HW #5 Solutions

$\{13.6, 13.13, 13.17, 13.19, 14.1, 14.3, 14.4, 14.5, 14.8(b), 14.9\}$

(13.6) See HW #4 solutions

(13.13) Prove $Cl(S)/int(S) = bd(S)$

pf ($\supseteq$) Let $x \in bd(S)$, we show $x \in Cl(S)/int(S)$.

Since $x \in bd(S)$, $\forall \varepsilon > 0$

$$N_\varepsilon(x) \cap S \neq \phi \quad \text{and} \quad N_\varepsilon(x) \cap (\mathbb{R} \setminus S) \neq \phi$$

Since $N_\varepsilon(x) \cap (\mathbb{R} \setminus S) \neq \phi$. every neighborhood of x contains a point in $\mathbb{R} \setminus S$, hence no neighborhood of x is entirely inside S

So $x \notin int(S)$.

Since $N_\varepsilon(x) \cap S \neq \phi$, every neighborhood of x contains a point of S

Hence $x \in Cl(S)$ [see paragraph below definition 13.16].

$\therefore$ $x \in Cl(S)/int(S)$

($\subseteq$) conversely, let $x \in Cl(S)/int(S)$

Then $x \in Cl(S)$ and $x \notin int(S)$.

But $x \in Cl(S) \Rightarrow \forall \varepsilon > 0$, $N_\varepsilon(x) \cap S \neq \phi$

and $x \notin int(S) \Rightarrow \forall \varepsilon > 0$, $N_\varepsilon(x) \cap (\mathbb{R} \setminus S) \neq \phi$

$\therefore$ $x \in bd(S)$

Hence $Cl(S)/int(S) = bd(S)$.

(13.17) Prove S' is closed. We closely follow Thm 13.7(b)'s method of proof.

We show $(S')' \subset S$ [so S' contains its accumulation pts].

Let $x \in (S')'$, then $\forall \varepsilon > 0$

$$N_\varepsilon^*(x) \cap S' \neq 0$$

We must show $N_\varepsilon^*(x) \cap S \neq 0$

To this end, let $y \in N_\varepsilon^*(s) \cap S'$. Since $y \in N_\varepsilon^*(s)$ and $N_\varepsilon^*(s)$ is open $\exists \varepsilon_0 > 0$ s.t.

$$N_{\varepsilon_0}(y) \subseteq N_\varepsilon^*(x)$$

(13.17 cont) But $y \in S'$ also, so every neighborhood of y intersects S non-trivially.
In particular, $\exists z \in N_{\varepsilon_2}(y) \cap S \subseteq N_\varepsilon^*(x) \cap S$, ie

$$N_\varepsilon^*(x) \cap S \neq \emptyset$$

Hence $x \in S'$

<u>Note</u>: This proof used the fact that a neighborhood is an open set.
This is a necessary condition to prove any of:

$$S' \text{ is closed}, \quad cl(S) \text{ is closed}, \quad \underbrace{int(S) \text{ is open etc}}_{\text{on your midterm}}$$

Try to think why...

(13.19) Since A is nonempty, let $x \in A$. Since A is open, $\exists \varepsilon > 0$ s.t.

$$N_\varepsilon(x) = (x-\varepsilon, x+\varepsilon) \subseteq A$$

Hence $y = x + \varepsilon/2 \in A$, but $x < y \Rightarrow \exists q \in \mathbb{Q}$ s.t.

$x < q < y$, hence $q \in A$.

So $q \in A \cap \mathbb{Q}$

$\therefore A \cap \mathbb{Q} \neq \emptyset$

(14.1) a) T, by def

b) T, Let $\{V_i\}$ be an open cover of $X = \{\text{finite set of pts}\} = \{x_i\}_{i=1}^{N}$

Then for each x_i, $\exists V_j$ s.t. $x_i \in V_j$

Since there are only finitely many x_i, only a finite number of V_j are needed to cover X

c) F, $[0,1]$ is a counterexample

d) T (at least in $\mathbb{R}$ by 14.4, false in general topological spaces)

e) F $[0,1) \cup (1,5]$ is a counterexample. Note it is not closed.

(14.3) a) $\bigcup_{n=1}^{\infty} (0, 3-1/n)$

b) $\bigcup_{n=1}^{\infty} (n - 1/4, n + 1/4)$

c) $\bigcup_{n=1}^{\infty} (1/n, 2)$

d) $\bigcup_{n=1}^{\infty} (-1, \sqrt{2} - \frac{1}{n}) \cup (\sqrt{2} + \frac{1}{n}, 3)$

(14.4) We prove this in $\mathbb{R}$ (I don't think it's true in general topological spaces)

Let $\{k_i\}$ be a collection of compact sets in $\mathbb{R}$.

Hence each k_i is closed and bounded.

By 13.11 a) arbitrary intersections of closed sets are closed

Moreover, since $k_1 \supset \bigcap_i k_i$ and k_1 is bounded then $\bigcap_i k_i$ is bounded

Hence $\bigcap_i k_i$ is closed and bounded, therefore compact.

(14.5) a) Since a finite union of closed sets is closed, $S_1 \cup S_2$ is closed.

Since S_1, S_2 are bounded, $\exists n, m \in \mathbb{N}$ s.t.

$S_1 \subset [-n, n]$ $S_2 \subset [-m, m]$

Then $S_1 \cup S_2$ is bounded by $\max\{n, m\}$

b) Take $S_n = [-n, n]$, each S_n is closed and bounded, but

$\bigcup_{n=1}^{\infty} S_n = \mathbb{R}$ which is unbounded.

(14.8) b) Since T is closed we must only show T is bounded.
But since $T \subseteq S$ and S is compact (hence bounded), T is bounded.

(14.9) Let $\varepsilon > 0$, then

$$\bigcup_{x \in \mathbb{R}} N_\varepsilon(x)$$ is an uncountable cover of $\mathbb{R}$

It has no finite subcover since any finite subcover would be bounded
It does however have a countable subcover, namely

$$\bigcup_{x \in \mathbb{Q}} N_\varepsilon(x)$$

This is an open cover of $\mathbb{R}$ by the density of $\mathbb{Q}$ in $\mathbb{R}$