Review of Surface Integrals

What is a parametrization?

It is a mapping from a flat domain to a curved space (e.g. curve, surface, solid)

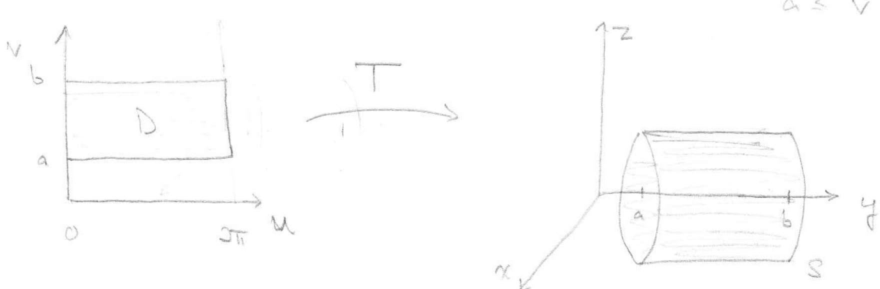
This mapping lets us associate functions defined on a curved space, to functions define on a flat space.

e.g. Let S be a cylinder of radius 1 along the y-axis

ie the set of points satisfying $x^2+z^2=1$

we can parameterize S by the mapping

$$T(u,v) = (\cos(u), v, \sin(u)) \qquad \left.\begin{array}{l} 0 \le u \le 2\pi \\ a \le v \le b \end{array}\right\} D = \text{domain}$$



Now say we have a function $f(x,y,z)$ defined on S

Essentially every point of S has a number associated to it

To calculate

$$\iint_S f(x,y,z)\, dS \approx \sum\sum f \Delta S$$

means to "sum up" all these numbers (and multiple by an infinitesimal surface area)

But what is an infinitesimal surface area?

What if instead, we tranferred all those numbers to the u-v plane?

$$\iint_D f(T(u,v)) \_\_\_\_ = \iint_S f(x,y,z)\, dS$$

and now note that an infinitesimal surface element has area $= dA \, \|T_u \times T_v\|$

ie a number on S has its value smeared out over a different area than on D.

So for example, say $f(x,y,z) = x^2 + z^2$

Then

$$\iint_S f\, dS = \iint_D f(T(u,v)) \, \|T_u \times T_v\| \, dA$$

$$= \int_a^b \int_0^{2\pi} du\, dv$$

$$= 2\pi(b-a)$$

$f(T(u,v)) = 1$

$$T_u \times T_v = \begin{vmatrix} i & j & k \\ -\sin(u) & 0 & \cos(u) \\ 0 & 1 & 0 \end{vmatrix}$$

$$= (-\cos(u), 0, -\sin(u))$$

$\|T_u \times T_v\| = 1$

Note that $T(u,v)$ hasn't stretched out any area (since $dA = dS$)

what if $T(u,v) = (\cos(2\pi u), v, \sin(2\pi v))$ $a \leq v \leq b$, $0 \leq u \leq 1$

was used?

The area would be stretched out by 2π!

Let's try this one:

Let S be the surface defined by the plane $x+y+z=1$

lying in the 1^{st} quadrant. Let $f(x,y,z) = z$

What is $\iint f\, dS$?

$$\iint_S f\, dS = \int_0^1 \int_0^{1-x} (1-x-y)\sqrt{3}\, dy\, dx = \int_0^1 \int_0^{1-z} z\sqrt{3}\, dy\, dz = \int_0^1 \int_0^{1-y} y\sqrt{3}\, dy\, dx$$

$$= \sqrt{3}/6$$

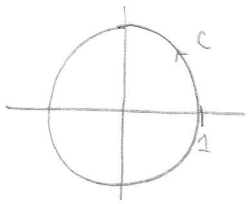
These correspond to the different parameterizations

$(x, y, 1-x-y)$, $(1-y-z, y, z)$, $(x, 1-x-z, z)$

respectively.

HW 3.

3)



Evaluate $\int_C y\,dx + 15x\,dy$

Here $F = (y, 15x) = (P, Q)$

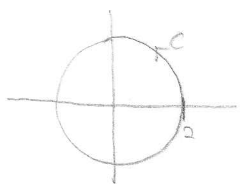
$ds = (dx, dy)$

$$\int_C y\,dx + 15x\,dy = \iint_D \left(\frac{\partial(15x)}{\partial x} - \frac{\partial(y)}{\partial y} \right) dA$$

$$= \iint_D 15 - 1 \; dA$$

$$= 14 \,(\text{area of } D) = 14\pi$$

5) Let $F = (-5y, 2x)$ Then $\int_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$



$$= \iint_D 2 - 5 \; dA$$

$$= -3\,(\pi 2^2)$$

$$= -12\pi$$

If you wanted to evaluate these path integrals directly you would need to parameterize the path like so:

3) Let $c(t) = (\cos(t), \sin(t))$ $\quad 0 \le t \le 2\pi$

Then

$$\int_C y\,dx + 15x\,dy = \int_0^{2\pi} \sin(t)\, \underbrace{d(\cos(t))}_{-\sin(t)\,dt} + 15\cos(t)\, \underbrace{d(\sin(t))}_{\cos(t)\,dt}$$

$$= \int_0^{2\pi} -\sin^2(t) + 15\cos^2(t)\; dt$$

$$= 14\pi$$

<u>Quiz</u> Let S be defined by the plane

$$2x + y + z = 2$$

lying in the 1st octant

Let $f(x,y,z) = x + y + z$

Compute

$$\iint_S f\, dS$$

<u>ans</u>:

$$\iint_S f\, dS = \iint_D (2-x)\sqrt{6}\, dA$$

$$= \sqrt{6}\int_0^1\int_0^{2-2x} (2-x)\, dy\, dx$$

$$= \sqrt{6}\int_0^1 2y - xy \Big|_0^{2-2x} dx$$

$$= \sqrt{6}\int_0^1 \underbrace{2(2-2x) - x(2-2x)}\, dx$$

$$= \sqrt{6}\int_0^1 2x^2 - 6x + 4\, dx$$

$$= \sqrt{6}\left(\frac{2}{3} - 3 + 4\right)$$

$$= \frac{5\sqrt{6}}{3}$$

$$T(x,y) = (x, y, 2-2x-y)$$

$$f(T(x,y)) = x + \cancel{y} + 2 - 2x - \cancel{y}$$

$$= 2 - x$$

$$T_x \times T_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= (2, -1, 1)$$

$$\|T_x \times T_y\| = \sqrt{6}$$