HW #4 Solutions

$\{12.3\,(g)\text{-}(\ell),\ 12.8,\ 12.9,\ 12.10,\ 12.13,\ 13.3,\ 13.5,\ \cancel{13.6},\ 13.8,\ 13.11\}$

12.3

g) $\sup = 1$, $\max$ = dne

h) $\sup = \max = 3/2$

i) $\sup$ & $\max$ dne

j) $\sup = 4$, $\max$ = dne

k) $\sup = \max = 1$ since $\bigcap(1-\frac{1}{n}, 1+\frac{1}{n}) = \{1\}$

ℓ) $\sup = 2$, $\max$ = dne since $\bigcup[\frac{1}{n}, 2-\frac{1}{n}] = (0,2)$

12.8 First we show $\inf(T) \le \inf(S)$

Let $a = \inf(T)$, $b = \inf(S)$

Then $\forall t \in T$, $a \le t$

Since $S \subseteq T$ $a \le s$ for $\forall s \in S$

Hence a is a lower bound for S

But b is the greatest lower bound of S

$\therefore a \le b$

The same argument shows $\sup(S) \le \sup(T)$

The last part to show is $\inf(S) \le \sup(S)$

But this is clear by taking some $s \in S$ (which is nonempty), then

$$\inf(S) \le s \le \sup(S)$$

$\Rightarrow \inf(S) \le \sup(S)$

Thus each inequality is proven

12.9 Let $S = \{n \in \mathbb{N} : y < n\}$

Note S is nonempty by the Archimedean principle

well-ordering principle

Hence S has a unique smallest element, call it N where $y < N$

Since $N-1 < N$, $N-1 \notin S$ since this would contradict the minimality of N

Hence $y \ge N-1$

$\therefore \exists! N \in \mathbb{N}$ s.t. $N-1 \le y < N$.

(12.10) a) We show there are infinitely many rationals in (x,y) with $x<y$.

Hence also in $[x,y]$.

Suppose there are only finitely many rationals $r_i \in (x,y)$ $i=1,\dots,n$.

Then $\exists$ a smallest rational, say r_j.

But $r_j > x$ since $r_j \in (x,y)$

And by the density of the rationals, $\exists r \in \mathbb{Q}$ s.t. [Thrm 12.12]

$$x < r < r_j < y$$

contradicting the minimality of r_j

$\therefore$ there are infinitely many rationals in (x,y).

b) Same as a) except you use Thrm 12.14.

(12.13) Let $S = \{q \in \mathbb{Q} : q < x\}$ Let $y = \sup(S)$.

We know S is bounded above by x.

Since y is the least upper bound, $y \le x$.

Suppose $y < x$.

Then by Thrm 12.12 $\exists r \in \mathbb{Q}$ s.t $y < r < x$

Hence $r \in S$. But this contradicts y being an upper bound of S.

Hence $y = x$.

(13.3) in book

(13.5) a) $\{\frac{1}{n} : n \in \mathbb{N}\}$ is neither

It is not closed since 0 is an accumulation pt, but $0 \neq \frac{1}{n}$ for any n

It is not open since it contains isolated pts.

b) $\mathbb{N}$ is closed since every pt is an isolated pt.

c) $\mathbb{Q}$ is neither

Not closed since, say $\sqrt{2}$, is an accum pt with $\sqrt{2} \notin \mathbb{Q}$

Not open since every neighborhood contains elements in $\mathbb{R} - \mathbb{Q}$

d) $\bigcap_1^\infty (0, \frac{1}{n}) = \emptyset$ which is both closed and open

e) $\{x : |x-5| \leq \frac{1}{2}\} = [5-\frac{1}{2}, 5+\frac{1}{2}]$ is closed since it is a closed interval

f) $\{x : x^2 > 0\} = (-\infty, 0) \cup (0, \infty)$ open since union of open sets

(13.6) a) $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$

b) $\mathbb{N}$

c) $\mathbb{R}$

d) $\emptyset$

e) $[5-\frac{1}{2}, 5+\frac{1}{2}]$

f) $\{x : x^2 \geq 0\} = \mathbb{R}$

(13.8) Suppose (a) $x \in \text{int}(S)$, we show $\sim$(b), $\sim$(c) ie $x \notin \text{int}(\mathbb{R}\setminus S)$, $x \notin \text{bd}(S)$

If $x \in \text{int}(S)$, then $\exists\, \varepsilon > 0$ s.t. $N_\varepsilon(x) \subset S$

Hence $N_\varepsilon(x) \cap (\mathbb{R}\setminus S) \subset S \cap (\mathbb{R}\setminus S) = \emptyset$

Hence $x \notin \text{bd}(S)$, so $\sim$(c).

In general for any set A, $\text{int}(A) \subset A$ [since $N_\varepsilon(x) \subset A \Rightarrow x \in A$]

Hence if $x \in \text{int}(\mathbb{R}\setminus S)$, then $x \in \mathbb{R}\setminus S \Rightarrow x \notin S \Rightarrow x \notin \text{int}(S)$

so $\sim$(b).

The proof of (b) $\Rightarrow$ $\sim$(a), $\sim$(c) is exactly the same [Take $S = \mathbb{R}\setminus S$]

Now suppose (c).

If $x \in \text{bd}(S)$ Then for $\forall \varepsilon > 0$,

$N_\varepsilon(x) \cap S \neq \emptyset$ and $N_\varepsilon(x) \cap (\mathbb{R}\setminus S) \neq \emptyset$

Hence no neighborhood of x is strictly contained in S or $\mathbb{R}\setminus S$

$\therefore$ $x \notin \text{int}(S)$ and $x \notin \text{int}(\mathbb{R}\setminus S)$

$\therefore$ $\sim$(b) and $\sim$(c).

(13.11) Let A be open, B closed then $\mathbb{R}\setminus A$ is closed and $\mathbb{R}\setminus B$ is open

$A\setminus B = A \cap (\mathbb{R}\setminus B)$ is open since it is a finite intersection of open sets. Similarly,

$B\setminus A = B \cap (\mathbb{R}\setminus A)$ is closed since it is an intersection of closed sets.

We must also show one of (a), (b) and (c) must hold

Let $x \in \mathbb{R}$, then either $x \in S$ or $x \in \mathbb{R}\setminus S$

If $x \in S$, then $x \in \text{int}(S)$ or $x \in \text{bd}(S)$, so either (a) or (c) hold.

If $x \in \mathbb{R}\setminus S$, then $x \in \text{int}(\mathbb{R}\setminus S)$ or $x \in \text{bd}(\mathbb{R}\setminus S)$, so either (b) or (c) hold

Hence one of (a), (b) and (c) hold.