$$=\frac{1}{2}\ln(x^2+2x+3)-\sqrt{2}\arctan\frac{x+1}{\sqrt{2}}+C.$$

(44) 设 $x=\tan t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，则 $x^2+1=\sec^2 t$，$dx=\sec^2 t dt$，于是

$$\begin{aligned}\int\frac{x^3+1}{(x^2+1)^2}dx &= \int\frac{\tan^3 t+1}{\sec^2 t}dt\\ &=\int\frac{\cos^2 t-1}{\cos t}d(\cos t)+\int\frac{1+\cos 2t}{2}dt\\ &=\frac{1}{2}\cos^2 t-\ln\cos t+\frac{t}{2}+\frac{1}{4}\sin 2t+C\\ &=\frac{1}{2}\cos^2 t-\ln\cos t+\frac{t}{2}+\frac{1}{2}\sin t\cos t+C.\end{aligned}$$

按 $\tan t=x$ 作辅助三角形(图 4−1)，便有

$$\cos t=\frac{1}{\sqrt{1+x^2}},\sin t=\frac{x}{\sqrt{1+x^2}},$$

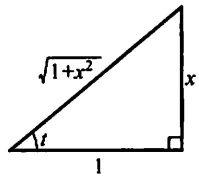

图 4−1

于是

$$\int\frac{x^3+1}{(x^2+1)^2}dx=\frac{1+x}{2(1+x^2)}+\frac{1}{2}\ln(1+x^2)+\frac{1}{2}\arctan x+C.$$

习题 4−3 分部积分法

求下列不定积分：

1. $\int x\sin x dx$.
2. $\int \ln x dx$.
3. $\int \arcsin x dx$.
4. $\int x e^{-x} dx$.
5. $\int x^2\ln x dx$.
6. $\int e^{-x}\cos x dx$.
7. $\int e^{-2x}\sin\frac{x}{2}dx$.
8. $\int x\cos\frac{x}{2}dx$.
9. $\int x^2\arctan x dx$.
10. $\int x\tan^2 x dx$.
11. $\int x^2\cos x dx$.
12. $\int t e^{-2t}dt$.
13. $\int \ln^2 x dx$.
14. $\int x\sin x\cos x dx$.
15. $\int x^2\cos^2\frac{x}{2}dx$.
16. $\int x\ln(x-1)dx$.
17. $\int (x^2-1)\sin 2x dx$.
18. $\int \frac{\ln^3 x}{x^2}dx$.

19. $\int e^{\sqrt[3]{x}}dx$.

20. $\int \cos \ln x dx$.

21. $\int (\arcsin x)^2 dx$.

22. $\int e^x \sin^2 x dx$.

23. $\int x\ln^2 x dx$.

24. $\int e^{\sqrt{3x+9}}dx$.

解 1. $\int x\sin x dx = -\int x d(\cos x) = -x\cos x + \int \cos x dx$

$= -x\cos x + \sin x + C.$

2. $\int \ln x dx = x\ln x - \int x \cdot \frac{1}{x}dx = x\ln x - x + C.$

3. $\int \arcsin x dx = x\arcsin x - \int x \cdot \frac{1}{\sqrt{1-x^2}}dx = x\arcsin x + \sqrt{1-x^2} + C.$

4. $\int xe^{-x}dx = -\int xde^{-x} = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C.$

5. $\int x^2\ln x dx = \frac{1}{3}\int \ln x d(x^3) = \frac{x^3\ln x}{3} - \frac{1}{3}\int x^3 \cdot \frac{1}{x}dx = \frac{x^3\ln x}{3} - \frac{x^3}{9} + C.$

6. $\int e^{-x}\cos x dx = -\int \cos x d(e^{-x}) = -e^{-x}\cos x + \int e^{-x}(-\sin x)dx$

$= -e^{-x}\cos x + \int \sin x d(e^{-x})$

$= -e^{-x}\cos x + e^{-x}\sin x - \int e^{-x}\cos x dx,$

故有

$$\int e^{-x}\cos x dx = \frac{e^{-x}(\sin x - \cos x)}{2} + C.$$

7. $\int e^{-2x}\sin \frac{x}{2}dx = -\frac{1}{2}\int \sin \frac{x}{2}d(e^{-2x})$

$= -\frac{1}{2}e^{-2x}\sin \frac{x}{2} + \frac{1}{2}\int e^{-2x} \cdot \frac{1}{2}\cos \frac{x}{2}dx$

$= -\frac{1}{2}e^{-2x}\sin \frac{x}{2} - \frac{1}{8}\int \cos \frac{x}{2}d(e^{-2x})$

$= -\frac{1}{2}e^{-2x}\sin \frac{x}{2} - \frac{1}{8}e^{-2x}\cos \frac{x}{2} + \frac{1}{8}\int e^{-2x} \cdot \left(-\frac{1}{2}\sin \frac{x}{2}\right)dx$

$= -\frac{1}{8}\left(4\sin \frac{x}{2} + \cos \frac{x}{2}\right)e^{-2x} - \frac{1}{16}\int e^{-2x}\sin \frac{x}{2}dx,$

故 $$\int e^{-2x}\sin \frac{x}{2}dx = -\frac{2}{17}\left(4\sin \frac{x}{2} + \cos \frac{x}{2}\right)e^{-2x} + C.$$

8. $\int x\cos \frac{x}{2}dx = 2\int xd\left(\sin \frac{x}{2}\right) = 2x\sin \frac{x}{2} - 2\int \sin \frac{x}{2}dx$

$$=2x\sin\frac{x}{2}+4\cos\frac{x}{2}+C.$$

9. $\int x^2\arctan x\mathrm{d}x=\frac{1}{3}\int \arctan x\mathrm{d}(x^3)=\frac{1}{3}x^3\arctan x-\frac{1}{3}\int\frac{x^3}{1+x^2}\mathrm{d}x$

$$=\frac{1}{3}x^3\arctan x-\frac{1}{3}\int\left(x-\frac{x}{1+x^2}\right)\mathrm{d}x$$

$$=\frac{1}{3}x^3\arctan x-\frac{1}{6}x^2+\frac{1}{6}\ln(1+x^2)+C.$$

10. $\int x\tan^2 x\mathrm{d}x=\int x(\sec^2 x-1)\mathrm{d}x=\int x\mathrm{d}(\tan x)-\frac{x^2}{2}$

$$=x\tan x+\ln|\cos x|-\frac{x^2}{2}+C.$$

11. $\int x^2\cos x\mathrm{d}x=\int x^2\mathrm{d}(\sin x)=x^2\sin x-\int 2x\sin x\mathrm{d}x$

$$=x^2\sin x+\int 2x\mathrm{d}(\cos x)$$

$$=x^2\sin x+2x\cos x-\int 2\cos x\mathrm{d}x$$

$$=x^2\sin x+2x\cos x-2\sin x+C.$$

12. $\int t\mathrm{e}^{-2t}\mathrm{d}t=-\frac{1}{2}\int t\mathrm{d}(\mathrm{e}^{-2t})=-\frac{1}{2}t\mathrm{e}^{-2t}+\frac{1}{2}\int \mathrm{e}^{-2t}\mathrm{d}t$

$$=-\frac{1}{2}t\mathrm{e}^{-2t}-\frac{1}{4}\mathrm{e}^{-2t}+C.$$

13. $\int \ln^2 x\mathrm{d}x=x\ln^2 x-\int 2\ln x\mathrm{d}x=x\ln^2 x-2x\ln x+\int 2\mathrm{d}x$

$$=x\ln^2 x-2x\ln x+2x+C.$$

14. $\int x\sin x\cos x\mathrm{d}x=\int -\frac{x}{4}\mathrm{d}(\cos 2x)=-\frac{x\cos 2x}{4}+\frac{1}{4}\int\cos 2x\mathrm{d}x$

$$=-\frac{x\cos 2x}{4}+\frac{\sin 2x}{8}+C.$$

15. $\int x^2\cos^2\frac{x}{2}\mathrm{d}x=\frac{1}{2}\int x^2(1+\cos x)\mathrm{d}x=\frac{1}{6}x^3+\frac{1}{2}\int x^2\mathrm{d}(\sin x)$

$$=\frac{1}{6}x^3+\frac{1}{2}x^2\sin x-\int x\sin x\mathrm{d}x$$

$$=\frac{1}{6}x^3+\frac{1}{2}x^2\sin x+\int x\mathrm{d}(\cos x)$$

$$=\frac{1}{6}x^3+\frac{1}{2}x^2\sin x+x\cos x-\int\cos x\mathrm{d}x$$

$$=\frac{1}{6}x^3+\frac{1}{2}x^2\sin x+x\cos x-\sin x+C.$$

16. $\int x\ln(x-1)\mathrm{d}x=\frac{1}{2}\int \ln(x-1)\mathrm{d}(x^2-1)$

$$=\frac{1}{2}(x^2-1)\ln(x-1)-\frac{1}{2}\int (x+1)\mathrm{d}x$$

$$=\frac{1}{2}(x^2-1)\ln(x-1)-\frac{1}{4}x^2-\frac{1}{2}x+C.$$

17. $\int (x^2-1)\sin 2x\mathrm{d}x=-\frac{1}{2}\int (x^2-1)\mathrm{d}(\cos 2x)$

$$=-\frac{1}{2}(x^2-1)\cos 2x+\int x\cos 2x\mathrm{d}x$$

$$=-\frac{1}{2}(x^2-1)\cos 2x+\frac{1}{2}\int x\mathrm{d}(\sin 2x)$$

$$=-\frac{1}{2}(x^2-1)\cos 2x+\frac{1}{2}x\sin 2x-\frac{1}{2}\int \sin 2x\mathrm{d}x$$

$$=-\frac{1}{2}\left(x^2-\frac{3}{2}\right)\cos 2x+\frac{1}{2}x\sin 2x+C.$$

18. $\int \frac{\ln^3 x}{x^2}\mathrm{d}x=\int -\ln^3 x\mathrm{d}\left(\frac{1}{x}\right)=-\frac{\ln^3 x}{x}-3\int \ln^2 x\mathrm{d}\left(\frac{1}{x}\right)$

$$=-\frac{\ln^3 x}{x}-3\left[\frac{\ln^2 x}{x}+2\int \ln x\mathrm{d}\left(\frac{1}{x}\right)\right]$$

$$=-\frac{\ln^3 x+3\ln^2 x+6\ln x+6}{x}+C.$$

19. $\int \mathrm{e}^{\sqrt[3]{x}}\mathrm{d}x\overset{x=u^3}{=\!=}\int 3u^2\mathrm{e}^u\mathrm{d}u=\int 3u^2\mathrm{d}(\mathrm{e}^u)=3u^2\mathrm{e}^u-\int 6u\mathrm{d}(\mathrm{e}^u)$

$$=(3u^2-6u+6)\mathrm{e}^u+C=3\mathrm{e}^{\sqrt[3]{x}}(x^{2/3}-2x^{1/3}+2)+C.$$

20. $\int \cos \ln x\mathrm{d}x\overset{x=\mathrm{e}^u}{=\!=}\int \mathrm{e}^u\cos u\mathrm{d}u$,

而 $\int \mathrm{e}^u\cos u\mathrm{d}u=\int \cos u\mathrm{d}(\mathrm{e}^u)=\mathrm{e}^u\cos u+\int \mathrm{e}^u\sin u\mathrm{d}u$

$$=\mathrm{e}^u\cos u+\int \sin u\mathrm{d}(\mathrm{e}^u)$$

$$=\mathrm{e}^u\cos u+\mathrm{e}^u\sin u-\int \mathrm{e}^u\cos u\mathrm{d}u,$$

因此 $\int \mathrm{e}^u\cos u\mathrm{d}u=\frac{\mathrm{e}^u(\cos u+\sin u)}{2}+C$，故有

$$\int \cos \ln x\mathrm{d}x=\frac{x(\cos \ln x+\sin \ln x)}{2}+C.$$

21. $\int (\arcsin x)^2\mathrm{d}x=x(\arcsin x)^2-\int \frac{2x\arcsin x}{\sqrt{1-x^2}}\mathrm{d}x$

$$=x(\arcsin x)^2+\int 2\arcsin x\,\mathrm{d}(\sqrt{1-x^2})$$

$$=x(\arcsin x)^2+2\sqrt{1-x^2}\arcsin x-2x+C.$$

22. $\int \mathrm{e}^x\sin^2 x\mathrm{d}x=\frac{1}{2}\int \mathrm{e}^x(1-\cos 2x)\mathrm{d}x=\frac{1}{2}\mathrm{e}^x-\frac{1}{2}\int \mathrm{e}^x\cos 2x\mathrm{d}x$，

$$\int \mathrm{e}^x\cos 2x\mathrm{d}x=\int \cos 2x\mathrm{d}(\mathrm{e}^x)=\mathrm{e}^x\cos 2x+2\int \mathrm{e}^x\sin 2x\mathrm{d}x$$

$$=\mathrm{e}^x\cos 2x+2\int \sin 2x\mathrm{d}(\mathrm{e}^x)$$

$$=\mathrm{e}^x\cos 2x+2\mathrm{e}^x\sin 2x-4\int \mathrm{e}^x\cos 2x\mathrm{d}x,$$

得 $\int \mathrm{e}^x\cos 2x\mathrm{d}x=\frac{\mathrm{e}^x\cos 2x+2\mathrm{e}^x\sin 2x}{5}+C$，因此有

$$\int \mathrm{e}^x\sin^2 x\mathrm{d}x=\frac{1}{2}\mathrm{e}^x-\frac{1}{5}\mathrm{e}^x\sin 2x-\frac{1}{10}\mathrm{e}^x\cos 2x+C.$$

23. $\int x\ln^2 x\mathrm{d}x=\int \ln^2 x\mathrm{d}\left(\frac{x^2}{2}\right)=\frac{x^2}{2}\ln^2 x-\int x\ln x\mathrm{d}x$

$$=\frac{x^2}{2}\ln^2 x-\int \ln x\mathrm{d}\left(\frac{x^2}{2}\right)=\frac{x^2}{2}\ln^2 x-\frac{x^2}{2}\ln x+\int \frac{x}{2}\mathrm{d}x$$

$$=\frac{x^2}{4}(2\ln^2 x-2\ln x+1)+C.$$

24. 设 $\sqrt{3x+9}=u$，即 $x=\frac{1}{3}(u^2-9)$，$\mathrm{d}x=\frac{2}{3}u\mathrm{d}u$，则

$$\int \mathrm{e}^{\sqrt{3x+9}}\mathrm{d}x=\int \frac{2}{3}u\mathrm{e}^u\mathrm{d}u=\int \frac{2}{3}u\mathrm{d}(\mathrm{e}^u)$$

$$=\frac{2}{3}u\mathrm{e}^u-\int \frac{2}{3}\mathrm{e}^u\mathrm{d}u=\frac{2}{3}u\mathrm{e}^u-\frac{2}{3}\mathrm{e}^u+C$$

$$=\frac{2}{3}\mathrm{e}^{\sqrt{3x+9}}(\sqrt{3x+9}-1)+C.$$

习题 4-4 有理函数的积分

求下列不定积分：

1. $\int \frac{x^3}{x+3}\mathrm{d}x$.
2. $\int \frac{2x+3}{x^2+3x-10}\mathrm{d}x$.
3. $\int \frac{x+1}{x^2-2x+5}\mathrm{d}x$.
4. $\int \frac{\mathrm{d}x}{x(x^2+1)}$.
5. $\int \frac{3}{x^3+1}\mathrm{d}x$.
6. $\int \frac{x^2+1}{(x+1)^2(x-1)}\mathrm{d}x$.

由于$\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\cos x)\mathrm{d}x$，因此结论成立.

6. 若 $f(x)$ 是连续的奇函数，证明$\int_0^x f(t)\mathrm{d}t$ 是偶函数；若 $f(x)$ 是连续的偶函数，证明$\int_0^x f(t)\mathrm{d}t$ 是奇函数.

证　记 $F(x) = \int_0^x f(t)\mathrm{d}t$，则有

$$F(-x) = \int_0^{-x} f(t)\mathrm{d}t \xlongequal{t=-u} -\int_0^x f(-u)\mathrm{d}u,$$

当 $f(x)$ 为奇函数时，$F(-x) = \int_0^x f(u)\mathrm{d}u = F(x)$，故$\int_0^x f(t)\mathrm{d}t$ 是偶函数.

当 $f(x)$ 为偶函数时，$F(-x) = -\int_0^x f(u)\mathrm{d}u = -F(x)$，故$\int_0^x f(t)\mathrm{d}t$ 是奇函数.

7. 计算下列定积分：

(1) $\int_0^1 x\mathrm{e}^{-x}\mathrm{d}x$；　　(2) $\int_1^{\mathrm{e}} x\ln x\mathrm{d}x$；

(3) $\int_0^{\frac{2\pi}{\omega}} t\sin\omega t\mathrm{d}t$（$\omega$ 为常数）；　　(4) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x}{\sin^2 x}\mathrm{d}x$；

(5) $\int_1^4 \frac{\ln x}{\sqrt{x}}\mathrm{d}x$；　　(6) $\int_0^1 x\arctan x\mathrm{d}x$；

(7) $\int_0^{\frac{\pi}{2}} \mathrm{e}^{2x}\cos x\mathrm{d}x$；　　(8) $\int_1^2 x\log_2 x\mathrm{d}x$；

(9) $\int_0^{\pi} (x\sin x)^2\mathrm{d}x$；　　(10) $\int_1^{\mathrm{e}} \sin(\ln x)\mathrm{d}x$；

(11) $\int_{\frac{1}{\mathrm{e}}}^{\mathrm{e}} |\ln x|\mathrm{d}x$；　　(12) $\int_0^1 (1-x^2)^{\frac{m}{2}}\mathrm{d}x$（$m \in \mathbf{N}^+$）；

(13) $J_m = \int_0^{\pi} x\sin^m x\mathrm{d}x$（$m \in \mathbf{N}^+$）.

解　(1) $\int_0^1 x\mathrm{e}^{-x}\mathrm{d}x = -\int_0^1 x\mathrm{d}(\mathrm{e}^{-x}) = -[x\mathrm{e}^{-x}]_0^1 + \int_0^1 \mathrm{e}^{-x}\mathrm{d}x$

$$= -\mathrm{e}^{-1} + [-\mathrm{e}^{-x}]_0^1 = 1 - \frac{2}{\mathrm{e}}.$$

(2) $\int_1^{\mathrm{e}} x\ln x\mathrm{d}x = \int_1^{\mathrm{e}} \frac{\ln x}{2}\mathrm{d}(x^2) = \left[\frac{1}{2}x^2\ln x\right]_1^{\mathrm{e}} - \int_1^{\mathrm{e}} \frac{x}{2}\mathrm{d}x = \frac{\mathrm{e}^2+1}{4}.$

(3) $\int_0^{\frac{2\pi}{\omega}} t\sin\omega t\mathrm{d}t = -\frac{1}{\omega}\int_0^{\frac{2\pi}{\omega}} t\mathrm{d}(\cos\omega t) = -\frac{1}{\omega}[t\cos\omega t]_0^{\frac{2\pi}{\omega}} + \frac{1}{\omega}\int_0^{\frac{2\pi}{\omega}} \cos\omega t\mathrm{d}t$

$$= -\frac{2\pi}{\omega^2} + \frac{1}{\omega^2}[\sin\omega t]_0^{\frac{2\pi}{\omega}} = -\frac{2\pi}{\omega^2}.$$

(4) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x}{\sin^2 x}dx = -\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} xd(\cot x) = [-x\cot x]_{\frac{\pi}{4}}^{\frac{\pi}{3}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cot x dx$

$$= -\frac{\pi}{3\sqrt{3}} + \frac{\pi}{4} + [\ln \sin x]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \left(\frac{1}{4} - \frac{\sqrt{3}}{9}\right)\pi + \frac{1}{2}\ln\frac{3}{2}.$$

(5) $\int_1^4 \frac{\ln x}{\sqrt{x}}dx = \int_1^4 2\ln x d\sqrt{x} = [2\sqrt{x}\ln x]_1^4 - \int_1^4 \frac{2}{\sqrt{x}}dx$

$$= 8\ln 2 - [4\sqrt{x}]_1^4 = 4(2\ln 2 - 1).$$

(6) $\int_0^1 x\arctan x dx = \frac{1}{2}\int_0^1 \arctan x d(x^2)$

$$= \left[\frac{1}{2}x^2\arctan x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{1+x^2}dx$$

$$= \frac{\pi}{8} - \frac{1}{2}[x - \arctan x]_0^1 = \frac{\pi}{4} - \frac{1}{2}.$$

(7) $\int_0^{\frac{\pi}{2}} e^{2x}\cos x dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos x d(e^{2x})$

$$= \frac{1}{2}[e^{2x}\cos x]_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\frac{\pi}{2}} e^{2x}\sin x dx$$

$$= -\frac{1}{2} + \frac{1}{4}\int_0^{\frac{\pi}{2}} \sin x d(e^{2x})$$

$$= -\frac{1}{2} + \frac{1}{4}[e^{2x}\sin x]_0^{\frac{\pi}{2}} - \frac{1}{4}\int_0^{\frac{\pi}{2}} e^{2x}\cos x dx,$$

因此有

$$\int_0^{\frac{\pi}{2}} e^{2x}\cos x dx = \frac{1}{5}(e^{\pi} - 2).$$

(8) $\int_1^2 x\log_2 x dx = \frac{1}{2}\int_1^2 \log_2 x d(x^2)$

$$= \frac{1}{2}[x^2\log_2 x]_1^2 - \frac{1}{2}\int_1^2 \frac{x}{\ln 2}dx$$

$$= 2 - \frac{1}{4\ln 2}[x^2]_1^2 = 2 - \frac{3}{4\ln 2}.$$

(9) $\int_0^{\pi} (x\sin x)^2 dx = \frac{1}{2}\int_0^{\pi} x^2(1 - \cos 2x)dx$

$$= \frac{\pi^3}{6} - \frac{1}{4}\int_0^{\pi} x^2 d(\sin 2x)$$

$$= \frac{\pi^3}{6} - \frac{1}{4}[x^2\sin 2x]_0^{\pi} + \frac{1}{2}\int_0^{\pi} x\sin 2x dx$$

$$= \frac{\pi^3}{6} - \frac{1}{4}\int_0^{\pi} x\mathrm{d}(\cos 2x)$$

$$= \frac{\pi^3}{6} - \frac{1}{4}[x\cos 2x]_0^{\pi} + \frac{1}{4}\int_0^{\pi}\cos 2x\mathrm{d}x$$

$$= \frac{\pi^3}{6} - \frac{\pi}{4}.$$

(10) $\int_1^{e}\sin(\ln x)\mathrm{d}x \xlongequal{x=e^u} \int_0^1 e^u \sin u\mathrm{d}u = [e^u\sin u]_0^1 - \int_0^1 e^u\cos u\mathrm{d}u$

$$= e\sin 1 - [e^u\cos u]_0^1 - \int_0^1 e^u\sin u\mathrm{d}u$$

$$= e(\sin 1 - \cos 1) + 1 - \int_0^1 e^u\sin u\mathrm{d}u,$$

所以 $\int_1^{e}\sin(\ln x)\mathrm{d}x = \frac{e}{2}(\sin 1 - \cos 1) + \frac{1}{2}.$

(11) $\int_{\frac{1}{e}}^{e} |\ln x| \mathrm{d}x = -\int_{\frac{1}{e}}^{1}\ln x\mathrm{d}x + \int_1^{e}\ln x\mathrm{d}x$

$$= -[x\ln x]_{\frac{1}{e}}^{1} + \int_{\frac{1}{e}}^{1}\mathrm{d}x + [x\ln x]_1^{e} - \int_1^{e}\mathrm{d}x$$

$$= 2 - \frac{2}{e}.$$

(12) $\int_0^1 (1-x^2)^{\frac{m}{2}}\mathrm{d}x \xlongequal{x=\sin u} \int_0^{\frac{\pi}{2}} \cos^{m+1} x\mathrm{d}x$

$$= \begin{cases} \frac{m}{m+1}\cdot\frac{m-2}{m-1}\cdot\cdots\cdot\frac{1}{2}\cdot\frac{\pi}{2}, & m\text{ 为奇数},\\ \frac{m}{m+1}\cdot\frac{m-2}{m-1}\cdot\cdots\cdot\frac{2}{3}, & m\text{ 为偶数},\end{cases}$$

$$= \begin{cases} \frac{1\cdot 3\cdot 5\cdot\cdots\cdot m}{2\cdot 4\cdot 6\cdot\cdots\cdot(m+1)}\cdot\frac{\pi}{2}, & m\text{ 为奇数},\\ \frac{2\cdot 4\cdot 6\cdot\cdots\cdot m}{1\cdot 3\cdot 5\cdot\cdots\cdot(m+1)}, & m\text{ 为偶数}.\end{cases}$$

(13) 由教材本节的例 6,可得

$$J_m = \int_0^{\pi} x\sin^m x\,\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi}\sin^m x\mathrm{d}x.$$

而 $$\int_0^{\pi}\sin^m x\mathrm{d}x \xlongequal{x=\frac{\pi}{2}+t} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^m t\mathrm{d}t$$

$$= 2\int_0^{\frac{\pi}{2}}\cos^m t\mathrm{d}t = 2\int_0^{\frac{\pi}{2}}\sin^m x\mathrm{d}x,$$

故 $$J_m = \pi\int_0^{\frac{\pi}{2}}\sin^m x\mathrm{d}x.$$