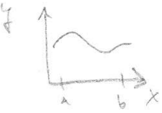
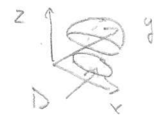
Surface Integrals Review

Before we always defined our functions on a flat, 1-D, or 2-D space

1-D $\quad y = f(x)$  $\displaystyle\int_a^b f(x)\,dx$

2-D $\quad z = f(x,y)$  $\displaystyle\iint_D f(x,y)\,dx\,dy$

What if we have a function defined on a curved space (like the surface of the earth)? These are path (1-D) and surface (2-D) integrals.

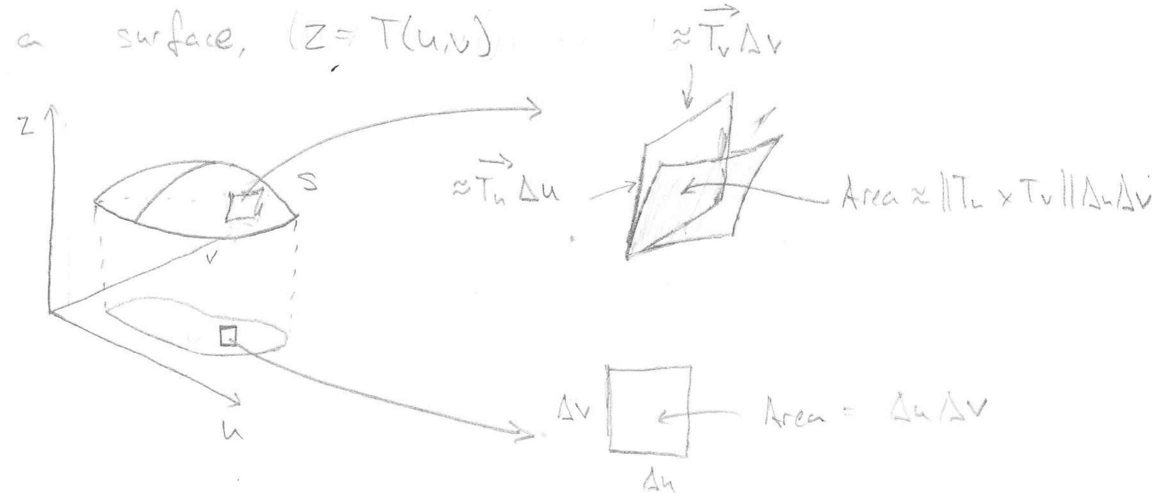
1-D $\quad c(t) \quad \displaystyle\int_C f\,ds = \int_{t_1}^{t_2} f(c(t))\,c'(t)\,dt$

2-D $\quad T(u,v) \quad \displaystyle\iint_S f\,dS = \iint_D f(T(u,v))\,\|T_u \times T_v\|\,dA$

In the 1-D case, the $c'(t)$ factor tells by how much the curve $c(t)$ distorts a small segment $\overline{\quad}_{\Delta t}$ (ie. $c(t+\Delta t) - c(t) \approx c'(t)\Delta t$)

In the 2-D case the $\|T_u \times T_v\|$ factor tells by how much the parameterization $T(u,v)$ distorts the small area segment $\square\,\Delta v$, Δu

So when given a surface, $(z = T(u,v)$



So here are the steps to solving $\iint_S f\, dS$

- If you have a parameterization $T(u,v)$ for S, then

$$\iint_S f\, dS = \int_{\substack{\text{range}\\ \text{of } v}} \int_{\substack{\text{range}\\ \text{of } u}} f(T(u,v))\, \|T_u \times T_v\|\, du\, dv$$

- If you don't have a parameterization for S, but instead S is a level curve of a function F, then

  i) calculate the unit normal of S by

$$n = \frac{\nabla F}{\|\nabla F\|}$$

  ii) $\iint_S f\, dS = \iint_D \frac{f}{|n \cdot v|}\, dA$ where D is the region defined by projecting S onto either the x-, y-, z-axis and v is the direction along which you projected, so v is either i, j, k respectively.

For surface integrals of vector fields we calculate instead

$$\iint_S \vec{F} \cdot d\vec{S}$$

If you have a parameterization you can solve this by calculating

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(T(u,v)) \cdot \vec{N}(u,v) \; \underbrace{dA}_{du\,dv}$$

where $N(u,v) = T_u \times T_v$

If you don't have a parameterization you solve this by noting

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(T(u,v)) \cdot \frac{\vec{N}(u,v)}{\|N(u,v)\|} \overbrace{\|N(u,v)\| \, dA}^{dS}$$

$$= \iint_S \vec{F}(T(u,v)) \cdot \vec{n} \; dS$$

as in step ii) before

$$= \iint_D \frac{\vec{F}(T(u,v)) \cdot \vec{n}}{|n \cdot v|} \, dA$$