# Math 34A Practice Final Solutions Fall 2007

**Problem 1** Find the derivatives of the following functions:

1. $f(x) = 3x^2 + 2e^{3x}$
2. $f(x) = \frac{x^2+1}{x}$
3. $f(x) = (x + 2a)^2$
4. Is the function $3t^2 - 4t - t^3$ increasing or decreasing when $t = 1$?
5. Find a nonzero function $f(x)$ such that $f'(x) - 4f(x) = 0$.

**Solution**

1. $f'(x) = 6x + 6e^{3x}$
2. $f'(x) = 1 - \frac{1}{x^2}$
3. $f'(x) = 2x + 4a$
4. $f'(x) = 6t - 4 - 3t^2$. So, $f'(1) = 6 - 4 - 3 = -1 < 0$, which means $f$ is decreasing.
5. $f(x) = e^{4x}$

**Problem 2**
**(a)** What is the equation for the line tangent to $f(x) = \sqrt{x}$ at $x = 1$?

**Solution** First, we find the slope of the line by finding $f'(1)$:

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x}} \\ \Rightarrow \quad f'(1) &= \frac{1}{2} \end{aligned}$$

Next, we find a point on the line by substituting $x = 1$ into $f(x)$:

$$(1, f(1)) = (1, 1)$$

Then we can use point-slope format to find the equation of the line:

$$\begin{aligned} y - 1 &= \frac{1}{2}(x - 1) \\ \Rightarrow \quad y &= \frac{1}{2}x + \frac{1}{2} \end{aligned}$$

**(b)** Use the tangent line approximation at $x = 1$ of $f(x)$ to approximate $\sqrt{1.1}$.

**Solution** To do this, we simply substitute $x = 1.1$ into the equation for line tangent to $f(x)$ at $x = 1$:

$$\begin{aligned} y &= \frac{1}{2}(1.1) + \frac{1}{2} \\ \Rightarrow \quad y &= 1.05 \end{aligned}$$

**Problem 3** If the half life of some element is 20 years, how long does it take until 1% of the element remains?

**Solution** Call the initial starting amount of the element $A_0$. Then according to the half life formula, the amount left of the element at year $t$ is given by

$$A(t) = A_0 \frac{1}{2}^{t/20}$$

Then to find when 1% of the element is left, we set $A(t)$ equal to $0.01 \cdot A_0$, and solve for $t$:

$$\begin{aligned} 0.01 \cdot A_0 &= A_0 \frac{1}{2}^{t/20} \\ \log 0.01 &= \frac{t}{20} \log 0.5 \\ t &= \frac{20 \log 0.01}{\log(0.5)} \end{aligned}$$

**Problem 4** A jet airliner flies at 300 mph for the first half hour and last half hour of a flight. The rest of the time it flies at 600 mph. How long does it take to fly from LA to NY, a distance of 2100 miles?

**Solution** Let T be the time, in hours, spent flying at 600 mph. In the middle of the flight, after the first half hour and before the last half hour, the plane travels a distance of $600T$ miles.
In the first half hour, the plane travels $300(\frac{1}{2}) = 150$ miles.
In the last half hour, the plane travels $300(\frac{1}{2}) = 150$ miles.
So the total distance traveled is $2100 = 150 + 600T + 150 = 300 + 600T$.
Solving this equation for T, we get

$$\begin{aligned} 2100 - 300 &= 600T \\ 1800 &= 600T \\ (\frac{1800}{600}) &= T \\ 3 &= T \end{aligned}$$

So the middle of the flight takes 3 hours. In total, the flight took $3+\frac{1}{2}+\frac{1}{2} = 4$ hours.

**Problem 5** Solve for x
**(a)** $5^x = 3 \cdot 9^x$
**Solution** First, rewrite the right hand side:

$$\begin{aligned} 3 \cdot 9^x &= 3 \cdot (3^2)^x \\ 3 \cdot (3^2)^x &= 3 \cdot (3^{2x}) \\ 3 \cdot (3^{2x}) &= 3^{2x+1} \end{aligned}$$

So now we have the equation $5^x = 3^{2x+1}$. Take the log of both sides and then solve for x:

$$\begin{aligned} log(5^x) &= log(3^{2x+1}) \\ xlog(5) &= (2x+1)log(3) \\ xlog(5) &= 2xlog(3) + log(3) \\ xlog(5) - 2xlog(3) &= log(3) \\ x(log(5) - 2log(3)) &= log(3) \\ x &= \frac{log(3)}{log(5) - 2log(3)} \end{aligned}$$

**(b)**
**Solution** $\ln(x^2) = \ln(x)$

$$\begin{aligned} 2\ln(x) &= \ln(x) \\ 2\ln(x) - \ln(x) &= 0 \\ \ln(x) &= 0 \\ x &= 1 \end{aligned}$$

**Problem 6** For $f(x) = x + x^2$, whats the average rate of change of the function from 1 to $1 + h$? Whats the instantaneous rate of change at $x = 1$?

**Solution** The rate of change formula in general is $\frac{f(x_2)-f(x_1)}{x_2-x_1}$. In our case $x_2 = 1 + h, x_1 = 1$, so

$$\begin{aligned} \frac{f(x_2) - f(x_1)}{x_2 - x_1} &= \frac{f(1+h) - f(1)}{1+h-1} \\ &= \frac{(1+h+(1+h)^2) - (1+1^2)}{h} \\ &= \frac{1+h+1+2h+h^2-2}{h} \\ &= \frac{3h+h^2}{h} \\ &= 3+h \end{aligned}$$

The instantaneous rate can be found in one of two ways, either we can take the limit as $h \to 0$ or take the derivative of $f(x)$ at $x = 1$. From the previous part, taking the limit gives the instantaneous rate to be 3. We can check this by noticing $f'(x) = 1 + 2x$ and $f'(1) = 1 + 2 * 1 = 3$.

**Problem 7** An object is dropped out of an airplane. The height of the object $t$ seconds after being dropped is $h(t) = 2000 - 5t^2$ meters.

1. What is the height of the plane when the object was dropped?
2. What is he velocity of the object at $t = 3$

3. What is the acceleration of the object?
4. When did the object hit the ground?
5. How fast was it going when it hit the ground?

**Solution**

1. The object is dropped at time = 0. At which its height is $h(0) = 2000 - 0 = 2000$.
2. The velocity is the derivative of position, in our case height. So $v(t) = -10t$. At time = 3, our velocity is $v(3) = -30$. The velocity is negative because we are falling.
3. The acceleration is the derivative of the velocity (or the second derivative of position). So $a(t) = -10$. Hence our acceleration is always -10, regardless of the time.
4. An object hits the ground when its height is 0. So we must solve $0 = 2000 - 5t^2$ which gives $t = 20$.
5. From the previous part we know when the object hits the ground. This is critical. It hits the ground after 20 seconds. So its velocity after 20 seconds is $v(20) = -10 * 20 = -200$. We're not done yet though, the question asks for speed, which is the magnitude of velocity. There's no such thing as a negative speed. The speed is the absolute value, or 200.

**Problem 8** A cylindrical metal can is to have no lid. It is to have a volume of $8\pi in^3$. What height minimizes the amount of metal used (ie surface area).
**Solution** Let $h$ = height of the can, and $r$ = radius of the can.
We try to minimize the surface area (SA), and the equation of surface area is $SA = 2\pi rh + \pi r^2$. We'd like to take the derivative of this function and set it equal to 0, however, the function is in two variables. We must eliminate one of them, so we use the extra information given in the problem.

$$V = 8\pi = \pi r^2 h$$
$$\Rightarrow h = \frac{8}{r^2}$$

Now we can plug this relationship into our area function to have it in terms of one variable.

$$\begin{aligned} SA &= 2\pi rh + \pi r^2 \\ &= 2\pi r(\frac{8}{r^2}) + \pi r^2 \\ &= 16\pi\frac{8}{r} + \pi r^2 \end{aligned}$$

Now we take the derivative and set it equal to 0.

$$\begin{aligned} 0 &= -16\pi r^{-2} + 2\pi r \\ &= \frac{-16\pi}{r^2} = 2\pi r \\ &\Rightarrow \frac{16\pi}{r^2} = 2\pi r \\ &\Rightarrow r^3 = 8 \\ &\Rightarrow r = 2. \end{aligned}$$

But we're not done yet. We meerly found the radius at which the height is minimized. But we know how the height related to the radius: $h = \frac{8}{r^2}$. So when $r = 2$ $h = 8/4 = 2$.

**Problem 9** A rectangular field will have one side made of a brick wall and the other three sides made of wooden fence. Brick wall costs \$20 per meter. Wooden fence costs \$40 for 4 meters. The area of the field is to be 2400 $m^2$. What length should the brick wall be to give the lowest total cost of the wall plus fence?

**Solution** Let $L$ = length of the rectangle, $H$ = height.
Then the area of the field = $LH = 2400$.
We need to figure out the cost of the fence in terms of these variables. Suppose the wall made of brick is along the length (this assumption won't matter in the end). Then

$$\begin{aligned} Cost &= \frac{40}{4}(L + H + H) + 20(L) \\ &= 30L + 20H \end{aligned}$$

Again this is an equation with two variables. We use the fact that $LH = 2400 \Rightarrow H = 2400/L$ to eliminate a variable. Plugging in gives,

$$\begin{aligned} Cost &= 30L + 20(\frac{2400}{L}) \\ &= 30L + 20 * 2400L^{-1} \end{aligned}$$

Taking the derivative and setting it equal to zero gives,

$$\begin{aligned} 0 &= 30 - 1 * 20 * 2400L^{-2} \\ &\Rightarrow 20 * 2400\frac{1}{L^2} = 30 \\ &\Rightarrow L^2 = \frac{20 * 2400}{30} = 1600 \\ &\Rightarrow L = 40 \end{aligned}$$

Since the brick wall was along the length, our answer is 40.