MATH 6A 19F with Prof. Pan

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Office Hour: W 10:00-11:00, South Hall 6432G
http://web.math.ucsb.edu/~danninglu/teaching/math_6A_F19/math_6A_F19.html

1 For lecture on 9/26

1. Find the vector pointing from $(0, 2, 3)$ to $(2, 4, 1)$.
2. Find the distance between the two points $(1, 2, 1)$ and $(3, 4, -1)$.
3. Find the dot product between the vectors $\langle 3, 3, -1\rangle$ and $\langle -2, 8, 2\rangle$
4. What is the angle between the vectors $\langle 1, 0, -2\rangle$ and $\langle 3, 5, 4\rangle$?
5. Find a real number a such that the vectors $\langle 3, a, 4\rangle$ and $\langle a - 3, 1, 1\rangle$ are perpendicular to each other.
6. Let $\alpha = \langle 3, -1, -1\rangle$ and $\beta = \langle 1, 2, 5\rangle$ be vectors. Find $\alpha \times \beta$.
7. Find the area of the parallelogram with vertices $(0, 0, 0)$, $(1, 3, 5)$, $(-2, -2, 1)$ and $(-1, 1, 6)$.
8. Find the area of the triangle with verticees $(1, 2, 3)$, $(4, 5, -1)$ and $(0, 3, 0)$.
9. Let $\overrightarrow{u} = \langle -3, 3, 2\rangle$, $\overrightarrow{v} = \langle -2, -4, 2\rangle$, and $\overrightarrow{w} = \langle 2, 3, 1\rangle$. Find
   (a) $\overrightarrow{u} \cdot (\overrightarrow{v} + \overrightarrow{w})$.
   (b) $\overrightarrow{u} \cdot (\overrightarrow{v} \times \overrightarrow{w})$.
   (c) $\overrightarrow{u} \times (\overrightarrow{v} + \overrightarrow{w})$.
   (d) $(\overrightarrow{u} \cdot \overrightarrow{v})\overrightarrow{w}$.
10. Find the parametric equation of the line passing points $(3, 4, 5)$ and $(1, 1, 1)$.
11. Find the parametric equation of the line segment with end points $(2, 5, -3)$ and $(1, -1, 1)$.
12. Find the plane that passes through the point $(1, 2, 3)$ and perpendicular with the vector $\langle -2, -3, -1\rangle$.
13. Find the plane that passes through the points $(1, 2, 3)$, $(4, 5, -1)$ and $(0, 3, 0)$.

2 For lecture on 10/1

1. Find the domain for the following functions.

   (a) $f(x, y) = \frac{\sqrt{x-2y}}{x}$.

   (b) $f(x, y) = \sqrt{\frac{x-2y}{x}}$.

   (c) $f(x, y) = \ln(16 - x^2 - y^2)$.

2. Find the limit of $f(x, y)$ as $(x, y) \to (0, 0)$, if it exists; or state the reason if it does not exist:

   (a) $f(x, y) = \frac{x^3y - xy^3 - x}{1 - xy}$.

   (b) $f(x, y) = \frac{2x^2 - y^2}{x^2 + 2y^2}$.

   (c) $f(x, y) = \frac{\sin(3x^2 + y^2)}{x^2 + 2y^2}$.

   (d) $f(x, y) = \frac{xy^3}{x^2 + y^6}$.

   (e) * $f(x, y) = (2x^2 + y^2)e^{\frac{1}{-y^2 - 2x^2}}$.

   (f) * $f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}$.

3. If $f(x, y, z) = (x - 2y, z + 3x)$ and $g(u, v) = (v, -u, \sqrt{u + v})$. Find $f \circ g$ and $g \circ f$, and state the domain for these two functions.

3 For Lecture on 10/3

1. Find the partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$.

   (a) $f(x, y) = x^y + y \sin x$

   (b) $f(x, y) = xe^{x^2+2xy-y^2}$

   (c) $f(x, y) = e^{xy} \cos x \ln(y - x^2)$

   (d) $f(x, y) = \arctan(x/y)$

2. Find the gradient for $f(x, y, z) = (y^2 + 1)e^{xyz}$.

3. Find the Jacobian for $F(x, y) = (y, -x \sin y, e^x)$.

4. Find the directional derivative of $f(x, y) = 3x^2 - 2xy$ in the direction of $(3, 4)$ at point (1,3).

5. Find linear approximation of $f(x, y) = e^{xy} - xy^3$ at point (1,0).

6. Find the equation of the plane tangent to the sphere $x^2 + y^2 + z^2 = 1$ at the point (x_0, y_0, z_0). For extra practice, do this in two ways, one by solving for z and consider the two cases $z_0 \geq 0$ and $z_0 < 0$, and the other by viewing the sphere as a level surface.

7. A hill has the shape of the graph $z = f(x, y) = 40 - x^2 - 2y^2$.

   (a) A ball is held at (1,2,31) on the hill. When it is released, it will move in the direction of the steepest descend. Give a unit vector $\vec{u}$ (in 3 dimension) in the direction of the ball when it is released.

   (b) A hiker is also at point (1,2,31), and wants to hike uphill. However, the hiker can not go in any paths that have slope bigger than 1. Assume that the angle between the direction of the path (in 2d, projected onto the xy-plane) and ∇f is θ. Give the range for θ.

8. Estimate $\cos(0.01)e^{0.02}$.

9. Use differentials to estimate the amount of tin in a closed tin can with diameter 3 inch and height 4 inch if the top and bottom is 0.02 inch thick and the side is 0.01 inch thick.

10. Suppose you need to know an equation of the tangent plane to a surface S at the point $P(2, 1, 3)$. You don't have an equation for S but you know that the curves

$$r_1(t) = (2 + 3t, 1 - t^2, 3 - 4t + t^2)$$

$$r_2(t) = (1 + t^2, 2t^3 - 1, 2t + 1)$$

    both lie on S. Find an equation of the tangent plane at P.

4 For Lecture on 10/8

1. Let $F(x, y, z) = (xy, z \sin y)$ and $G(u, v) = (u + v, u^2, e^v)$. Find the Jacobians DF, DG and $D(F \circ G)$.

2. Find the tangent plane of the function $u(s, t)$ given by $u = x^4 y + y^2 z^3$ and $x = se^t$, $y = s^2 e^{-t}$ and $z = s \sin t$ at point $(s, t) = (1, 0)$.

3. * If $c(t)$ is a differentiable vector-valued function with the restriction that length of $c(t)$ remain constant, prove that $c(t)$ is perpendicular to $c'(t)$.

5 No problem set for Lecture on 10/10

6 For lecture on 10/15

1. Find the maximal and minimal of the functions in the given region.
   (a) $f(x, y) = x^2 + xy + y^2 + y$, $-1 \le x \le 1$, $-1 \le y \le 1$.
   (b) $f(x, y) = e^x \cos y$, $-1 \le x \le 0$, $-\pi/2 \le y \le 5\pi/2$.
   (c) $f(x, y) = (x - y)(1 - xy)$, $0 \le x \le 2$, $0 \le y \le 2$.

2. Find the absolute maximum and minimum values of $f(x, y) = x^2 + y^2 - 2x$ on the set D, which is the closed triangular region with vertices $(2, 0)$, $(0, 2)$ and $(0, -2)$.

3. Use Lagrange multipliers to find the extreme values of the function subject to the given constraint.
   (a) $f(x, y) = x^2 - y^2$; $x^2 + y^2 = 1$.
   (b) $f(x, y) = xe^y$; $x^2 + 2y^2 = 2$.
   (c) $f(x, y, z) = \ln(x^2 + 1) + \ln(y^2 + 1) + \ln(z^2 + 1)$; $x^2 + y^2 + z^2 = 12$.

4. Given function $f(x, y) = x^2 + y^2 + 4x - 4y$ and the region D given by $x^2 + y^2 \le 9$. Find the maximum and minimum of the function on the region D.

5. Find the points on the surface $y^2 = 9 + xz$ that are closest to the origin. (As an exercise, use two methods to solve this question. You can view z as a function of x and y, or you can use the Lagrange Multipliers.)

6. *Find the points on both the plane $x + y + 2z = 2$ and the paraboloid $z = x^2 + y^2$ that are nearest to and farthest from the origin.

7. **Find the maximum and minimum values of
$$f(x, y, z) = ye^{x-z}$$
subject to the restrictions $9x^2 + 4y^2 + 36z^2 = 36$ and $xy + yz = 1$.

8. ***
   (a) Find the maximum value of
$$f(x_1, ..., x_n) = \sqrt[n]{x_1 x_2 ... x_n}$$
given that $x_1, x_2, ..., x_n$ are positive numbers and $x_1 + x_2 + ... + x_n = c$, where c is a constant.
   (b) Deduce from part (a) that if $x_1, x_2, ..., x_n$ are positive numbers, then
$$\sqrt[n]{x_1 x_2 ... x_n} \le \frac{x_1 + x_2 + ... + x_n}{n}.$$
Under what circumstances the equality holds?

7 For lecture on 10/17

1. Let $\vec{F}(x, y, z) = (\sin x, \cos z, \tan y)$ be a vector field. Find the following:
   (a) $\nabla \cdot \vec{F}$.
   (b) $\nabla \times \vec{F}$.
   (c) The Jacobian of $\vec{F}$.
   (d) If $\vec{u}(t)$ is a curve in 3-dimension space, with $\vec{u}(0) = (1, 2, 3)$ and $\vec{u}'(0) = (4, 5, 6)$, find $(\vec{F} \circ \vec{u})'(0)$.
2. Find the divergence and curl for the vector field $F(x, y) = (e^x \sin y, -\cos x)$.
3. Find and classify all local maxima and minima of the function
$$f(x, y) = x^4 + y^4 - 4xy - 38.$$
4. Find the absolute maximum and minimum of f in region D, where
$$f(x, y) = \sqrt[3]{x^3 - 48x + y^3 - 12y},$$
$$D = \text{Closed triangle region with vertices } (-2, 3), (5, 3), (5, -4).$$

8 For lecture on 10/23

1. (reference Midterm practice question 5)Let γ be the graph of $y = e^{-t^2}.t \in [-2, 2]$ in $\mathbb{R}^2$.
   (a) Give a parametric equation for γ.
   (b) Setup the integral of the length of γ. (You don't need to evaluate the integral).
   (c) Let S be the surface of revolution generated by rotating γ around the x-axis. Write a parametric equation for S.
   (d) Give the tangent plane of S at point $P(1, \frac{1}{2e}, \frac{-\sqrt{3}}{2e})$.
2. Given $A(2, 1, 7)$, $B(4, 0, 5)$, $C(1, 0, 1)$, find the parametric equation of the triangle ABC in $\mathbb{R}^3$.

9 For lecture on 10/31

1. Compute $\int_{\mathbf{c}} f\, ds$.

   (a) $f(x,y,z) = 2xy - z$, $\mathbf{c}(t) =< 2\sin t, 2\cos t, 7t >$, $0 \le t \le 2\pi$.

   (b) $f(x,y) = x^3y^{30}$, $\mathbf{c}$ is the unit circle in $\mathbb{R}^2$.

   (c) $f(x,y,z) = x + 2y - z^2$, $\mathbf{c}$ consists of the path $t\mathbf{i} + t^2\mathbf{j}$ from (0,0,0) to (1,1,0), followed by the straight line to $(1,-1,1)$.

   (d) $^*f(x,y) = x^3 + y^3$, $\mathbf{c}$ is the part of the curve $x^{2/3} + y^{2/3} = 1$ in the first quadrant.

   (e) Find $\int_C (x^5y^{18} + x^7y^{16} + 1)ds$, where C is the upper half of the unit circle $x^2 + y^2 = 9$.

2. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

   (a) C is the line segment from $(2,3)$ to $(0,3)$ and $\mathbf{F} = \langle x, -y\rangle$.

   (b) C is the line segment from $(5,0,2)$ to $(5,3,4)$ and $\mathbf{F} = \langle z, -y, x\rangle$.

   (c) C is the curve on $y = e^x$ from $(2, e^2)$ to $(0,1)$ and $\mathbf{F} = \langle x^2, -y\rangle$.

   (d) C is part of the circle of radius 3 centred at origin, from $(3,0)$ to $(0,3)$ and $\mathbf{F} = \langle 1, -y\rangle$.

   (e) C is part of the curve $x = \cos(y)$ from $(1, 2\pi)$ to $(1,0)$ and $\mathbf{F} = \langle y, 2x\rangle$.

3. * Find $\int_C \cos(x+z)dx + 2yze^{y^2z}dy + (\cos(x+z) + y^2e^{y^2z})dz$, where C is the *Slinky* curve, given by $\vec{c}(t) = (\sin(40t), (2+\cos(40t))\sin(t), (2+\cos(40t))\cos(t))$ with $0 \le t \le \pi$.

10 For lecture on 11/5

1. State which of the following sets are connected and/or simply-connected.

   (a) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \neq 1\}$;

   (b) $\mathbb{R}^3 - \{(x, y, z) : x^2 + y^2 = 1, z = 1\}$;

   (c) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 > 1\}$;

   (d) $\mathbb{R}^3 - \{(\cos t, \sin t, t) : 0 \leq t \leq \pi\}$;

   (e) $\mathbb{R}^3 - \{(\cos t, \sin t, t) : t \in \mathbb{R}\}$;

   (f) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z < 1\}$.

2. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = (y^2 \cos x, 2y \sin x)$, C is any path starting at (1,1) and ending at (1,3).

3. Determine whether $\mathbf{F}$ is a gradient vector field, and if so, find a function f such that $\mathbf{F} = \nabla f$.

   (a) $\mathbf{F} = (e^x \sin y + x^2, e^x \cos y + y^2)$;

   (b) $\mathbf{F} = (4x^2 - 4y^2 + x, 7xy + \ln y)$;

   (c) $\mathbf{F} = (xy^2 + 3x^2 \ln x + x^2, x^3/y)$;

   (d) $\mathbf{F} = (yz + e^x \sin z, xz + y^2 - e^y, xy + e^x \cos z)$;

   (e) $\mathbf{F} = (y \cos(xy), x \cos(xy) - z \sin y, \cos y)$.

4. Let $\mathbf{F} = (-y/(x^2 + y^2), x/(x^2 + y^2), 1$.

   (a) Check that $\nabla \times \mathbf{F} = \mathbf{0}$ in its domain.

   (b) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by $\mathbf{r}(t) = (\cos t, \sin t, 0)$, with $0 \leq t \leq 2\pi$.

   (c) Based on your calculation, is $\mathbf{F}$ a gradient vector field? Explain your judgement.

11 For lecture on 11/7

1. Evaluate the integrals.

   (a) $\int_0^2 \int_0^x (x + 2y)dydx$.

   (b) $\int_0^1 \int_{1-x}^{1+x} (24x^2 + 4y)dydx$.

   (c) $\iint_D xydA$ where D is the triangle with vertices (0,0),(6,0),(0,1).

2. Compute the solid under the graph of $f(x, y) = 3 + 2x^2 + 7y$ over the rectangle $R = \{(x, y)|1 \leq x \leq 3, 0 \leq y \leq 1\}$.

3. Reverse order of integration.

   (a) $\int_0^1 \int_x^{2x} e^{y-x}dydx$.

   (b) $\int_0^{2\sqrt{3}} \int_{y^2/6}^{\sqrt{16-y^2}} 1dxdy$.

   (c) $\int_0^7 \int_{x^2-6x}^{x} f(x, y)dydx$.

   (d) $\int_1^2 \int_x^{x^3} f(x, y)dydx + \int_2^8 \int_x^8 f(x, y)dydx$.

4. Evaluate the integral by reversing the order of integration.

$$\int_0^1 \int_{7y}^7 e^{x^2} dxdy.$$

12 For lecture on 11/12

1. Evaluate $\iint_D xy dA$, where D is the triangle region with vertices $(0,1)$, $(1,2)$, $(4,1)$.

2. Find Volume of solid.

   (a) Tetrahedron in first octant bounded by coordinate planes and $z = 7 - 3x - 2y$.

   (b) Use a double integral to determine the volume of the region bounded by $z = 6 - 5x^2$ and the planes $y = 2x$, $y = 2$,$x = 0$ and the xy-plane.

   (c) Solid inside both the sphere $x^2 + y^2 + z^2 = 3$ and above paraboloid $2z = x^2 + y^2$.

13 For lecture on 11/14

1. Convert from Cartesian coordinate to polar coordinate before integration.

   (a) Find $\iint_D xy\,dxdy$ where D is the region bounded by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 1$.

   (b) Find the volume of the solid bounded by the paraboloid $z = 4-x^2-y^2$ and the xy-plane.

   (c) Find the volume inside the sphere and outside the cylinder $x^2+y^2 = 9$.

2. Evaluate $\iint_D xydA$, where D is the triangle region with vertices $(0, 1)$, $(1, 2)$, $(4, 1)$.

3. Evaluate $\iint_R \arctan \frac{y}{x} dA$, where $R = \{(x, y)|1 \le x^2 + y^2 \le 4, 0 \le y \le x\}$.

4. Evaluate $\iint_D xydA$, where D is the interior of the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$, by using the change of variables $x = 3r\cos\theta$, $y = 4r\sin\theta$.

14 For lecture on 11/19

1. Evaluate the integral $\iiint_E f dV$ with f and E given below. You may need to draw the region for your integral.

   (a) $f = xy + z^2$. $E = \{(x, y, z)|0 \le x \le 2, 0 \le y \le 1, 0 \le z \le 3\}$.

   (b) $f = y$. $E = \{(x, y, z)|0 \le x \le 3, 0 \le y \le x, x - y \le z \le x + y\}$.

   (c) $f = \sin y$. E is the solid below $z = x$ and above the triangle region with vertices $(0, 0, 0), (0, \pi, 0), (\pi, 0, 0)$.

   (d) $f = x - y$. E is enclosed by the surfaces $z = x^2 - 1$, $z = 1 - x^2$, $y = 0$ and $y = 2$.

   (e) $f = xz$. E is the tetrahedron with vertices $(0, 0, 0), (0, 1, 0), (0, 0, 2), (3, 0, 0)$.

2. Sketch the solid and function that is being integrated by the formula given below.

   (a) $\int_{-\pi/2}^{\pi/2} \int_0^2 \int_0^{r^2} z dz dr d\theta$.

   (b) $\int_0^2 \int_0^{2\pi} \int_0^r rz \sin\theta dz d\theta dr$.

3. Evaluate the integral $\iiint_E f dV$ with f and E given below. You may need to draw the region for your integral.

   (a) $f = \sqrt{x^2 + y^2}$. E is the solid that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.

   (b) $f = z$. E is enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

   (c) $f = x + y + z$. E is the solid in the first octant that lies under the paraboloid $y = 4 - x^2 - z^2$.

4. Sketch the solid and function that is being integrated by the formula given below.

   (a) $\int_0^{\pi/6} \int_0^{\pi/2} \int_0^3 \rho d\rho d\theta d\phi$.

   (b) $\int_0^{\pi/4} \int_0^{2\pi} \int_0^{\sec\phi} \rho \cos\theta \sin\phi d\rho d\theta d\phi$.

5. Evaluate the integral $\iiint_E f dV$ with f and E given below. You may need to draw the region for your integral.

   (a) $f = y^2 z^2$. E is the solid that lies above the cone $\phi = \pi/3$ and below the sphere $\rho = 1$.

   (b) $f = xe^{x^2+y^2+z^2}$. E is the solid in the first octant and between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 8$.

6. Evaluate $\iiint f\, dV$, where $f(x, y, z) = x - 2y + 3z$ and V is the tetrahedron with vertices (0,1,0), (1,0,1), (0,0,5) and (2,1,7).

15 For lecture on 11/21

Evaluate the surface integral $\iint_S f\, dS$.

1. $f = xyz$, S is the triangle with vertices (0,1,0), (1,0,1), (2,3,0).
2. $f = z^2$, S is the cone $z = \sqrt{3x^2 + 3y^2}$ with $1 \leq x^2 + y^2 \leq 4$.
3. $f = 2z - y + x^2$, S is the side of cylinder, whose rotation axis is the x-axis, with radius 1 and $1 \leq x \leq 2$.
4. $f = xy + 3z$, S is part of the sphere centered at original point with radius 3 in the first octant.
5. $f = 1$, S is part of the cone $z = \sqrt{x^2 + y^2}$, contained within the cylinder $y^2 + z^2 \leq 49$.
6. $f = \frac{xy}{(y^2+z^2)\sqrt{4x^2(y^2+z^2)+1}}$, S is the surface of revolution, generated by $y = e^{-x^2}$ on the xy-plane rotating about the x-axis, in the first octant and with $z \geq 1/e$.

16 For lecture on 12/3

1. Evaluate the surface integral $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F} = (x^2y, xy^3, 2xyz)$ and S is the surface given by $z = 2x^2y$, $0 \leq x \leq 1$, $0 \leq y \leq 2$, oriented downwards.

2. Compute $\iint_S \vec{F} \cdot d\vec{S}$, where $\vec{F} = (2x + 3y, -4y - 3z, 4z)$ and S consists of the parabolid $z = x^2 + y^2$, $0 \leq z \leq 1$ oriented upwards, and the disk $x^2 + y^2 <= 1$, $z = 1$ oriented downwards.
(a) Compute directly. (b) By using divergence theorem.