

7. $\int \frac{x dx}{(x+1)(x+2)(x+3)}$. 8. $\int \frac{x^5+x^4-8}{x^3-x} dx$.
9. $\int \frac{dx}{(x^2+1)(x^2+x)}$. 10. $\int \frac{1}{x^4-1} dx$.
11. $\int \frac{dx}{(x^2+1)(x^2+x+1)}$. 12. $\int \frac{(x+1)^2}{(x^2+1)^2} dx$.
13. $\int \frac{-x^2-2}{(x^2+x+1)^2} dx$. 14. $\int \frac{dx}{3+\sin^2 x}$.
15. $\int \frac{dx}{3+\cos x}$. 16. $\int \frac{dx}{2+\sin x}$.
17. $\int \frac{dx}{1+\sin x+\cos x}$. 18. $\int \frac{dx}{2\sin x-\cos x+5}$.
19. $\int \frac{dx}{1+\sqrt[3]{x+1}}$. 20. $\int \frac{(\sqrt{x})^3-1}{\sqrt{x}+1} dx$.
21. $\int \frac{\sqrt{x+1}-1}{\sqrt{x+1}+1} dx$. 22. $\int \frac{dx}{\sqrt{x}+\sqrt{x}}$.
23. $\int \sqrt{\frac{1-x}{1+x}} dx$. 24. $\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}}$.

- 解 1. $\int \frac{x^3}{x+3} dx = \int \left(x^2 - 3x + 9 - \frac{27}{x+3} \right) dx$
 $= \frac{1}{3}x^3 - \frac{3}{2}x^2 + 9x - 27\ln|x+3| + C$.
2. $\int \frac{2x+3}{x^2+3x-10} dx = \int \frac{d(x^2+3x-10)}{x^2+3x-10} = \ln|x^2+3x-10| + C$.
3. $\int \frac{x+1}{x^2-2x+5} dx = \int \frac{x-1}{(x-1)^2+4} dx + \frac{1}{2} \int \frac{1}{\left(\frac{x-1}{2}\right)^2+1} dx$
 $= \frac{1}{2} \ln(x^2-2x+5) + \arctan \frac{x-1}{2} + C$.
4. $\int \frac{dx}{x(x^2+1)} = \int \left(\frac{1}{x} - \frac{x}{x^2+1} \right) dx = \ln|x| - \frac{1}{2} \int \frac{d(x^2+1)}{x^2+1}$
 $= \ln|x| - \frac{1}{2} \ln(x^2+1) + C$.
5. $\int \frac{3}{1+x^3} dx = \int \frac{3}{(1+x)(x^2-x+1)} dx = \int \left(\frac{1}{1+x} + \frac{2-x}{x^2-x+1} \right) dx$
 $= \ln|1+x| - \frac{1}{2} \int \frac{(x^2-x+1)'}{x^2-x+1} dx + \frac{3}{2} \int \frac{1}{x^2-x+1} dx$
 $= \ln|1+x| - \frac{1}{2} \ln(x^2-x+1) + \sqrt{3} \int \frac{1}{\left(\frac{2x-1}{\sqrt{3}}\right)^2+1} d\left(\frac{2x-1}{\sqrt{3}}\right)$

$$= \ln|1+x| - \frac{1}{2} \ln(x^2 - x + 1) + \sqrt{3} \arctan \frac{2x-1}{\sqrt{3}} + C.$$

$$\begin{aligned} 6. \int \frac{x^2+1}{(x+1)^2(x-1)} dx &= \int \left[\frac{1}{2(x-1)} + \frac{1}{2(x+1)} - \frac{1}{(x+1)^2} \right] dx \\ &= \frac{1}{2} \ln|x-1| + \frac{1}{2} \ln|x+1| + \frac{1}{x+1} + C \\ &= \frac{1}{2} \ln|x^2-1| + \frac{1}{x+1} + C. \end{aligned}$$

$$\begin{aligned} 7. \int \frac{x dx}{(x+1)(x+2)(x+3)} &= \int \left[-\frac{1}{2(x+1)} + \frac{2}{x+2} - \frac{3}{2(x+3)} \right] dx \\ &= -\frac{1}{2} \ln|x+1| + 2 \ln|x+2| - \frac{3}{2} \ln|x+3| + C. \end{aligned}$$

$$\begin{aligned} 8. \int \frac{x^5+x^4-8}{x^3-x} dx &= \int \left(x^2+x+1 + \frac{8}{x} - \frac{3}{x-1} - \frac{4}{x+1} \right) dx \\ &= \frac{x^3}{3} + \frac{x^2}{2} + x + 8 \ln|x| - 3 \ln|x-1| - 4 \ln|x+1| + C. \end{aligned}$$

$$\begin{aligned} 9. \int \frac{dx}{(x^2+1)(x^2+x)} &= \int \left[\frac{1}{x} - \frac{1}{2(x+1)} - \frac{1+x}{2(x^2+1)} \right] dx \\ &= \ln|x| - \frac{1}{2} \ln|x+1| - \frac{1}{2} \arctan x - \frac{1}{4} \int \frac{d(x^2+1)}{x^2+1} \\ &= \ln|x| - \frac{1}{2} \ln|x+1| - \frac{1}{2} \arctan x - \frac{1}{4} \ln(x^2+1) + C. \end{aligned}$$

$$\begin{aligned} 10. \int \frac{1}{x^4-1} dx &= \int \frac{1}{(x-1)(x+1)(x^2+1)} dx \\ &= \frac{1}{4} \int \frac{1}{x-1} dx - \frac{1}{4} \int \frac{1}{x+1} dx - \frac{1}{2} \int \frac{1}{x^2+1} dx \\ &= \frac{1}{4} \ln \left| \frac{x-1}{x+1} \right| - \frac{1}{2} \arctan x + C. \end{aligned}$$

$$\begin{aligned} 11. \int \frac{dx}{(x^2+1)(x^2+x+1)} &= \int \left(\frac{-x}{x^2+1} + \frac{x+1}{x^2+x+1} \right) dx \\ &= -\frac{\ln(x^2+1)}{2} + \frac{1}{2} \int \frac{d(x^2+x+1)}{x^2+x+1} + \\ &\quad \frac{1}{2} \int \frac{1}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} dx \\ &= -\frac{\ln(x^2+1)}{2} + \frac{\ln(x^2+x+1)}{2} + \frac{1}{\sqrt{3}} \arctan \frac{2x+1}{\sqrt{3}} + C. \end{aligned}$$

$$12. \int \frac{(x+1)^2}{(x^2+1)^2} dx = \int \frac{x^2+1}{(x^2+1)^2} dx + \int \frac{2x dx}{(x^2+1)^2}$$

$$= \arctan x - \frac{1}{x^2+1} + C.$$

$$\begin{aligned} 13. \int \frac{-x^2-2}{(x^2+x+1)^2} dx &= \int \left[-\frac{1}{x^2+x+1} + \frac{x-1}{(x^2+x+1)^2} \right] dx \\ &= -\int \frac{1}{x^2+x+1} dx + \frac{1}{2} \int \frac{d(x^2+x+1)}{(x^2+x+1)^2} - \frac{3}{2} \int \frac{1}{(x^2+x+1)^2} dx, \end{aligned}$$

令 $u = x + \frac{1}{2}$, 并记 $a = \frac{\sqrt{3}}{2}$, 则

$$\begin{aligned} \int \frac{1}{(x^2+x+1)^2} dx &= \int \frac{1}{(u^2+a^2)^2} du = \frac{1}{2a^2} \left[\frac{u}{u^2+a^2} + \int \frac{1}{u^2+a^2} du \right] \\ &= \frac{u}{2a^2(u^2+a^2)} + \frac{1}{2a^2} \int \frac{1}{u^2+a^2} du, \end{aligned}$$

由此得

$$\begin{aligned} &\int \frac{1}{x^2+x+1} dx + \frac{3}{2} \int \frac{1}{(x^2+x+1)^2} dx \\ &= \int \frac{1}{u^2+a^2} du + \frac{3}{2} \left[\frac{u}{2a^2(u^2+a^2)} + \frac{1}{2a^2} \int \frac{1}{u^2+a^2} du \right] \\ &= \frac{3u}{4a^2(u^2+a^2)} + \left(\frac{3}{4a^2} + 1 \right) \int \frac{1}{u^2+a^2} du \\ &= \frac{3u}{4a^2(u^2+a^2)} + \frac{1}{a} \left(\frac{3}{4a^2} + 1 \right) \arctan \frac{u}{a} + C_1 \\ &= \frac{2x+1}{2(x^2+x+1)} + \frac{4}{\sqrt{3}} \arctan \frac{2x+1}{\sqrt{3}} + C_1, \end{aligned}$$

因此有

$$\begin{aligned} \int \frac{-x^2-2}{(x^2+x+1)^2} dx &= -\frac{1}{2(x^2+x+1)} - \frac{2x+1}{2(x^2+x+1)} - \\ &\quad \frac{4}{\sqrt{3}} \arctan \frac{2x+1}{\sqrt{3}} + C \\ &= -\frac{x+1}{x^2+x+1} - \frac{4}{\sqrt{3}} \arctan \frac{2x+1}{\sqrt{3}} + C. \end{aligned}$$

$$\begin{aligned} 14. \int \frac{dx}{3+\sin^2 x} &= -\int \frac{d(\cot x)}{3\csc^2 x+1} \xrightarrow{u=\cot x} \int \frac{du}{3u^2+4} \\ &= -\frac{1}{2\sqrt{3}} \arctan \frac{\sqrt{3}u}{2} + C \\ &= -\frac{1}{2\sqrt{3}} \arctan \frac{\sqrt{3}\cot x}{2} + C. \end{aligned}$$

15. 令 $u = \tan \frac{x}{2}$, 则

$$\begin{aligned}\int \frac{dx}{3 + \cos x} &= \int \frac{1}{3 + \frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2} du = \int \frac{1}{2+u^2} du \\ &= \frac{1}{\sqrt{2}} \arctan \frac{u}{\sqrt{2}} + C = \frac{1}{\sqrt{2}} \arctan \frac{\tan \frac{x}{2}}{\sqrt{2}} + C.\end{aligned}$$

16. 令 $u = \tan \frac{x}{2}$, 则

$$\begin{aligned}\int \frac{dx}{2 + \sin x} &= \int \frac{1}{2 + \frac{2u}{1+u^2}} \cdot \frac{2}{1+u^2} du = \int \frac{1}{u^2 + u + 1} du \\ &= \int \frac{1}{\left(u + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} du = \frac{2}{\sqrt{3}} \arctan \frac{2u+1}{\sqrt{3}} + C \\ &= \frac{2}{\sqrt{3}} \arctan \frac{2 \tan \frac{x}{2} + 1}{\sqrt{3}} + C.\end{aligned}$$

17. 令 $u = \tan \frac{x}{2}$, 则

$$\begin{aligned}\int \frac{dx}{1 + \sin x + \cos x} &= \int \frac{1}{1 + \frac{2u}{1+u^2} + \frac{1-u^2}{1+u^2}} \cdot \frac{2}{1+u^2} du \\ &= \int \frac{du}{1+u} = \ln |1+u| + C = \ln \left| 1 + \tan \frac{x}{2} \right| + C.\end{aligned}$$

18. 令 $u = \tan \frac{x}{2}$, 则

$$\begin{aligned}\int \frac{dx}{2 \sin x - \cos x + 5} &= \int \frac{1}{\frac{4u}{1+u^2} - \frac{1-u^2}{1+u^2} + 5} \cdot \frac{2}{1+u^2} du \\ &= \int \frac{1}{3u^2 + 2u + 2} du \\ &= \frac{1}{3} \int \frac{1}{\left(u + \frac{1}{3}\right)^2 + \left(\frac{\sqrt{5}}{3}\right)^2} d\left(u + \frac{1}{3}\right) \\ &= \frac{1}{\sqrt{5}} \arctan \frac{3u+1}{\sqrt{5}} + C \\ &= \frac{1}{\sqrt{5}} \arctan \frac{3 \tan \frac{x}{2} + 1}{\sqrt{5}} + C.\end{aligned}$$

19. 令 $u = \sqrt[3]{x+1}$, 即 $x = u^3 - 1$, 则

$$\begin{aligned}\int \frac{dx}{1+\sqrt[3]{x+1}} &= \int \frac{3u^2}{1+u} du = \int \left(3u - 3 + \frac{3}{1+u}\right) du \\ &= \frac{3}{2}u^2 - 3u + 3\ln|1+u| + C \\ &= \frac{3}{2}\sqrt[3]{(x+1)^2} - 3\sqrt[3]{x+1} + 3\ln|1+\sqrt[3]{x+1}| + C.\end{aligned}$$

$$\begin{aligned}20. \int \frac{(\sqrt{x})^3 - 1}{\sqrt{x} + 1} dx &= \int \left(x - \sqrt{x} + 1 - \frac{2}{\sqrt{x} + 1}\right) dx \\ &= \frac{x^2}{2} - \frac{2}{3}x\sqrt{x} + x - \int \frac{4t}{t+1} dt \\ &= \frac{x^2}{2} - \frac{2}{3}x\sqrt{x} + x - 4\int \left(1 - \frac{1}{t+1}\right) dt \\ &= \frac{x^2}{2} - \frac{2}{3}x\sqrt{x} + x - 4\sqrt{x} + 4\ln(\sqrt{x}+1) + C.\end{aligned}$$

21. 令 $u = \sqrt{x+1}$, 即 $x = u^2 - 1$, 则

$$\begin{aligned}\int \frac{\sqrt{x+1} - 1}{\sqrt{x+1} + 1} dx &= \int \frac{u-1}{u+1} \cdot 2u du = 2\int \left(u - 2 + \frac{2}{u+1}\right) du \\ &= u^2 - 4u + 4\ln|u+1| + C \\ &= x - 4\sqrt{x+1} + 4\ln(\sqrt{x+1}+1) + C.\end{aligned}$$

22. 令 $u = \sqrt[4]{x}$, 即 $x = u^4$, 则

$$\begin{aligned}\int \frac{dx}{\sqrt{x} + \sqrt[4]{x}} &= \int \frac{1}{u^2 + u} \cdot 4u^3 du = 4\int \left(u - 1 + \frac{1}{u+1}\right) du \\ &= 2u^2 - 4u + 4\ln|u+1| + C \\ &= 2\sqrt{x} - 4\sqrt[4]{x} + 4\ln(\sqrt[4]{x}+1) + C.\end{aligned}$$

23. 方法一

令 $u = \sqrt{\frac{1-x}{1+x}}$, 即 $x = \frac{1-u^2}{1+u^2}$, 则

$$\begin{aligned}\int \sqrt{\frac{1-x}{1+x}} \cdot \frac{dx}{x} &= \int u \cdot \frac{1+u^2}{1-u^2} \cdot \frac{-4u}{(1+u^2)^2} du = \int \frac{-4u^2}{(1-u^2)(1+u^2)} du \\ &= \int \left(\frac{2}{1+u^2} - \frac{1}{1-u} - \frac{1}{1+u}\right) du \\ &= 2\arctan u + \ln|1-u| - \ln|1+u| + C \\ &= 2\arctan\sqrt{\frac{1-x}{1+x}} + \ln\left|\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right| + C.\end{aligned}$$

方法二

$$\begin{aligned}\int \sqrt{\frac{1-x}{1+x}} \frac{dx}{x} &= \int \frac{1-x}{x\sqrt{1-x^2}} dx \stackrel{x=\sin u}{=} \int \frac{1-\sin u}{\sin u} du \\ &= \int \csc u du - \int du = \ln|\csc u - \cot u| - u + C \\ &= \ln \frac{1-\sqrt{1-x^2}}{|x|} - \arcsin x + C.\end{aligned}$$

$$24. \int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}} = \int \frac{1}{x^2-1} \sqrt[3]{\frac{x+1}{x-1}} dx,$$

令 $u = \sqrt[3]{\frac{x+1}{x-1}}$, 即 $x = \frac{u^3+1}{u^3-1}$, 得到

$$\begin{aligned}\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}} &= \int \frac{u}{\left(\frac{u^3+1}{u^3-1}\right)^2 - 1} \cdot \frac{-6u^2}{(u^3-1)^2} du = -\frac{3}{2} \int du \\ &= -\frac{3}{2}u + C = -\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}} + C.\end{aligned}$$

习题 4-5

积分表的使用

利用积分表计算下列不定积分:

1. $\int \frac{dx}{\sqrt{4x^2-9}}$

2. $\int \frac{1}{x^2+2x+5} dx.$

3. $\int \frac{dx}{\sqrt{5-4x+x^2}}$

4. $\int \sqrt{2x^2+9} dx.$

5. $\int \sqrt{3x^2-2} dx.$

6. $\int e^{2x} \cos x dx.$

7. $\int x \arcsin \frac{x}{2} dx.$

8. $\int \frac{dx}{(x^2+9)^2}.$

9. $\int \frac{dx}{\sin^3 x}.$

10. $\int e^{-2x} \sin 3x dx.$

11. $\int \sin 3x \sin 5x dx.$

12. $\int \ln^3 x dx.$

13. $\int \frac{1}{x^2(1-x)} dx.$

14. $\int \frac{\sqrt{x-1}}{x} dx.$

15. $\int \frac{1}{(1+x^2)^2} dx.$

16. $\int \frac{1}{x\sqrt{x^2-1}} dx.$

17. $\int \frac{x}{(2+3x)^2} dx.$

18. $\int \cos^6 x dx.$