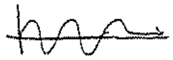
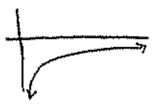
HW #7 Solutions

$\{18.2,\ 18.3(c),\ 18.5,\ 18.7,\ 18.8,\ 19.4\ (a)(b),\ 19.7,\ 19.9,\ 19.15\}$

18.2 a) False. 

b) False. 

c) False. A sequence of rationals approaching an irrational will not converge in $\mathbb{Q}$ however the points will be getting close to each other.

18.3 c) $s_1 = 1$, $s_{n+1} = 1/4(2s_n + 5)$ $\forall n$

$s_2 = 1/4(7) = 7/4 > 1$ so sequence seems to be increasing

Assume $s_k \geq s_{k-1}$ for some $k \in \mathbb{N}$, then

$$s_{k+1} = 1/4(2s_k + 5) \geq 1/4(2s_{k-1} + 5) = s_k$$

Hence by induction $\{s_k\}$ is monotonically inc. ✓

Suppose $s_k < 100$ for some $k \in \mathbb{N}$, then

$$s_{k+1} = 1/4(2 \cdot 100 + 5) = \frac{205}{4} < 100$$

Hence s_k is bounded by induction. ✓

To find the limit, we solve

$$s = 1/4(2s + 5) \Rightarrow s = \tfrac{1}{2}s + \tfrac{5}{4} \Rightarrow \tfrac{1}{2}s = 5/4 \Rightarrow s = 10/4$$

So $\lim_{k \to \infty} s_k = 5/2$ ✓

18.5 a) Counter-example: $a_n = x^2$, $b_n = -x$. Then $a_n + b_n = x^2 - x = x(x-1)$, see below.

b) Counterexample: $a_n = x$, $b_n = x - 1$, see above.

(18.7) $s_1 = \sqrt{6}$, $s_{n+1} = \sqrt{6+s_n}$

To show $\{s_n\}$ converges we show it is monotone and bounded

Pf $s_1 = \sqrt{6}$, $s_2 = \sqrt{6+\sqrt{6}} > s_1$

Assume $s_k > s_{k-1}$ for some $k \in \mathbb{N}$, then

$$s_{k+1} = \sqrt{6+s_k} > \sqrt{6+s_{k-1}} = s_k$$

So by induction $\{s_n\}$ is monotone

Assume $s_k < 100$ for some k, then

$$s_{k+1} = \sqrt{6+s_k} < \sqrt{106} < 11 < 100$$

Hence by induction $\{s_n\}$ is bounded

$\therefore$ s_n converges

To find its limit, solve

$$s = \sqrt{6+s} \implies s^2 = 6+s \implies s^2 - s + 6 = 0$$

So $s \in \{-3, 2\}$

Since $s_k \geq 0$ $\forall k$, $s = 2$.

(18.8) $s_1 = k$, $s_{n+1} = \sqrt{4s_n - 1}$

Note this sequence is monotone, since for some n

if $s_n > s_{n-1}$, then $s_{n+1} = \sqrt{4s_n - 1} > \sqrt{4s_{n-1} - 1} = s_n$, so $\{s_n\}$ is increasing

if $s_n < s_{n-1}$, then similarly $\{s_n\}$ is decreasing

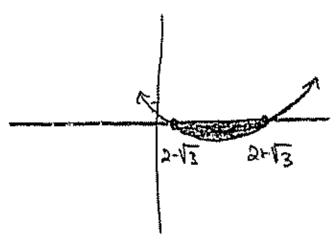
Hence to know which values of k s_n will be increasing we need to find out what value makes $s_2 > s_1$, ie we need to solve for k:

$$\sqrt{4k-1} > k$$

So $4k - 1 > k^2 \implies k^2 - 4k + 1 < 0 \implies (k-(2+\sqrt{3}))(k-(2-\sqrt{3})) < 0$

To make sure the LHS is negative we need

$$2-\sqrt{3} < k < 2+\sqrt{3}$$



Similarly $\{s_n\}$ will be monotone dec. for

$k < 2-\sqrt{3}$ and $k > 2+\sqrt{3}$

19.4 a) $S = \{0\}$, $\limsup = \liminf = 0$

b) $S = \mathbb{N}$ $\quad \lim\sup = \infty$ $\quad \liminf = 0$

19.7 a) True, by Thm 19.7 every bounded sequence has a convergent sequence and every convergent sequence is Cauchy (18.10)

b) False, $s_n = n$

c) True, take $r_n = \frac{s_n}{2} + c(-1)^n$, $t_n = \frac{s_n}{2} + c(-1)^{n+1}$ where $c > 0$

19.9 Suppose $\lim s_n = r$ and $r \neq s$

If $r > s$ then, since r is a subsequential limit

$s = \limsup (s_n) \geq r > s$, contradiction

Similarly if $r < s$

Hence r must be equal to s

19.15 a) Let $s = \liminf (s_n)$, $t = \liminf (t_n)$

Then for $\forall \varepsilon > 0$, $\exists N_1$ s.t. $\forall n \geq N_1$ $\quad s_n > s - \varepsilon$ and $s_n < s + \varepsilon$

Similarly, $\exists N_2$ s.t. $\forall n \geq N_2$ $\quad t_n > t - \varepsilon$ and $t_n < t + \varepsilon$

Let $N_\varepsilon = \max\{N_1, N_2\}$, then for $n \geq N_\varepsilon$ [note that N depends on ε]

$$s + t - 2\varepsilon < s_n + t_n < s + t + 2\varepsilon$$

Now let $\varepsilon = 1/k$, then for each k, $\exists n_k > N_{1/k}$ s.t.

$$|(s_{n_k} + t_{n_k}) - (s + t)| < 2/k$$

as $k \to \infty$,

$$s_{n_k} + t_{n_k} \to s + t$$

Hence $s + t \in \{\text{set of subsequential limits of } (s_n + t_n)\}$

Since $\liminf (s_n + t_n)$ is the infimum of this set (ie greatest lower bound)

$$\liminf (s_n + t_n) \geq s + t = \liminf (s_n) + \liminf (t_n)$$

b) Take $s_n = (-1)^n$, $t_n = (-1)^{n+1}$, note $s_n + t_n \equiv 0$ $\forall n$

while $s = t = -1$