# MATH 6A WORKSHEET 3 

DANNING LU

## 1. Limit and Continuity

1. Find the limit of $f(x, y)$ as $(x, y) \rightarrow(0,0)$, if it exists; or state the reason if it does not exist:
(1) $f(x, y)=\frac{x^{3} y-x y^{3}-x}{1-x y}$
(2) $f(x, y)=\frac{x y}{\left(x^{2}+y^{2}\right)^{3 / 2}}$
(3) $f(x, y)=\frac{\sin (3 x-2 y+x y)}{3 x-2 y+x y}$
(4) $f(x, y)=\frac{2 x^{2}-y^{2}}{x^{2}+2 y^{2}}$
(5) $f(x, y)=(2 x-y) e^{\frac{1}{y-2 x}}$
(6) $f(x, y)=\frac{x y}{\sqrt{x^{2}+y^{2}}}$
(7) $f(x, y)=\frac{\sin \left(3 x^{2}+y^{2}\right)}{x^{2}+2 y^{2}}$

Answers:
(1) Limit exists and equal to 1 , since the denominator does not vanish at $(x, y)=$ $(0,0)$.
(2) Answer is not given since this is a homework problem.
(3) Let $u=3 x-2 y+x y$ then the limit will equal to $\lim _{u \rightarrow 0} \frac{\sin u}{u}=1$.
(4) Answer is not given since this is a homework problem.
(5) Let $u=2 x-y$ then the limit will equal to $\lim _{u \rightarrow 0} u e^{-1 / u}$, which does not exist since $\lim _{u \rightarrow 0+} u e^{-1 / u}=0 \cdot 0=0$ and $\lim _{u \rightarrow 0-} u e^{-1 / u}=\lim _{w \rightarrow-\infty} \frac{e^{-w}}{w}=$ $\lim _{w \rightarrow-\infty} \frac{-e^{-w}}{1}=-\infty$, which does not agree. Here we use the substitution $w=1 / u$.
(6) When we switch to polar coordinates, we have

$$
\lim _{(x, y) \rightarrow(0,0)} \frac{x y}{\sqrt{x^{2}+y^{2}}}=\lim _{r \rightarrow 0} \frac{(r \cos \theta)(r \sin \theta)}{r}=\lim _{r \rightarrow 0} r \sin \theta \cos \theta=0 .
$$

(7) When we restrict to the $x$-axis, we have

$$
\lim _{y=0, x \rightarrow 0} \frac{\sin \left(3 x^{2}+y^{2}\right)}{x^{2}+2 y^{2}}=\lim _{x \rightarrow 0} \frac{\sin \left(3 x^{2}\right)}{x^{2}}=\lim _{u \rightarrow 0+} \frac{\sin u}{u / 3}=3
$$

We are using substitution $u=3 x^{2}$.
When we restrict to the line $y=x$, we have

$$
\lim _{y=x, x \rightarrow 0} \frac{\sin \left(3 x^{2}+y^{2}\right)}{x^{2}+2 y^{2}}=\lim _{x \rightarrow 0} \frac{\sin \left(4 x^{2}\right)}{3 x^{2}}=\lim _{u \rightarrow 0+} \frac{\sin u}{3 u / 4}=4 / 3
$$

We are using substitution $u=4 x^{2}$.
Since the two limits does not agree, we know that the limit does not exist.
2. Find all points where the vector valued function

$$
\mathbf{F}(x, y, z)=\left(\frac{x}{x^{2}+y^{2}+z^{2}}, \frac{y}{x^{2}+y^{2}+z^{2}}\right)
$$

is not continuous.
Answer: Since the denominator $x^{2}+y^{2}+z^{2}$ only vanish at the origin, so we know that the vector valued function must be continuous at any points other than the origin. Now let's consider the origin point. If you restrict on the $x$-axis, then we have

$$
\lim _{y=z=0, x \rightarrow 0} \frac{x}{x^{2}+y^{2}+z^{2}}=\lim _{x \rightarrow 0} \frac{1}{x}= \pm \infty
$$

while if you restrict on the $y$-axis, then we have

$$
\lim _{x=z=0, y \rightarrow 0} \frac{x}{x^{2}+y^{2}+z^{2}}=\lim _{y \rightarrow 0} \frac{0}{y^{2}}=0 .
$$

Since the limits does not agree, that means at least the first component of $\mathbf{F}$ is not continuous, which means that itself can not be continuous at the origin.
$3^{*}$. Let $\mathbf{x}_{0}$ be a point in $\mathbb{R}^{3}$, and a be a vector in $\mathbb{R}^{3}$. Find all points in $\mathbb{R}^{3}$ such that the following two vector valued functions $\mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ is continuous, respectively.

$$
\begin{aligned}
\mathbf{F}(\mathrm{x}) & =\frac{\mathrm{x}-\mathrm{x}_{0}}{\left\|\mathrm{x}-\mathrm{x}_{0}\right\|} \\
\mathbf{G}(\mathrm{x}) & =\frac{\mathbf{x} \times \mathbf{a}}{\|\mathbf{x} \times \mathbf{a}\|}
\end{aligned}
$$

Answer:
(1) For $\mathbf{F}(\mathbf{x})$, it is clear that the function is smooth at points away from $\mathbf{x}_{0}$, since the denominator would not be zero. However, when $\mathbf{x} \rightarrow \mathbf{x}_{0}$, the length of $\mathbf{F}(\mathbf{x})$ is always 1, but the direction of the vector varies rapidly. (In other words, if you consider $\mathbf{x}=\mathbf{x}_{0}+t \mathbf{v}$ ) for some unit vector $\mathbf{v}$ and $t$ as a scalar, $\left.\mathbf{F}\left(\mathbf{x}_{0}+t \mathbf{v}\right)\right)=\left\{\begin{array}{rc}\mathbf{v}, & \text { when } t>0 \\ -\mathbf{v}, & \text { when } t<0\end{array}\right.$. . Hence we conclude that it is not continuous at $\mathbf{x}=\mathbf{x}_{0}$ but continuous elsewhere.
(2) Similarly, if $\mathbf{a} \neq 0, \mathbf{G}(\mathbf{x})$ is continuous anywhere but when $\mathbf{x} \times \mathbf{a}=0$, i.e., when x is a scalar multiplication of $\mathbf{a}$.
$4^{* *}$. Let $f(x, y)$ be a function such that $\lim _{t \rightarrow 0} f(\lambda t, \mu t)=f(0,0)$ for any real numbers $\lambda, \mu$. Is this sufficient to say that $f(x, y)$ is continuous at point $(0,0)$ ?
Answer: No. Consider the function $f(x, y)=\frac{x^{2} y}{x^{4}+y^{2}}$ and let $f(0,0)=0$. If we restrict the function on any line that passes through the origin, i.e., we have following three possibilities:
(1) If $y=k x$ where $k \neq 0$, we have

$$
\lim _{y=k x, x \rightarrow 0} \frac{x^{2} y}{x^{4}+y^{2}}=\lim _{x \rightarrow 0} \frac{k x^{3}}{x^{4}+k^{2} x^{2}}=\lim _{x \rightarrow 0} \frac{k x}{x^{2}+k^{2}}=\frac{0}{k^{2}}=0 .
$$

(2) If $y=0$ we have

$$
\lim _{y=0, x \rightarrow 0} \frac{x^{2} y}{x^{4}+y^{2}}=\lim _{x \rightarrow 0} \frac{0}{x^{4}}=0 .
$$

(3) If $x=0$ we have

$$
\lim _{x=0, y \rightarrow 0} \frac{x^{2} y}{x^{4}+y^{2}}=\lim _{y \rightarrow 0} \frac{0}{y^{2}}=0 .
$$

It seems like that all the limits agree with each other, however, if we restrict the function on $y=x^{2}$, we have

$$
\lim _{y=x^{2}, x \rightarrow 0} \frac{x^{2} y}{x^{4}+y^{2}}=\lim _{x \rightarrow 0} \frac{x^{4}}{x^{4}+x^{4}}=\frac{1}{2} .
$$

## 2. Partial Derivatives

1. Find the partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ for each of the following functions:
(1) $f(x, y)=x^{y}+y \sin x$
(2) $f(x, y)=x e^{x^{2}+2 x y-y^{2}}$
(3) $f(x, y)=e^{x y} \cos x \ln \left(y-x^{2}\right)$
(4) $f(x, y)=\arctan (x / y)$

What do you discover?
Answer:
(1)

$$
\begin{gather*}
\frac{\partial f}{\partial x}=\left(2 x^{2}+2 x y-1\right) e^{x^{2}+2 x y-y^{2}}  \tag{2}\\
\frac{\partial f}{\partial y}=\left(2 x^{2}-2 x y\right) e^{x^{2}+2 x y-y^{2}} \\
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\left(4 x-2 y+4 x^{3}-4 x y^{2}\right) e^{x^{2}+2 x y-y^{2}} \tag{3}
\end{gather*}
$$

$$
\begin{gathered}
\frac{\partial f}{\partial x}=y e^{x y} \cos (x) \ln \left(y-x^{2}\right)+e^{x y} \sin (x) \ln \left(y-x^{2}\right)-e^{x y} \cos x \frac{2 x}{y-x^{2}} \\
\frac{\partial f}{\partial y}=x e^{x y} \cos (x) \ln \left(y-x^{2}\right)+e^{x y} \cos (x) \frac{1}{y-x^{2}} \\
\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=e^{x y} \cos (x) \ln \left(y-x^{2}\right)+x y e^{x y} \cos (x) \ln \left(y-x^{2}\right)-x e^{x y} \sin (x) \ln \left(y-x^{2}\right) \\
+e^{x y} \cos (x) \frac{y-2 x^{2}}{y-x^{2}}-e^{x y} \sin (x) \frac{1}{y-x^{2}}+e^{x y} \cos (x) \frac{2 x}{\left(y-x^{2}\right)^{2}}
\end{gathered}
$$

(4) Answer is not given since this is a homework problem.

## Quizzes

NAME:
PERM: $\qquad$
Show your work. Partial points might be awarded. No calculators. You DO need to simplify your answers.
(1) Plane $\alpha$ contains points $(1,3,5),(0,1,0),(-2,4,1)$. Find the equation of $\alpha$. (Please simplify the result as much as possible.)

Answer: By finding vectors on the plane (I'll choose $\langle 1,2,5\rangle$ and $\langle-2,3,1\rangle$ ) and do cross product, we will get a normal vector (I'll get $\langle-13,-11,7\rangle$ and some other people might get $\langle 13,11,-7\rangle$ based on different choices of vectors). Now according to $0=\mathbf{n} \cdot\left(\mathbf{x}-\mathbf{x}_{0}\right)$ we will have the answer $0=-13 x-11 y+7 z+11$. (I recommend to let ( $0,1,0$ ) to be $\mathbf{x}_{0}$ since it will simplify calculation.)
(2) Let $\beta$ be a plane that has equation $x-y+3 z=10$. What's the angle between $\alpha$ and $\beta$ ?
(Hint: the angle between two planes is exactly the angle between the two normal vectors of the planes. You can use inverse trigonometric functions to express your answer.)

Answer: The normal vector for $\alpha$ is $n_{\alpha}=\langle-13,-11,7\rangle$, and the normal vector for $\beta$ is just $n_{\beta}=\langle 1,-1,3\rangle$. Now we use the fact that $n_{\alpha} \cdot n_{\beta}=\left\|n_{\alpha}\right\| \cdot\left\|n_{\beta}\right\| \cdot \cos \theta$, we conclude that

$$
\theta=\arccos \frac{n_{\alpha} \cdot n_{\beta}}{\left\|n_{\alpha}\right\| \cdot\left\|n_{\beta}\right\|}=\arccos \frac{-13(1)-11(-1)+7(3)}{\sqrt{13^{2}+11^{2}+7^{2}} \sqrt{1^{2}+1^{2}+3^{2}}}=\arccos \frac{19}{\sqrt{3729}} .
$$

Note 1: cross product would also work but it would be harder.
Note 2: some people may get $\arccos \frac{-19}{\sqrt{3729}}$ which is also fine. In fact, when two planes intersect you always get two angles, which are supplementary angles. People usually use the angle that is less than or equal to $\pi / 2$.

