X, which are the "classical" points. □

Exercise 3.6.L. If $X = \operatorname{Spec} A$, show that $[\mathfrak{q}]$ is a specialization of $[\mathfrak{p}]$ if and only if $\mathfrak{p} \subset \mathfrak{q}$. Hence show that $V(\mathfrak{p}) = \overline{\{[\mathfrak{p}]\}}$.

Proof: $[\mathfrak{q}]$ is a specialization of $[\mathfrak{p}]$, by definition, if and only if $\mathfrak{q} \in \overline{\{\mathfrak{p}\}}$, if and only if ($\forall I$ such that $\mathfrak{p} \in V(I)$, $\mathfrak{q} \in V(I)$), if and only if ($\forall I$ such that $I \subset \mathfrak{p}$, $I \subset \mathfrak{q}$), if and only if $\mathfrak{p} \subset \mathfrak{q}$.

$V(\mathfrak{p}) = \{[\mathfrak{q}] \in \operatorname{Spec} A : \mathfrak{p} \subset \mathfrak{q}\} = \{[\mathfrak{q}] \in \operatorname{Spec} A : [\mathfrak{q}] \text{ is a specialization of } [\mathfrak{p}]\} = \overline{\{[\mathfrak{p}]\}}$. □

Exercise 3.6.M. Verify that $[(y - x^2)] \in \mathbb{A}^2$ is a generic point for $V(y - x^2)$.

Proof: By Exercise 3.6.L, $\overline{[(y - x^2)]} = V((y - x^2)) = V(y - x^2)\}$. □

Exercise 3.6.N. Suppose p is a generic point for the closed subset K. Show that it is near every point q of K (every neighborhood of q contains p), and not near any point r not in K (there is a neighborhood of r not containing p).

Proof: If $q \in K$, and there is a neighborhood U of q that does not contain p, then U^c is a closed set containing p, thus must contain K, which leads to contradiction, since $q \in K$.

K^c is a neighborhood of r not containing p. □

Exercise 3.6.O. Show that every point x of a topological space X is contained in an irreducible component of X.

Proof: Consider the partially ordered set $\mathcal{S}$ of irreducible closed subsets of X containing x. By Zorn's lemma, $\{\overline{\{x\}}\}$ is a totally ordered subset of $\mathcal{S}$, thus contained in a maximal totally ordered subset $\{Z_\alpha\}$. $\cup Z_\alpha$ must be irreducible, since if U, V are closed subsets of X such that $U \cup V = \cup Z_\alpha$, then for any α, $U \cap Z_\alpha = Z_\alpha$ or $V \cap Z_\alpha = Z_\alpha$, and thus one of them must equal to the union. By our choice, Z_α is an irreducible component. □

Exercise 3.6.P. Show that $\mathbb{A}^2_{\mathbb{C}}$ is a Noetherian topological space: any decreasing sequence of closed subsets of $\mathbb{A}^2_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[x, y]$ must eventually stabilize. Show that $\mathbb{C}^2$ with the classical topology is not a Noetherian topological space.

Proof: As described in 3.4.3, the closed sets of $\operatorname{Spec} \mathbb{C}[x, y]$ are the union of finitely many of "curves" and finitely many of closed points, if not the whole space. Hence the descending chain must stabilize, since if there's one element in the chain that is not the whole set, then it has finitely many closed subsets.

$\mathbb{C}^2$ in the classical topology is clearly not Noetherian, since the chain $\{B(0, 1/n)\}$ is a counterexample. □

Exercise 3.6.Q. Show that every connected component of a topological space X is the union of irreducible components of X. Show that any subset of X that is simultaneously open and closed must be the union of some of the connected components of X. If X is a Noetherian topological space, show that the union of any subset of the connected components of X is always open and closed in X. (In particular, connected components of Noetherian topological spaces are always open, which is not true for more general topological spaces, see Remark 3.6.13.)

Proof:

1. Let U be a connected component of X. Then every point of U belongs to an irreducible component, which must be connected, hence it must be a subset of U. Then U is the union of all such irreducible components.

2. Let S be simultaneously open and closed in X. Then every point p of S belongs to a connected component $T_p$which must be contained in S. (If not, then both $T_p \cap S$ and $T_p - S$ are simultaneously open and closed in T_p, which make it not connected.) Then S is the union of all such connected components.

3. We first proof that X should be the union of finitely many irreducible components. If not, let $\mathcal{C}$ be the set of all closed subsets of X that is union of infinitely many irreducible components. It is clearly not empty, since $X \in \mathcal{C}$. For any $C \in \mathcal{C}$, C can not be irreducible, and so $C = A \cup B$ with A, B closed and strictly subset of C, one of

which must contain infinitely many components. Hence we can find a decreasing and non-stabilizing chain of closed sets, which contradicts with the Noetherian hypothesis of X.

So the union of any subset of the connected components of X must be the union of irreducible components of X, which must be finitely many of them. Since irreducible components are closed, so shall their finite union. Hence the union must be closed, and open since it's complement is closed.

□

Exercise 3.6.R. Show that a ring A is Noetherian if and only if every ideal of A is finitely generated.

Proof: $\Rightarrow$: If there is an ideal I that is not finitely generated, then we can find an ascending chain of ideal $\{I_i\}$ such that $I_0 = (0)$, and I_{i+1} generated by I_i and an element in $I - I_i$. This chain of ideal will not stabilize, which contradicts the Noetherian hypothesis.

$\Leftarrow$: If there is a chain of ideals $\{I_i\}$ that does not stabilize, then $\cup I_i$ is an ideal that is not finitely generated, since otherwise the generators will be contained in $\cup_{i=0}^{N} I_i$ for some $N >> 0$, which makes $I_k = I_N$ for all $k > N$, contradiction! □

Exercise 3.6.S. If A is Noetherian, show that $\operatorname{Spec} A$ is a Noetherian topological space. Describe a ring A such that $\operatorname{Spec} A$ is not a Noetherian topological space.

Proof: If $\operatorname{Spec} A$ is not Noetherian, then there exist a strictly descending sequence of closed subsets $V(I_1) \supset V(I_2) \supset ...$, which gives a strictly ascending sequence of ideals of A: $I_1 \subset I_2 \subset ...$, which contradicts the Noetherian hypothesis of A.

$\operatorname{Spec} k[x_1, x_2, ...]$ is not Noetherian since $V(x_1) \supset V(x_1, x_2) \supset ...$ is strictly descending. □

Exercise 3.6.T. Show that every open subset of a Noetherian topological space is quasicompact. Hence if A is Noetherian, every open subset of $\operatorname{Spec} A$ is quasicompact.

Proof: Assume that U is open in a Noetherian topological space X, along with an open cover $\{U_\alpha\}$. Let $\mathcal{U}$ be the set of unions of finite subsets of $\{U_\alpha\}$. $\mathcal{U}$ must have a maximal element, since otherwise we can obtain a strictly increasing sequence of open sets. Assume $A = U_1 \cup ... \cup U_n$ is a maximal element. Then $A \supset U$, since otherwise there exist $x \in (U - A)$, and so we can find $x \in U_\beta \subset \{U_\alpha\}$, and thus $A \cup U_\beta$ is strictly larger than A, which leads to contradiction. □

Exercise 3.6.U. Show that if M is a Noetherian A-module, then any submodule of M is a finitely generated A-module.

Proof: If M' is a submodule of M that is not finitely generated, then we can pick an increasing sequence of submodules of M as follows: M_0 is the zero submodule, and M_{i+1} is generated by M_i and one element of $M - M_i$. This sequence can not stabilize, since all M_i's are finitely generated, thus strictly contained in M', but that contradicts with the Noetherian hypothesis. □

Exercise 3.6.V. If $0 \to M' \to M \to M'' \to 0$ is exact, show that M' and M'' are Noetherian if and only if M is Noetherian.

Proof: We first proof that if $\phi : M \to M''$ with $\ker \phi = M'$, then $N \subset N' \subset M$, and $N \cap M' = N' \cap M'$ and $\phi(N) = \phi(N')$ implies $N = N'$. In fact, assume $x \in (N' - N)$. If $\phi(x) \notin \phi(N)$ then we are done since then $\phi(N) \neq \phi(N')$. Now assume $\phi(x) \in \phi(N)$. Then there exist $y \in N$ such that $\phi(x) = \phi(y)$. So $(x - y) \in \ker \phi = M'$. Then $(x - y) \in N' \cap M'$, but $(x - y) \notin N$, which makes $N' \cap M' \neq N \cap M'$, contradiction!

$\Leftarrow$: Given an ascending submodules $\{M_i\}$ of M, we obtain ascending submodules $\{M_i \cap M'\}$ of M' and $\{\phi(M_i)\}$ of M'', which will all stabilize by Noetherian hypothesis. And thus by our previous result, $\{M_i\}$ will also stabilize.

$\Rightarrow$: Submodules of M' are also submodules of M, so M being Noetherian guarantees M' being Noetherian. Submodules of M'' will pull back to submodules of M, preserving ordering, and so M being Noetherian guarantees M'' being Noetherian. $\square$

Exercise 3.6.W. Show that if A is a Noetherian ring, then $A^{\oplus n}$ is a Noetherian A-module.

Proof: Proof by induction. It clearly works for $n = 1$. For $n > 1$, consider the exact sequence

$$0 \to A \to A^{\oplus n} \to A^{\oplus(n-1)} \to 0.$$

A and $A^{\oplus(n-1)}$ are Noetherian by induction hypothesis, and hence $A^{\oplus n}$ is Noetherian by Exercise 3.6.V. $\square$

Exercise 3.6.X. Show that if A is a Noetherian ring and M is a finitely generated A-module, then M is a Noetherian module. Hence by Exercise 3.6.U, any submodule of a finitely generated module over a Noetherian ring is finitely generated.

Proof: We have the short exact sequence

$$0 \to M' \to A^{\oplus n} \to M \to 0,$$

where $A^{\oplus n}$ has basis the generators of M, the second map as natural inclusion, and M' as the kernel. By Exercise 3.6.W, $A^{\oplus n}$ is Noetherian, and by Exercise 3.6.V, so shall M. $\square$

7 The function$I(\cdot)$, taking subsets of $\operatorname{Spec} A$ to ideals of A

Exercise 3.7.A. Let $A = k[x, y]$. If $S = \{[(y)], [(x, y-1)]\}$ (see Figure 3.10), then $I(S)$ consists of those polynomials vanishing on the y-axis, and at the point $(1, 0)$. Give generators for this ideal.

Proof: $I(S) = (xy, y(y-1))$. In fact, since $V((xy, y(y-1))) = V(y) \cup V(x, y-1) = S$, $I(S) = I(V((xy, y(y-1)))) = \sqrt{(xy, y(y-1))} = (xy, y(y-1))$ by Exercise 3.7.D. $\square$

Exercise 3.7.B. Suppose $S \subset \mathbb{A}^3_{\mathbb{C}}$ is the union of the three axes. Give generators for the ideal $I(S)$.

Proof: $I(S) = (xyz)$. In fact, since $V(xyz) = V(x) \cup V(y) \cup V(z)$ is the union of the three axes, $I(S) = I(V((xyz))) = \sqrt{(xyz)} = (xyz)$ by Exercise 3.7.D. □

Exercise 3.7.C. Show that $V(I(S)) = \bar{S}$. Hence $V(I(S)) = S$ for a closed set S.

Proof: By definition $V(I(S))$ is closed, and contains S, hence contains $\bar{S}$. However, $\bar{S} = V(J)$ contains S, so we must have $J \subset I(S)$, which ensures $\bar{S} = V(J) \supset V(I(S))$. □

Exercise 3.7.D. Prove that if $J \subset A$ is an ideal, then $I(V(J)) = \sqrt{J}$.

Proof: $f \in I(V(J))$, if and only if f vanishing on $V(J)$ by definition, if and only if $f \in \sqrt{J}$ by Exercise 3.4.J. □

Exercise 3.7.E. Show that $V(\cdot)$ and $I(\cdot)$ give a bijection between irreducible closed subsets of $\operatorname{Spec} A$ and prime ideals of A. From this conclude that in $\operatorname{Spec} A$ there is a bijection between points of $\operatorname{Spec} A$ and irreducible closed subsets of $\operatorname{Spec} A$ (where a point determines an irreducible closed subset by taking the closure). Hence each irreducible closed subset of $\operatorname{Spec} A$ has precisely one generic point any irreducible closed subset Z can be written uniquely as $\overline{\{z\}}$.

Proof: For a prime ideal $\mathfrak{p}$, $I(V(\mathfrak{p})) = \sqrt{\mathfrak{p}} = \mathfrak{p}$, and hence the map V from prime ideals to closed subsets of A is an injective map with a left inverse I. Moreover, the image of V is all irreducible closed subsets, since if $V(\mathfrak{p}) = V(I) \cup V(J)$, with $V(I), V(J) \subsetneqq V(\mathfrak{p})$, we have $I, J \supsetneqq \mathfrak{p}$, and thus $\exists x, y$ such that $x \in I - \mathfrak{p}$, $y \in J - \mathfrak{p}$, but then $xy \in IJ \subset \sqrt{IJ} = \mathfrak{p}$ which leads to contradiction.

Moreover, $\overline{\{[\mathfrak{p}]\}} = V(I(\{\mathfrak{p}\})) = V(\mathfrak{p})$, which means that the point $[\mathfrak{p}]$ is mapped bijectively to the prime ideal $\mathfrak{p}$, which corresponed to $V(\mathfrak{p}) = \overline{\{[\mathfrak{p}]\}}$.

□

Exercise 3.7.F. A prime ideal of a ring A is a minimal prime ideal (or more simply, minimal prime) if it is minimal with respect to inclusion. (For example, the only minimal prime of $k[x, y]$ is (0).) If A is any ring, show that the irreducible components of $\operatorname{Spec} A$ are in bijection with the minimal prime ideals of A. In particular, $\operatorname{Spec} A$ is irreducible if and only if A has only one minimal prime ideal; this generalizes Exercise 3.6.C.

Proof: By Exercise 3.7.E there is a bijection between prime ideals of A and irreducible closed subsets of $\operatorname{Spec} A$, and if $\mathfrak{p} \subset \mathfrak{q}$ then $V(\mathfrak{p}) \supset V(\mathfrak{q})$. The result follows. □

Exercise 3.7.G. What are the minimal prime ideals of $k[x, y]/(xy)$ (where k is a field)?

Proof: Claim: any prime ideal $\mathfrak{p} \subset k[x, y]$ containing (xy) must contain (x) or (y). We will prove this by contradiction. In fact, if $(x) \not\subset \mathfrak{p}$, then there exist $f \in \mathfrak{p}$ such that $f \notin (x)$, which means that $f(x, y) = xg(x, y) + h(y)$. Thus $yf = xyg + yh \in \mathfrak{p}$. Since $(xy) \in \mathfrak{p}$, $yh \in \mathfrak{p}$. Assuming $(y) \notin \mathfrak{p}$, we have $h \in \mathfrak{p}$ since $\mathfrak{p}$ is prime. Hence there exist $u(y) \in \mathfrak{p}$, such that $u(y)$ is an irreducible polynomial of y. Since $u(y) \neq y$, we know that $u(y) = yv(y) + c$ where $v(y) \in k[y]$ and $c \in k$. Then $xu - xyv = cx \in \mathfrak{p}$, which leads to contradiction.

Hence we conclude that the minimal prime ideals are (x) and (y). □