MATH 6A WORKSHEET 4 ANSWER

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(1) Find the equation of the plane tangent to the sphere $x^2 + y^2 + z^2 = 1$ at the point (x_0, y_0, z_0). For extra practice, do this in two ways, one by solving for z and consider the two cases $z_0 \geq 0$ and $z_0 < 0$, and the other by viewing the sphere as a level surface.

Answer:

(a) Method 1: Assume that $z_0 > 0$. (It will be similar if $z_0 < 0$.) Then we can write $z = \sqrt{1 - x^2 - y^2}$. Let $f(x, y) = \sqrt{1 - x^2 - y^2}$, we have $f_x = \frac{-x}{\sqrt{1-x^2-y^2}}$ and $f_y = \frac{-y}{\sqrt{1-x^2-y^2}}$. Hence the linear approximation is $dz = -\frac{x_0}{\sqrt{1-x_0^2-y_0^2}}dx - \frac{y_0}{\sqrt{1-x_0^2-y_0^2}}dy$. So the equation for the tangent plane is

$$z - z_0 = -\frac{x_0}{\sqrt{1 - x_0^2 - y_0^2}}(x - x_0) - \frac{y_0}{\sqrt{1 - x_0^2 - y_0^2}}(y - y_0).$$

(b) Method 2: Let $G(x, y, z) = x^2+y^2+z^2$. Then the sphere is the level surface $G = 1$. By the properties of gradient, we know that $\mathbf{n} = \nabla G(x_0, y_0, z_0) = \langle 2x_0, 2y_0, 2z_0 \rangle$ is a normal vector for the tangent plane. Hence the equation for the tangent plane can be expressed by $\mathbf{n} \cdot (\langle x, y, z\rangle - \langle x_0, y_0, z_0\rangle) = 0$, i.e.,

$$2x_0(x - x_0) + 2y_0(y - y_0) + 2z_0(z - z_0) = 0.$$

You can verify that they are the same equation by using the fact that $z_0 = \sqrt{1 - x_0^2 + y_0^2}$ and $x_0^2 + y_0^2 + z_0^2 = 1$.

(2) A soccer player is running in the field, where his position is described by the following equation:

$$r(t) = \langle 16 \sin^3 t, 13 \cos t - 5 \cos(2t)\rangle.$$

Find the velocity and acceleration of the soccer player.

Answer:

$$v(t) = r'(t) = \langle 48 \sin^2 t \cos t, -13 \sin t + 10 \sin(2t)\rangle.$$

$$a(t) = v'(t) = \langle 96 \sin t \cos^2 t - 48 \sin^3 t, -13 \cos t + 20 \cos(2t)\rangle.$$

(3) A skier is on the mountain with equation

$$h = 100 - 0.4y^2 - 0.3x^2,$$

where h denotes the height.

(a) The skier is located at the point with xy-coordinates $(1, 1)$, and wants to ski downhill along the steepest possible path. In which direction (indicated by a vector $\langle a, b\rangle$ in the xy-plane) should the skier begin skiing?

(b) The skier begins skiing in the direction given by the xy-vector (a, b) you found in part (3a) so the skier heads in a direction in space given by the vector (a, b, c). Find the value of c.

(c) A hiker located at the same point on the mountain decides to begin hiking downhill in a direction given by a vector in the xyplane that makes an angle θ with the vector (a, b) you found in part (3a). How big should θ be if the hiker wants to head downhill along a path whose slope is at most 0.5 (in absolute value)?

Answer:

(a) Since gradient is steepest climb, steepest descend should be negative gradient.

$$\langle a, b\rangle = \nabla h(1, 1) = \langle .8, .6\rangle$$

(b)

$$c = \nabla h(1, 1) \cdot \langle .8, .6\rangle = -1.$$

(c) If θ is the angle between the path of hiker (denoted by u) with the path of skier, then $\pi - \theta$ will be the angle between the path of hiker and the gradient. Since we have

$$\frac{\partial h}{\partial u} = \nabla f \cdot u = ||\nabla f|| \cdot ||u|| \cdot \cos(\pi - \theta)$$

So we need

$$-.5 \leqslant \cos(\pi - \theta) < 0,$$

which gives $\pi/3 \leqslant \theta < \pi/2$.

(4) Estimate $\cos(0.01)e^{0.02}$.

Answer: Let $f(x, y) = \cos xe^y$. Then $f_x = -\sin xe^y$ and $f_y = \cos xe^y$. The linear approximate at $(0, 0)$ is

$$f(\Delta x, \Delta y) \approx 1 + 0 \cdot (\Delta x - 0) + 1 \cdot (\Delta y - 0).$$

Make $\Delta x = 0.01$ and $\Delta y = 0.02$ and we get $\cos(0.01)e^{0.02} \approx 0.02$.

(5) Use differentials to estimate the amount of tin in a closed tin can with diameter 3 inch and height 4 inch if the top and bottom is 0.02 inch thick and the side is 0.01 inch thick.

Answer: We know that the volume of the tin can is expressed by $V = \pi r^2 h$, where r and h denote the radius and height, respectively. Hence we know by linear approximation,

$$dV = (2\pi rh)dr + (\pi r^2)dh.$$

By substituting $r = 1.5$, $h = 4$, $dr = 0.01$, $dh = 0.02 \times 2 = 0.04$ we get volume of tin is $dV = 0.66in^3$.

(6) The gas law for a fixed mass m of an ideal gas at absolute temperature T, pressure p and volume V is given below:

$$pV = nRT,$$

where R is a constant. Show that

$$\frac{\partial p}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial p} = -1.$$

Answer:

Since

$$p = \frac{nRT}{V}$$
$$\frac{\partial p}{\partial V} = -\frac{nRT}{V^2};$$

Since

$$V = \frac{nRT}{p}$$
$$\frac{\partial V}{\partial T} = \frac{nR}{p};$$

Since

$$T = \frac{pV}{nR}$$
$$\frac{\partial T}{\partial p} = \frac{V}{nR}.$$

Hence we have

$$\frac{\partial p}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial p} = -\frac{nRT}{V^2}\frac{nR}{p}\frac{V}{nR} = -\frac{n^2R^2TV}{nRpV^2} = -\frac{nRT}{pV} = -1.$$

(7) Suppose you need to know an equation of the tangent plane to a surface S at the point $P(2, 1, 3)$. You don't have an equation for S but you know that the curves

$$r_1(t) = (2 + 3t, 1 - t^2, 3 - 4t + t^2)$$
$$r_2(u) = (1 + u^2, 2u^3 - 1, 2u + 1)$$

both lie on S. Find an equation of the tangent plane at P.

Note: typo here: should be $r_2(u)$ instead of $r_2(t)$.

Answer: Since $r_1(0) = r_2(1) = (2, 1, 3)$, and $r_1'(0) = \langle 3, 0, -4\rangle$, $r_2'(1) = \langle 2, 6, 2\rangle$, we know that these two vectors all lie in the tangent plane. Thus we can get a normal vector

$$n = \langle 3, 0, -4\rangle \times \langle 2, 6, 2\rangle = \langle 24, -14, 18\rangle.$$

So the equation for the tangent plane is $24(x - 2) - 14(y - 1) + 18(z - 3) = 0$, which simplifies to $12x - 7y + 9z = 44$.

(8) Find the tangent plane of the function $u(s,t)$ given by $u = x^4y + y^2z^3$ and $x = se^t$, $y = s^2e^{-t}$ and $z = s\sin t$ at point $(s,t) = (1,0)$.

Answer: When $(s,t) = (1,0)$, we have $x = 1$, $y = 1$ and $z = 0$. $\frac{\partial u}{\partial x}(1,1,0) = 4$, $\frac{\partial u}{\partial y}(1,1,0) = 1$ and $\frac{\partial u}{\partial z}(1,1,0) = 0$. $u(1,1,0) = 1$

$$\frac{\partial u}{\partial s}(1,0) = \frac{\partial u}{\partial x}(1,1,0)\frac{\partial x}{\partial s}(1,0)+\frac{\partial u}{\partial y}(1,1,0)\frac{\partial y}{\partial s}(1,0)+\frac{\partial u}{\partial z}(1,1,0)\frac{\partial z}{\partial s}(1,0) = 4\cdot1+1\cdot2+0\cdot0 = 6$$

$$\frac{\partial u}{\partial t}(1,0) = \frac{\partial u}{\partial x}(1,1,0)\frac{\partial x}{\partial t}(1,0)+\frac{\partial u}{\partial y}(1,1,0)\frac{\partial y}{\partial t}(1,0)+\frac{\partial u}{\partial z}(1,1,0)\frac{\partial z}{\partial t}(1,0) = 4\cdot1+1\cdot(-1)+0\cdot0 = 3.$$

So the tangent plane is given by $u = 1 + 6(s-1) + 3(t-0)$, or $u = 6s + 3t - 5$.

(9) (a) Write down the transition function $\mathbf{\Phi}$ from spherical coordinate to Cartesian coordinate. (i.e., this is a function that you input the spherical coordinate and get Cartesian coordinate)
(b) Find the derivative $D\mathbf{\Phi}$.
(c) A curve has the form $(1+2t, \pi/2 - t, \pi + 3t)$ in spherical coordinate. Find the starting point of the curve when $t = 0$ in Cartesian coordinate, and find the velocity vector at $t = 0$ expressed in Cartesian coordinate.
(d) Do the same for cylindrical coordinate.

Answer:

(a) Spherical coordinate:

(i)

$$\mathbf{\Phi}(\rho,\theta,\phi) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$$

(ii)

$$D\mathbf{\Phi} = \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\sin\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix}$$

(iii)

$$D\mathbf{\Phi}(1,\pi/2,\pi) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

so the velocity vector equals

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}.$$

(b) Cylinder coordinate:

(i)

$$\mathbf{\Phi}(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$$

(ii)

$$D\mathbf{\Phi} = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

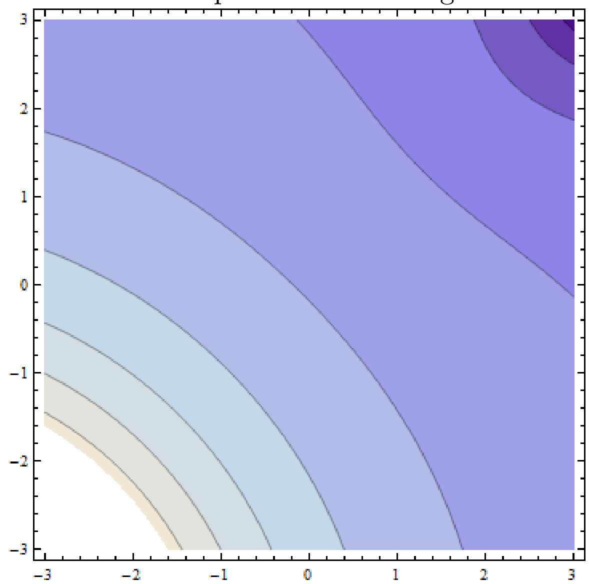
(iii)

$$D\mathbf{\Phi}(1, \pi/2, \pi) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

so the velocity vector equals

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

(10) The contour map of a function is given below.



Draw the gradient at point (-2,0). Estimate the partial derivatives at the same point. (Assume that the difference between adjacent lines is 1, and dark shadow represent lower value.)

Answer: gradient should be perpendicular to the curve, pointing to the lower left. I would guess $-1 < f_x < -0.7$ and $f_y \approx -0.5$.

1. Quizzes

NAME:________________________
PERM:_________________________

This time you can just write your final answer. However, you still need to simplify them.

Let

$$f(x,y) = \cos(x^2 - y)\ln(y^3 - x)$$
$$g(x,y,z) = \frac{e^{x^2 z}}{xyz}$$

Find partial derivatives:

(1)
$$f_x = -2x\sin(x^2 - y)\ln(y^3 - x) - \frac{\cos(x^2 - y)}{y^3 - x}$$

(2)
$$f_y = \sin(x^2 - y)\ln(y^3 - x) + \frac{3y^2\cos(x^2 - y)}{y^3 - x}$$

(3)
$$g_x = \frac{e^{x^2 z}(2x^2 z - 1)}{x^2 yz}$$

(4)
$$g_y = -\frac{e^{x^2 z}}{xy^2 z}$$

(5)
$$g_z = \frac{e^{x^2 z}(x^2 z - 1)}{xyz^2}$$