HW #2 Solutions

$\{5.4, 5.5, 5.11, 5.23\ (a),(b), 5.25, 10.4, 10.12, 10.13, 10.16, 10.27, 10.28\}$

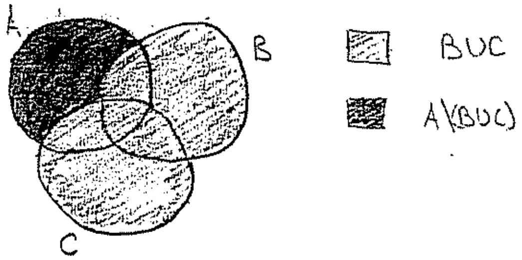
5.4 a) $A \cap B = \{2,4\}$ b) $A \cup B = \{1,2,3,4,6,8\}$ c) $A \setminus B = \{6,8\}$

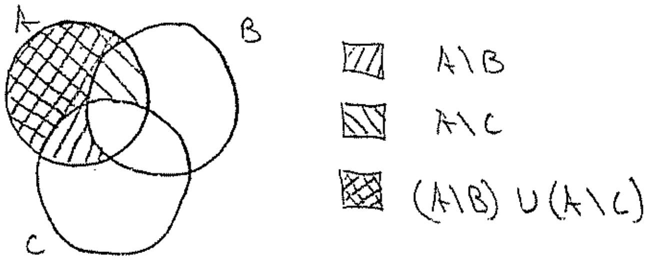
d) $B \cap C = \phi$ e) $B \setminus C = \{1,2,3,4\}$ f) $(B \cup C) \setminus A = \{1,3,5,7\}$

g) $(A \cap C) \setminus B = \{6\}$ h) $C \setminus (A \cup B) = \{5,7\}$

5.5 a) in book

b)




5.11 i) if $x \in A$ and $A \subseteq B$, then $x$ is also an element of $B$. Hence $x \in A \cap B$.

ii) $x \in A \cap B$ so $x$ is also in $B$.

5.23 $A$ and $B$ are subsets of $U$.

a) Prove $A \setminus B = (U \setminus B) \setminus (U \setminus A)$

pf Though this can be proved directly, we first simplify the right hand side

$$(U \setminus B) \setminus (U \setminus A) = (U \setminus B) \cap (U \setminus (U \setminus A)) \quad \text{by } 5.11$$
$$= (U \setminus B) \cap A \quad \text{by } 5.13\ (c).$$

Now let $x \in A \setminus B$. (We show $x \in U \setminus B$ and $x \in A$, hence it is in their intersection.) Then $x \in A$ and $x \notin B$. Since $A \setminus B \subset U$, $x \in U$, But $x \notin B$, $x \in U \Rightarrow x \in U \setminus B$.

Hence $x \in A$ and $x \in U \setminus B$ so $x \in U \setminus B \cap A$

$\therefore A \setminus B \subseteq (U \setminus B) \setminus (U \setminus A)$

(5.23 cont.) Now let $x \in U \setminus B \cap A$.

This means $x \in U \setminus B$ and $x \in A$.

But $x \in U \setminus B \Rightarrow x \notin B$

Finally $x \in A, x \notin B \Rightarrow x \in A \setminus B$

$\therefore (U \setminus B) \setminus (U \setminus A) \subseteq A \setminus B$

Hence the sets are equal.

b) Prove $U \setminus (A \setminus B) = (U \setminus A) \cup B$

($\subseteq$) Let $x \in U \setminus (A \setminus B)$

Then $x \in U$ and $x \notin A \setminus B$

Now the elements in $A \setminus B$ are in $A$ and not in $B$

To be outside this set $x$ is either outside $A$ or inside $B$ $[\sim(A \wedge \sim B) = \sim A \vee B]$.

If $x \notin A$ then $x \in U \setminus A$ since $x \in U$

Otherwise $x \in B$. Since one or the other happens

$x \in (U \setminus A) \cup B$

$\therefore U \setminus (A \setminus B) \subseteq (U \setminus A) \cup B$

($\supseteq$) Now let $x \in (U \setminus A) \cup B$

Then either $x \in U \setminus A$ or $x \in B$ (or both)

If $x \in U \setminus A$, then $x \in U$ and $x \notin A$

Since we already know $x \in U$ all we need to do is show $x \notin A \setminus B$

But $x \notin A \Rightarrow x \notin A \setminus B \subset A$.

Now suppose $x \in B$, then $x \in U$ also and all we need to show is $x \notin A \setminus B$

But $B \cap (A \setminus B) = \phi$ [5.13 b)], hence $x \in B \Rightarrow x \notin A \setminus B$

$\therefore (U \setminus A) \cup B \subseteq U \setminus (A \setminus B)$

Hence the sets are equal.

(5.25) a) $\bigcup_{n=1}^{\infty} [1, 1+\frac{1}{n}] = [1,2]$

$\bigcap_{n=1}^{\infty} [1, 1+\frac{1}{n}] = \{1\}$ since $1 \in [1, 1+\frac{1}{n}]$ for $\forall n$, while $\forall \varepsilon > 0$, $\exists N$ s.t.

$\frac{1}{N} < \varepsilon$. Hence $1+\varepsilon \notin [1, 1+\frac{1}{N}]$

b) $\bigcup_{1}^{\infty} (1, 1+\frac{1}{n}) = (1,2)$

$\bigcap_{1}^{\infty} (1, 1+\frac{1}{n}) = \phi$ since $1 \notin (1, 1+\frac{1}{n})$ and $\forall \varepsilon > 0$, $\exists N$ s.t. $1+\varepsilon \notin (1, 1+\frac{1}{N})$

c) $\bigcup_{\substack{x \in \mathbb{R} \\ x > 2}} [2, x] = [2, \infty)$, $\bigcap_{\substack{x \in \mathbb{R} \\ x > 2}} [2, x] = \{2\}$ just as in a)

d) $\underbrace{[0,3] \cup (1,5)}_{[0,5)} \cup [2,4) = [0,5)$

$\underbrace{[0,3] \cap (1,5)}_{(1,3]} \cap [2,4) = [2,3]$

(10.4) Prove $\sum_{i=1}^{n} i^3 = \frac{1}{4} n^2 (n+1)^2 \quad \forall n \in \mathbb{N}$

pf Let $n=1$, then

$\sum_{i=1}^{1} i^3 = 1 = \frac{1}{4}(1)(2)^2 = 1$ ✓

Suppose the equality it true for $n < k$, we prove it is true for $n = k$.

$$\sum_{i=1}^{k} i^3 = \sum_{i=1}^{k-1} i^3 + k^3$$

$$= \frac{1}{4}(k-1)^2(k)^2 + \frac{4k^3}{4} \quad \text{by inductive hypothesis}$$

$$= \frac{k^4 - 2k^3 + k^2 + 4k^3}{4}$$

$$= \frac{k^4 + 2k^3 + k^2}{4}$$

$$= \frac{1}{4}(k+1)^2 k^2$$ ✓

Hence the equality is true for $\forall n \in \mathbb{N}$

(16.12) Prove $\sum_{i=1}^{n} i2^{i-1} = (n-1)2^n + 1$ for $\forall n \in \mathbb{N}$

pf Let $n=1$, then

$$\sum_{i=1}^{1} i2^{i-1} = 1, \quad (1-1)2^1 + 1 = 1 \checkmark$$

Suppose true for $n < k$, prove for $n = k$.

$$\sum_{i=1}^{k} i2^{i-1} = \sum_{i=1}^{k-1} i2^{i-1} + k2^{k-1}$$

$$= (k-2)2^{k-1} + 1 + k2^{k-1} \quad \text{by inductive hypothesis}$$

$$= 2^k \underbrace{\left[\frac{k-2}{2} + \frac{k}{2}\right]}_{\frac{2k-2}{2}} + 1$$

$$= (k-1)2^k + 1 \checkmark$$

Hence the equality is true for $\forall n \in \mathbb{N}$

(16.13) Prove $5^{2n} - 1$ is a multiple of 8 for $\forall n \in \mathbb{N}$

pf Let $n=1$, then $5^{2 \cdot 1} - 1 = 24 = 8 \cdot 3 \checkmark$

Suppose true for $n \leq k$ prove for $n = k+1$

$$5^{2(k+1)} - 1 = 5^2(5^{2k} - 1) + 5^2 - 1 \quad \text{by hint in book}$$

$$= 5^2(8 \cdot c) + \underbrace{24}_{8 \cdot 3} \quad \text{for some } c \in \mathbb{Z}$$

$$= 8\left[5^2 c + 3\right] \checkmark$$

Hence equality is true for $\forall n \in \mathbb{N}$

(16.16) Prove $\sum_{i=1}^{n} (3i-1) = \frac{1}{2} n(3n+1)$

pf for $n=1$, $\sum_{i=1}^{1} (3i-1) = 2$ and $\frac{1}{2}(1)(3 \cdot 1 + 1) = 2$ ✓

Assume true for $n < k$, prove for $n = k$

$$\sum_{i=1}^{k} (3i-1) = \sum_{i=1}^{k-1} (3i-1) + 3k - 1$$

$$= \frac{1}{2}(k-1)(3k-2) + 3k - 1 \quad \text{by inductive hypothesis}$$

$$= \frac{1}{2}(3k^2 - 5k + 2) + \frac{6k-2}{2}$$

$$= \frac{1}{2}(3k^2 + k) = \frac{1}{2}k(3k+1) \checkmark$$

(10.27) $g_i = \#$ of gifts on $i^{th}$ day $= g_{i-1} + i$ , $g_1 = 1$.

$t_n = \#$ of gifts in first $n$ days.

Show $t_n$ is of the form $\frac{n(n+a)(n+b)}{c}$ $a, b, c \in \mathbb{N}$

pf First

$$t_n = \sum_{i=1}^{n} g_i \quad \text{by definition}$$

We find another expression for $g_i$ inductively.

Note:

$$\begin{aligned} g_n &= g_{n-1} + n \\ &= g_{n-2} + (n-1) + n \\ &\ \vdots \\ &= \overset{=1}{g_1} + 2 + 3 + \cdots + n \\ &= \sum_{i=1}^{n} i \\ &= \tfrac{1}{2} n(n+1) \end{aligned}$$

So

$$\begin{aligned} t_n &= \sum_{i=1}^{n} \tfrac{1}{2} i(i+1) = \tfrac{1}{2}\left[\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right] \\ &= \tfrac{1}{2}\left[\tfrac{1}{6}\big(n(n+1)(2n+1)\big) + \tfrac{1}{2} n(n+1)\right] \quad \text{by problem 10.3} \\ &= \tfrac{1}{2}\left[ n(n+1)\Big[\underbrace{\tfrac{2n+1}{6} + \tfrac{1}{2}}_{\frac{2n+4}{6}}\Big]\right] \\ &= \frac{n(n+1)(n+2)}{6} \end{aligned}$$

so $a = 1, b = 2, c = 6$.

10.28 a) Show $\binom{n}{r}+\binom{n}{r-1}=\binom{n+1}{r}$

pf $\binom{n}{r}+\binom{n}{r-1}=\dfrac{n!}{r!(n-r)!}+\dfrac{n!}{(r-1)!(n-r+1)!}$

$$=\frac{n!}{r(r-1)!(n-r)!}+\frac{n!}{(r-1)!(n-r)!(n-r+1)}$$

$$=\frac{n!}{(r-1)!(n-r)!}\underbrace{\left[\frac{1}{r}+\frac{1}{n-r+1}\right]}_{\frac{n-r+1+r}{(n-r+1)r}=\frac{n+1}{r(n-r+1)}}$$

$$=\frac{(n+1)!}{r!(n+1-r)!}=\binom{n+1}{r}$$

b) Prove $(a+b)^n=\sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i$ for all $n\in\mathbb{N}$

Let $n=1$, then $\sum_{i=0}^{1}\binom{1}{i}a^{1-i}b^i=\binom{1}{0}a^1b^0+\binom{1}{1}a^0b^1$

$$=a+b$$

Suppose true for $n<k$, show true for $n=k$.

$$(a+b)^k=(a+b)^{k-1}(a+b)=(a+b)\sum_{i=0}^{k-1}\binom{k-1}{i}a^{k-i-1}b^i$$

$$=\underbrace{\sum_{i=0}^{k-1}\binom{k-1}{i}a^{k-i}b^i}_{a^k+\sum_{1}^{k-1}\binom{k-1}{i}a^{k-i}b^i}+\underbrace{\sum_{i=0}^{k-1}\binom{k-1}{i}a^{k-i-1}b^{i+1}}_{\sum_{i=1}^{k-1}\left[\binom{k-1}{i-1}a^{k-i}b^i\right]+b^k}$$

$$=a^k+b^k+\sum_{i=1}^{k-1}\underbrace{\left[\binom{k-1}{i}+\binom{k-1}{i-1}\right]}_{\binom{k}{i}\text{ by part (a).}}a^{k-i}b^i$$

$$=a^k+b^k+\sum_{i=1}^{k-1}\binom{k}{i}a^{k-i}b^i$$

$$=\sum_{i=0}^{k}\binom{k}{i}a^kb^i$$

Hence equality is true for ~~∀~~ $n\in\mathbb{N}$.