HW # 6 Solutions

$\{16.2,\ 16.6\,(c)(d),\ 16.7\,(a\text{-}d),\ 16.8\,(b\text{-}e),\ 16.10,\ 16.13,\ 17.3,\ 17.8,\ 17.15\}$

(16.2) a) True. $s_n \to 0 \Rightarrow \forall \varepsilon > 0,\ \exists N$ s.t. for $n \geq N$ $|s_n| < \varepsilon \Rightarrow -\varepsilon < \boxed{s_n < \varepsilon}$

b) False. s_n can converge to any negative number and still satisfy that

$\forall \varepsilon > 0 \quad \exists N$ s.t. for $n \geq N \quad s_n < \varepsilon$

so s_n need not go to 0.

c) False. Notice a_n need not go to 0

d) True Theorem 16.14

(16.6) c) pv $\lim \frac{\sin(n)}{n} = 0$

pf Let $\varepsilon > 0$, take $N = \frac{2}{\varepsilon}$, then for $n \geq N$

$$\left| \frac{\sin(n)}{n} - 0 \right| \leq \frac{1}{n} \leq \frac{1}{N} \leq \frac{\varepsilon}{2} < \varepsilon$$

d) pv $\lim \frac{n+2}{n^2-3} = 0$. Let $\varepsilon > 0$,

pf: $\left| \frac{n+2}{n^2-3} \right| = \frac{n+2}{n^2-3} < \frac{2n}{\frac{1}{2}n^2}$ if $n > 2$ $\left[\text{note } n^2 > 3,\ \frac{1}{2}n^2 < n^2-3 \text{ if } n > 2;\ 2n > n+2\right.$

$$= \frac{4}{n}$$

if we take $N = \max\{2, \frac{4}{\varepsilon}\}$, then $\frac{4}{n} \leq \frac{4}{N} \leq \varepsilon$ for all $n \geq N$

Hence $\left| \frac{n+2}{n^2-3} \right| < \varepsilon \quad \forall n \geq N$

(16.7) a) $0 \leq \lim \frac{1}{1+3n} < \frac{1}{3} \lim \left(\frac{1}{n}\right) = 0 \Rightarrow \lim \left(\frac{1}{1+3n}\right) = 0$

b) just like 16.6 d)

c) $\left| \frac{6n^2+5}{2n^2-3n} - 3 \right| = \left| \frac{\cancel{6n^2}+5-\cancel{6n^2}+6n}{2n^2-3n} \right| < \frac{7n}{n^2} = \frac{7}{n}$ if $n > 5$

Note $7n = 5+6n \Rightarrow n = 5 \Rightarrow$ when $n > 5$, $7n > 5+6n$ and

$n^2 = 2n^2-3n \Rightarrow n = 3 \Rightarrow$ when $n > 3$, $n^2 < 2n^2-3n$

So take $N = \max\{5, 7/\varepsilon\}$ to show convergence

d) $0 \leq \lim \frac{\sqrt{n}}{n-1} \leq \lim \frac{1}{\sqrt{n}} = 0$ by practice 16.4

(16.8) Instead of negating the def. of convergence (which is tricky) we use the fact [Thrm 16.14] that if s_n converges, its limit is unique. Hence by the contrapositive, if we can find two different limits of s_n then the sequence cannot converge.

b) $b_n = (-1)^n$ so $b_{2n} = 1$, $b_{2n+1} = -1$ $\forall n = 0, 1, \dots$

hence as $n \to \infty$ along the even numbers $b_n \to 1$

$n \to \infty$ along the odd numbers $b_n \to -1$

$\therefore$ b_n does not converge.

c) $c_n = \cos(\frac{n\pi}{3})$ so $c_{3n} = \cos(n\pi) = (-1)^n$

and so by b) $c_{6n} = 1$, $c_{6n+1} = -1$ and c_n does not converge

(16.10) a) decimal expansions of π

b) $\{\pi/n\}_{n=1}^{\infty}$

(16.13) Let $\varepsilon > 0$. Since $a_n \to b$, $\exists N_1$ s.t. $\forall n \geq N$ $|a_n - b| < \varepsilon$ $\Rightarrow$ $-\varepsilon < a_n - b$

Since $c_n \to b$ $\exists N_2$ s.t. $\forall n \geq N$, $|c_n - b| < \varepsilon$ $\Rightarrow$ $c_n - b < \varepsilon$

Now let $N = \max\{N_1, N_2\}$, then for $\forall n \geq N$

$$b - \varepsilon < a_n \leq b_n \leq c_n \leq b + \varepsilon$$

Hence

$$-\varepsilon < b_n - b < \varepsilon \Rightarrow |b_n - b| < \varepsilon \quad \forall n \geq N$$

$\therefore$ $b_n \to b$

17.3 a) $$\lim \frac{3n^2+4n}{7n^2-5n} = \lim \frac{\frac{1}{n^2}(3n^2+4n)}{\frac{1}{n^2}(7n^2-5n)} = \lim \frac{(3+\frac{4}{n})}{(7+\frac{5}{n})} = \frac{\lim(3+\frac{4}{n})}{\lim(7+\frac{5}{n})}$$

$$= \frac{3}{7}$$

b) $$\lim \frac{n^4+13}{2n^5+3} = \lim \frac{\frac{1}{n^5}(n^4+13)}{\frac{1}{n^5}(2n^5+3)} = \frac{\lim \frac{1}{n} + \frac{13}{n^5}}{\lim (2+\frac{3}{n^5})} = \frac{0}{2} = 0$$

17.8 a) $s_n = 1$

b) $t_n = n$ since $\lim \frac{n}{n+1} = \frac{\lim 1}{\lim(1+\frac{1}{n})} = 1$ yet $t_n \to \infty$

17.15 a) $$\lim(\sqrt{n+1} - \sqrt{n}) = \lim\left(\frac{(\sqrt{n+1}-\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}(\sqrt{n+1}+\sqrt{n})\right)$$

$$= \lim\left(\frac{1}{\sqrt{n+1}+\sqrt{n}}\right) \leq \lim\left(\frac{1}{\sqrt{n}}\right) = 0$$

b) $$\lim(\sqrt{n^2+1} - n) = \lim\left(\frac{(\sqrt{n^2+1}-n)}{\sqrt{n^2+1}+n}(\sqrt{n^2+1}+n)\right)$$

$$= \lim\left(\frac{1}{\sqrt{n^2+1}+n}\right) < \lim\left(\frac{1}{n}\right) = 0$$

c) $$\lim(\sqrt{n^2+n} - n) = \lim\left(\frac{(\sqrt{n^2+n}-n)}{\sqrt{n^2+n}+n}(\sqrt{n^2+n}+n)\right)$$

$$= \lim\left(\frac{n}{\sqrt{n^2+n}+n}\right)$$

$$= \lim\left(\frac{1}{\sqrt{1+1/n}+1}\right)$$

$$= \frac{1}{1+\lim\sqrt{1+1/n}}$$

$$= \frac{1}{2}$$ which is straight forward to check.