HW #3 Solutions

$\{11.4, 11.6, 11.7, 11.10, 11.11 (b)(c)\}$

(11.4) We prove by contradiction

Suppose $x \geq 0$, $x \leq \varepsilon \ \forall \varepsilon > 0$ and $x \neq 0$

Then $x \neq 0$ and $x \geq 0 \Rightarrow x > 0$

Choose $\varepsilon = \frac{x}{2}$

Then $\varepsilon > 0$, but $x \nleq \varepsilon$ Contradiction!

$\therefore x = 0$.

(11.6) Prove a) $||x|-|y|| \leq |x-y|$ $\quad$ [or $-|x-y| \leq |x|-|y| \leq |x-y|$ $(\maltese)$]

pf <u>Useful Trick</u> $|x| = |x-y+y| \leq |x-y| + |y|$

Here $|x|-|y| \leq |x-y|$ which is the RHS of $(\maltese)$

Similarly $|y| \leq |y-x| + |x|$

So

$|y|-|x| \leq |y-x| \Rightarrow |x|-|y| \geq -|x-y|$ which is LHS of $(\maltese)$.

This proves part a)

b) If $|x-y| < c$ then $|x| < |y| + c$

pf $|x| \leq |x-y| + |y| < c + |y|$

c) If $|x-y| < \varepsilon \ \forall \varepsilon > 0$, Then $x = y$.

pf Proof by contradiction,

Suppose $|x-y| < \varepsilon \ \forall \varepsilon > 0$ and $x \neq y$

Since $x \neq y$, $|x-y| > 0$. Say $|x-y| = k$.

Then take $\varepsilon = \frac{k}{2} > 0$ making

$|x-y| > \varepsilon$ Contradiction

$\therefore x = y$

(11.7) We prove this by induction

Clearly $|x_1| \le |x_1|$ and by the triangle inequality $|x_1+x_2| \le |x_1|+|x_2|$

This takes care of cases $n=1$ and 2. Suppose we have the inequality for $n<k$. Then for $n=k$

$$|x_1+\cdots+x_{k-1}+x_k| \le |x_1+\cdots+x_{k-1}|+|x_k| \quad \text{by triangle inequality}$$
$$\le |x_1|+\cdots+|x_{k-1}|+|x_k| \quad \text{by inductive hypothesis}$$

This completes the induction.

(11.10) Case 1: $a=0$

Then $a^2+1=1$

Claim: $1>0$

pf Suppose $1<0$, then

$1 \cdot 1 > 0$ [since negative #'s reverse inequality when multiplied]

Hence $1>0$ contradiction!

$\therefore 1>0$

$\therefore a^2+1>0$

Case 2 $a>0$

Then $a^2>0$ by same reasoning as before

Hence $a^2+1>0+1$ so $a^2+1>0$

Case 3 $a<0$

Then $a^2>0$ so $a^2+1>1$.

(11.11) b) $\{-x^3,\ 3-x,\ 5,\ x+2,\ x^2\}$ (smallest $\rightarrow$ biggest)

since for instance $-x^3>3-x$ since $-x^3-3+x=-1<0$

c) $\left\{\frac{x+1}{x^2-2},\ \frac{x+2}{x^2-1},\ \frac{x^2-2}{x+1},\ \frac{x^2+2}{x-1}\right\}$ (smallest $\rightarrow$ biggest)

since for instance

$$\frac{x+1}{x^2-2}-\frac{x+2}{x^2-1} = \frac{(x^2-1)(x+1)-(x+2)(x^2-2)}{(x^2-2)(x^2-1)} = \frac{-x^2+x+5}{x^4-3x^2+2} = -1<0$$