MATH 3B WORKSHEET 4 ANSWER

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1. Quick Review

Table of Indefinite integrals:

(1) (C is a constant) $\int Cf(x)dx = C\int f(x)dx$
(2) $\int[f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$
(3) (k is a constant) $\int kdx = kx + C$
(4) $\int x^n dx = \begin{cases} \dfrac{x^{n+1}}{n+1} + C, & \texttt{if } n \neq -1; \\ \ln|x| + C, & \texttt{if } n = -1. \end{cases}$
(5) $\int e^x dx = e^x + C$
(6) (b is a constant) $\int b^x dx = \frac{b^x}{\ln b} + C$
(7) $\int \sin x dx = -\cos x + C$
(8) $\int \cos x dx = \sin x + C$
(9) $\int \sec^2 x dx = \tan x + C$
(10) $\int \csc^2 x dx = -\cot x + C$
(11) $\int \sec x \tan x dx = \sec x + C$
(12) $\int \csc x \cot x dx = -\csc x + C$
(13) $\int \frac{1}{x^2+1} dx = \arctan x + C$
(14) $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$

Some useful trigonometry identities:

(1) definitions of $\tan, \cot, \sec, \csc$:
(2) Three Pythagorean identities: $\sin^2 x + \cos^2 x = 1$, $\tan^2 x + 1 = \sec^2 x$, $1 + \cot^2 x = \csc^2 x$.
(3)

	$-x$	$\frac{\pi}{2} - x$	$\pi - x$	$x + \frac{\pi}{2}$	$x - \frac{\pi}{2}$
sin	$-\sin x$	$\cos x$	$\sin x$	$\cos x$	$-\cos x$
cos	$\cos x$	$\sin x$	$-\cos x$	$-\sin x$	$\sin x$
tan	$-\tan x$	$\cot x$	$-\tan x$	$-\cot x$	$-\cot x$
cot	$-\cot x$	$\tan x$	$-\cot x$	$-\tan x$	$-\tan x$
sec	$\sec x$	$\csc x$	$-\sec x$	$-\csc x$	$\csc x$
csc	$-\csc x$	$\sec x$	$\csc x$	$\sec x$	$-\sec x$

(4) $\sin(x+y) = \sin x \cos y + \sin y \cos x$
(5) $\sin(x-y) = \sin x \cos y - \sin y \cos x$
(6) $\cos(x+y) = \cos x \cos y - \sin x \sin y$
(7) $\cos(x-y) = \cos x \cos y + \sin x \sin y$

(8) $\sin(2x) = 2\sin x \cos x$
(9) $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

2. Practice Problems

2.1. Find the Integrals.

(1) $\int (3 + \frac{4}{5}x^4 + \frac{7}{6}x^7)dx$

$$= 3x + \frac{4x^5}{25} + \frac{7x^8}{48} + C$$

(2) $\int (u+1)(u^2+1)du$

$$= \int (u^3 + u^2 + u + 1)du$$

$$= \frac{u^4}{4} + \frac{u^3}{3} + \frac{u^2}{2} + u + C$$

(3) $\int \frac{1+x+x^2}{\sqrt{x}}dx$

$$= \int (x^{-1/2} + x^{1/2} + x^{3/2})dx$$

$$= 2\sqrt{x} + \frac{2x^{3/2}}{3} + \frac{2x^{5/2}}{5} + C$$

(4) $\int \left(x^2 + 1 + \frac{1}{1+x^2}\right) dx$

$$= \frac{x^3}{3} + x + \arctan x + C$$

(5) $\int \left(\frac{1+r}{r}\right)^2 dr$

$$= \int \frac{1 + 2r + r^2}{r^2} dr$$

$$= \int \left(\frac{1}{r^2} + \frac{2}{r} + 1\right) dr$$

$$= -\frac{1}{r} + 2\ln|r| + r + C$$

(6) $\int_0^1 (x^{10} + 10^x)dx$

$$= \left[\frac{x^{11}}{11} + \frac{10^x}{\ln 10}\right]_0^1$$

$$= (1/11 + 10/\ln 10) - (0 + 1/\ln 10)$$

$$= 1/11 + 9/\ln 10$$

(7) $\int_0^{\pi/4} \sec\theta \tan\theta d\theta$

$$= [\sec\theta]_0^{\pi/4}$$

$$= \sqrt{2} - 1$$

(8) $\int_{\pi/6}^{\pi/4} \frac{1+\cos^2\theta}{\cos^2\theta} d\theta$

$$\int_{\pi/6}^{\pi/4} \left(\sec^2\theta + 1\right) d\theta$$

$$= [\tan\theta + \theta]_{\pi/6}^{\pi/4}$$

$$= (1 + \pi/4) - (1/\sqrt{3} + \pi/6)$$

$$= 1 - 1/\sqrt{3} + \pi/12$$

(9) * $\int \cot^2 x dx$

$$= \int (\csc^2 x - 1) dx$$

$$= -\cot x - x + C$$

(10) $\int_0^{\pi/3} \frac{\sin\theta + \sin\theta\tan^2\theta}{\sec^2\theta} d\theta$

$$= \int_0^{\pi/3} \frac{\sin\theta(1 + tan^2\theta)}{\sec^2\theta} d\theta$$

$$= \int_0^{\pi/3} \frac{\sin\theta\sec^2\theta}{\sec^2\theta} d\theta$$

$$= \int_0^{\pi/3} \sin\theta d\theta$$

$$= [-\cos\theta]_0^{\pi/3}$$

$$= -1/2 + 1$$

$$= 1/2$$

(11) $\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dr}{\sqrt{1-r^2}}$

$$= [\arcsin x]_{\sqrt{2}/2}^{\sqrt{3}/2}$$

$$= \pi/3 - \pi/4$$

$$= \pi/12$$

(12) $\int_0^{3\pi/2} |\sin x| dx$

First, we need to determine the intervals. By solving $\sin x = 0$ while $0 < x < 3\pi/2$ we have only one zero point $x = \pi$. For $0 < x < \pi$, $\sin x > 0$ and for $\pi < x < 3\pi/2$, $\sin x < 0$.

Besides, $\int \sin x dx = -\cos x + C$. Hence we know that

$$\int_0^{3\pi/2} |\sin x| dx = [-\cos x]_0^{\pi} + [-\cos x]_{3\pi/2}^{\pi}$$

$$= (1 - (-1)) + (1 - 0)$$

$$= 3.$$

(13) $\int_0^4 |(x-1)(x-2)(x-3)|dx$
First, we need to determine the intervals. By solving $(x-1)(x-2)(x-3) = 0$ while $0 < x < 4$ we have three zero points $x = 1, 2, 3$. For $0 < x < 1$, $(x-1)(x-2)(x-3) < 0$, for $1 < x < 2$, $(x-1)(x-2)(x-3) > 0$, for $2 < x < 3$, $(x-1)(x-2)(x-3) < 0$, and for $3 < x < 4$, $(x-1)(x-2)(x-3) > 0$.
Besides, $\int(x-1)(x-2)(x-3)dx = \int(x^3-6x^2+11x-6)dx = \frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x + C$. Hence we know that

$$\int_0^4 |(x-1)(x-2)(x-3)|dx = \left(\int_1^0 + \int_1^2 + \int_3^2 + \int_3^4\right)(x-1)(x-2)(x-3)dx$$

$= (0 - (-9/4)) + (-2 - (-9/4)) + (-2 - (-9/4)) + (0 - (-9/4))$
$= 9/4 + 1/4 + 1/4 + 9/4$
$= 5.$

2.2. **The Net Change Theorem.**

(1) The current in a wire is defined as the derivative of the charge: $I(t) = Q'(t)$. What does $\int_a^b I(t)dt$ represent?
Answer: The total charge going through the wire during the time period $a \leqslant t \leqslant b$.
(2) If oil leaks from a tank at a rate of $r(t) = 100e^{-0.01t}$ gallons per minute at time t in minutes. How much oil will leak in the first two hours?
Answer: $\int_0^1 20100e^{-0.01t}dt = [-10000e^{-0.01t}dt]_0^{120} = 10000(1 - e^{-1.2})(\approx 6988.06)$.
(3) If x is measured in feet and $f(x)$ is measured in newtons, what are the units for $\int_0^{100} f(x)dx$? For $f'(x)$? For $\int_0^{100} x^2 f(x)dx$?
Answer: newton·feet; newton/feet; newton·feet^3.
(4) A ball is having velocity $v(t) = \sqrt{3}\sin t + 2$ in feet per second, where t is measured in seconds. The ball is starting at $s(0) = 5$. Where is the ball at $t = \frac{19\pi}{6}$? What's the total distance travelled during this period?
Answer: The final location of the ball is given by

$$s(\frac{19\pi}{6}) = s(0) + \int_0^{19\pi/6} (\sqrt{3}\sin t + 2)dt$$

$= 5 + [-\sqrt{3}\cos t + 2t]_0^{19\pi/6}$
$= 5 + (-\sqrt{3} \cdot (-\sqrt{3}/2) + 19\pi/3) - (-\sqrt{3})$
$= 13/2 + 19\pi/3 + \sqrt{3}.$

The total distance travelled during this period is given by

$$\int_0^{19\pi/6} |\sqrt{3}\sin t + 2|dt. \quad (*)$$

Since $\sqrt{3}\sin t + 2$ is always large than zero, so we have

$$(*) = \int_0^{19\pi/6} \sqrt{3}\sin t + 2dt = 3/2 + 19\pi/3 + \sqrt{3}$$

as previous calculated.

Note: Actually there is a typo in this question. I meant $v(t) = 2\sin t + \sqrt{3}$. For exercise please do this problem again with the new $v(t)$.
Answer: $7 + \sqrt{3} + \frac{19\pi}{2\sqrt{3}}$; $6 + \sqrt{3} + \frac{5\sqrt{3}\pi}{2}$.