MATH 3B WORKSHEET 2 ANSWER

DANNING LU

1. Quick Review

(1) State the definition of Riemann sum.

(2) State the definition of net area.

2. Practice Problems

2.1. **Find the Riemann sum for arbitrary n.** Note: There are multiple correct answers for each problem this problem set, as you can choose any endpoints you like. The answers showing here are assuming right endpoints.

(1) $f(x) = x^2$, $0 \leqslant x \leqslant 3$.

$$\sum_{i=1}^{n} \frac{3}{n} \left(\frac{3i}{n}\right)^2$$

(2) $f(x) = 3x - \frac{2}{x^2}$, $3 \leqslant x \leqslant 4$.

$$\sum_{i=1}^{n} \frac{1}{n} \left(3\left(3 + \frac{i}{n}\right) - \frac{2}{(3 + \frac{i}{n})^2}\right)$$

(3) $f(x) = \frac{2x}{x^2+1}$, $1 \leqslant x \leqslant 3$.

$$\sum_{i=1}^{n} \frac{2}{n} \frac{2\left(1 + \frac{2i}{n}\right)}{\left(1 + \frac{2i}{n}\right)^2 + 1}$$

(4) $f(x) = x^2 + \sqrt{1 + 2x}$, $4 \leqslant x \leqslant 7$.

$$\sum_{i=1}^{n} \frac{3}{n} \left(\left(4 + \frac{3i}{n}\right)^2 + \sqrt{1 + 2\left(4 + \frac{3i}{n}\right)}\right)$$

(5) $f(x) = \sqrt{\sin x}$, $0 \leqslant x \leqslant \pi$.

$$\sum_{i=1}^{n} \frac{\pi}{n} \sqrt{\sin \frac{\pi i}{n}}$$

2.2. **Find the region whose area is evaluated by the given limit.** Note: There are multiple correct answers for each question in this problem set, as explained in the class.

(1) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{3n} \cdot \left(\frac{3i}{n}\right)^3$

$$f(x) = \frac{1}{9}x^3,\ 0 \leqslant x \leqslant 3$$

(2) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1+\frac{3i}{n}}$

$$f(x) = \sqrt{x},\ 1 \leqslant x \leqslant 4$$

(3) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{64n^2}{27i^3}$

$$f(x) = \left(\frac{4}{3x}\right)^3,\ 0 \leqslant x \leqslant 1$$

(4) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan\frac{i\pi}{4n}$

$$f(x) = \tan x,\ 0 \leqslant x \leqslant \frac{\pi}{4}$$

2.3. **Use geometric results to evaluate the net area.**

(1) $y = 3x$, $a \leqslant x \leqslant b$, when
   (a) $0 < a < b$:

$$A = \frac{1}{2}(b-a)(3a+3b)$$

   Hint: The graph looks like a trapezoid.
   (b) $a < 0 < b$:

$$A = \frac{3}{2}b^2 - \frac{3}{2}(-a)^2$$

   Hint: The graph looks like two triangles. Notice that you need to do subtraction since you are caring about net areas.
   (c) $a < b < 0$:

$$A = -\frac{1}{2}(b-a)(-3a-3b)$$

   Hint: The graph looks like a trapezoid, but notice that the net area is the negation of the area of the trapezoid. Be careful with signs.
(2) $y = -\sqrt{1-x^2}$:
   (a) $-1 \leqslant x \leqslant 1$:

$$A = -\pi/2$$

   (b) $\frac{1}{2} \leqslant x \leqslant 1$:

$$A = \frac{\sqrt{3}}{8} - \frac{\pi}{6}$$

   Hint: The area is actually a sector of angle $\pi/3$ minus a triangle.

(c) *$-1 \leqslant x \leqslant a$, where $-1 < a < 1$:

$$A = -\left(\frac{1}{2}a\sqrt{1-a^2} + \frac{1}{2}(\pi - \arccos a)\right)$$

Hint: Same idea, but with careful calculation.

2.4. ***Estimate the deviation of Riemann sum.** Let A be the area under the graph of an increasing continuous function f from a to b, and let L_n and R_n be the approximations to A with n subintervals using left and right endpoints, respectively.

(1) How are A, L_n, and R_n related? Give a ranking for them for all the n's.

$$L_1 < L_2 < ... < L_n < L_{n+1} < ... < A < ... < R_{n+1} < R_n < ... < R_2 < R_1$$

(2) Show that

$$R_n - L_n = \frac{b-a}{n}[f(b) - f(a)].$$

Try to explain the geometric meaning of this equation.
Hint: Write down the expression for R_n and L_n, and then subtract them. Try to cancel out the same terms when you do the subtraction.
Geometric meaning: The n rectangles representing $R_n - L_n$ can be reassembled to form a single rectangle who has width Δx and length $f(b) - f(a)$.

(3) Deduce that

$$R_n - A < \frac{b-a}{n}[f(b) - f(a)].$$

Hint: Since $A > L_n$ we have $R_n - A < R_n - L_n$. Then we apply the previous result.

(4) If we want to find the area under the curve $y = 2^x$ from 1 to 3, use the above results to find a value n such that the Riemann sum has a maximum deviation of 0.0001.

$$(3-1)(2^3 - 2^1)/0.0001 = 140000$$