Math 3B Final Review

- Fundamental Thrm of Calculus

ex. ① Let $g(x) = \int_0^x \sin t^2 \, dt$

Then

$g'(x) = \sin(x^2)$

② Let $g(x) = \int_5^{\ln(x)} \sin(t^2)\,dt$ — can be any # doesn't matter.

derivative

Then

$g'(x) = \sin(x^2)\underbrace{\left(\frac{1}{x}\right)}$

- Techniques of Integration
  - Substitution rule (udu)
  - Integration by parts (IBP): $\int u\,dv = uv - \int v\,du$ indefinite integral

    $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$ definite integral
  - Trig Substitution

How to recognize one technique from another?

- Use subst. rule when dividing by the derivative of something in the integrand will cancel everything else
- Use IBP when the derivative of u and the integral of dv are easier to evaluate
- Use trig sub to make expressions involving $\sqrt{\pm a \pm x^2}$ easier to evaluate.

Examples: (* means worked out in detail later)

hard!

*① $\int x^4 \ln(x)\,dx$

② $\int \frac{\ln(x)}{x}\,dx$

*③ $\int \frac{\sqrt{1+\ln(x)}}{x\ln(x)}\,dx$

*④ $\int \frac{x}{x-2}\,dx$

⑤ $\int \sin(x)\cos(\cos(x))\,dx$

⑥ $\int e^x\sqrt{1+e^x}\,dx$

⑦ $\int x\sin(x)\,dx$

*⑧ $\int e^x \sin(x)\,dx$

*⑨ $\int \cos(x)\ln(\sin(x))\,dx$

⑩ $\int \frac{\ln(\tan(x))}{\sin(x)\cos(x)}\,dx$

*⑪ $\int \tan^3(x)\,dx$

⑫ $\int \sin^3(x)\cos^2(x)\,dx$

⑬ $\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

*⑭ $\int \frac{x^2}{\sqrt{x^2-4}}\,dx$

⑮ $\int \frac{1}{x^2\sqrt{1+x^2}}\,dx$

Technique

① Notice the derivative of $\ln(x)$ is easy and the integral of x^4 isn't bad either. They combine to give $\left(\frac{1}{x}\right)\left(\frac{1}{5}x^5\right)$ which is simple to integrate ... so think IBP!

② This seems similar to ① and can be solved the same way, except there is an easier way too! Notice the derivative of $\ln(x)$ is $\frac{1}{x}$ and dividing by $\frac{1}{x}$ is like multiplying by x which cancels the denominator ... so think udu!

③ IBP would only make this uglier, Trig sub isn't applicable, lets try udu! I will work this out later

④ Wouldn't it be nice if we had $\int \frac{x-2}{x}\,dx$ instead? We can "flip the integral over" by using udu and solve for the original variable.

⑤ What's the deriv. of $\cos(x)$? Right! use udu

⑥ What's the deriv of $1+e^x$? " "

⑦ Deriv. of x = easy! Int of $\sin(x)$ = easy! Use IBP!

⑧ This is IBP 2 times. We'll work it out

(9) Can we use udu? Deriv of sin(x) = cos(x). the integral reduces to $\int \ln(y)\,dy$ But how do we do this? Deriv of $\ln(x)$ = easy, so is integral of dy. Try IBP $u = \ln(y)$ $dv = dy$!

(10) Looks like (9) but much easier to solve! Notice

$$\frac{d}{dx}\ln(\tan(x)) = \frac{1}{\tan(x)}\sec^2(x)\,dx = \frac{\cos(x)}{\sin(x)}\frac{1}{\cos^2(x)}\,dx = \frac{1}{\sin(x)\cos(x)}\,dx$$

Look familiar? Use udu! $u = \ln(\tan(x))$

(11) Remember the rules (pg 486) if power of tan is odd break it up into $\tan^2(x)$'s $\longrightarrow$ $(\sec^2(x)-1)$'s and let $u = \sec(x)$

(12) Remember the rules (pg 484) look to odd powers trig function (here $\sin(x)$) and let u = the other one (here $u = \cos(x)$)

(13), (14), (15)

When you see

$\sqrt{1-x^2}$	use $x = \sin(\theta)$	recall $1-\sin^2(\theta) = \cos^2(\theta)$
$\sqrt{1+x^2}$	use $x = \tan(\theta)$	recall $1+\tan^2(\theta) = \sec^2(\theta)$
$\sqrt{x^2-1}$	use $x = \sec(\theta)$	recall $\sec^2(\theta) - 1 = \tan^2(\theta)$

exception: $\int \frac{x}{\sqrt{1-x^2}}\,dx$ is much easier since udu works!

Let's get our hands dirty now!

Doing these in detail - - -

① $\int x^4 \ln(x)\,dx$ let $u = \ln(x)$ $du = \frac{1}{x}dx$

$dv = x^4 dx$ $v = \frac{1}{5}x^5$

$= \frac{1}{5}x^5 \ln(x) - \int \left(\frac{1}{5}x^5\right)\left(\frac{1}{x}\right)dx$

$= \frac{1}{5}x^5 \ln(x) - \frac{1}{5}\int x^4\,dx$

$= \boxed{\frac{1}{5}x^5 \ln(x) - \frac{1}{25}x^5 + C}$

③ $\int \frac{\sqrt{1+\ln(x)}}{x\ln(x)}\,dx$ let $u = \sqrt{1+\ln(x)}$ [note: you can also let $u = 1+\ln(x)$ but will need to make another substitution, see pg 503 ex. 9]

$du = \frac{1}{2}(1+\ln(x))^{-1/2}\frac{dx}{x}$

$= \frac{dx}{2ux}$

$= \int \frac{u}{x\ln(x)}(2ux\,du)$

$= 2\int \frac{u^2}{\ln(x)}\,du$ but $\ln(x) = u^2 - 1$

$= 2\int \frac{u^2}{u^2-1}\,du$

$= 2\int \frac{(u^2-1)+1}{u^2-1}$ Common trick when exponent powers are same top & bottom

$= 2\int \left(1 + \frac{1}{u^2-1}\right)du$

$= 2u + 2\int \frac{1}{u^2-1}\,du$ note $\frac{1}{u^2-1} = \frac{1}{(u+1)(u-1)} = \frac{A}{u+1} + \frac{B}{u-1} \Rightarrow$ $A(u-1) + B(u+1) = 1$

$u(A+B) - A + B = 1$

$\Rightarrow A+B = 0 \Rightarrow A = -B \Rightarrow A = -1/2$

$-A+B = 1 \Rightarrow 2B = 1 \Rightarrow B = 1/2$

$= 2u + 2\int \frac{-1/2}{u+1} + \frac{1/2}{u-1}\,du$

$= \boxed{2u - \ln(u+1) + \ln(u-1) + C}$

④ $\int \frac{x}{x-2}\,dx$ $u = x-2 \Rightarrow x = u-2$

$du = dx$

$= \int \frac{u-2}{u}\,du$ This is a common "flip the fraction over" trick using u-sub

$= \int 1 - \frac{2}{u}\,du$

$\boxed{= u - 2\ln(u) + C}$

⑧ $\int e^x \sin(x)dx$ $\qquad u=\sin(x) \quad du=\cos(x)dx$
$\qquad dv=e^x dx \quad v=e^x$

$= e^x\sin(x) - \underbrace{\int e^x\cos(x)dx}_{\cos(x)e^x - \int e^x(-\sin(x))dx}$ $\qquad u=\cos(x) \quad du=-\sin(x)dx$
$\qquad dv=e^x dx \quad v=e^x$

putting it together gives

$$\underline{\int e^x \sin(x)dx} = e^x\sin(x) - e^x\cos(x) - \underline{\int e^x\sin(x)dx}$$

same!

$\Rightarrow 2\int e^x\sin(x)dx = e^x\sin(x) - e^x\cos(x)$

so $\boxed{\int e^x\sin(x)dx = \frac{1}{2}e^x\left(\sin(x)-\cos(x)\right)+C}$

⑨ $\int \cos(x)\ln(\sin(x))dx$ $\qquad u=\sin(x)$
$\qquad du=\cos(x)dx \Rightarrow dx=\frac{du}{\cos(x)}$

$= \int \cancel{\cos(x)}\ln(u)\left(\frac{du}{\cancel{\cos(x)}}\right)$

$= \int \ln(u)du$ $\qquad$ This is a IBP see pg 477 ex 2

$= u\ln(u)-u+C$

$= \boxed{\sin(x)\ln(\sin(x))-\sin(x)+C}$

⑪ $\int \tan^3(x)dx$

$= \int \tan^2(x)\tan(x)dx$

$= \int(\sec^2(x)-1)\tan(x)dx$

$= \int\sec^2(x)\tan(x)dx - \int\tan(x)dx$ $= \boxed{\frac{1}{2}\tan^2(x)+\ln|\sec(x)|+C}$

$\int \sec^2(x)\tan(x)dx$ $\qquad u=\tan(x)$
$\qquad du=\sec^2(x)dx$

$\int u\,du$

$= \frac{1}{2}u^2+C = \frac{1}{2}\tan^2(x)+C$

$\int\tan(x)dx = \int\frac{\sin(x)}{\cos(x)}dx$ $\qquad u=\cos(x)$
$\qquad du=-\sin(x)dx$

$= -\int\frac{1}{u}du$

$= -\ln(u)+C = -\ln(\cos(x))+C$ or $\ln\left(\frac{1}{\cos(x)}\right)=\ln(\sec(x))$

(14) $\displaystyle\int \frac{x^2}{\sqrt{x^2-4}}\,dx$ $x=2\sec(\theta)$, $dx=2\sec(\theta)\tan(\theta)$ $\sqrt{x^2-4} \longrightarrow \sqrt{4\sec^2\theta-4} = 2\sqrt{\sec^2\theta-1} = 2\tan(\theta)$

$$= \int \frac{4\sec^2(\theta)}{2\tan(\theta)}\,d\theta = 2\int \frac{\sec^2(\theta)}{\tan(\theta)}\,d\theta$$

but deriv of tan is $\sec^2$!

$u=\tan(\theta)$

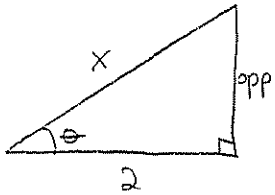
$du=\sec^2(\theta)\,d\theta$

$$= 2\int \frac{1}{u}\,du$$

$$= 2\ln(u)+C$$

$$= 2\ln(\tan(\theta))+C$$

$$= \boxed{2\ln\left(\frac{\sqrt{x^2-4}}{2}\right)+C}$$



$\sec(\theta) = \frac{x}{2} = \frac{1}{\cos(\theta)} = \frac{hyp}{adj}$

what is $\tan(\theta) = ?$

$2^2+opp^2=x^2 \Rightarrow opp=\sqrt{x^2-4}$

$\tan(\theta) = \frac{opp}{adj} = \frac{\sqrt{x^2-4}}{2}$

Rational Functions

- First Step - if deg(numerator) < deg(denominator) divide!

ex. $\dfrac{x^3-4x-10}{x^2-x-6}$

$$\begin{array}{r} x+1 \\ x^2-x-6\,\overline{)\,x^3-4x-10} \\ -\;x^3-x^2-6x \\ \hline x^2+2x-10 \\ -\;x^2-x-6 \\ \hline 3x-4 \end{array}$$

← remainder since $\deg(3x-4)=1<\deg(x^2-x-6)=2$

$$= \boxed{x+1+\frac{3x-4}{x^2-x-6}}$$

- Cases

1) Simplest: denom. breaks up into non-repeating linear factors

ex. $\dfrac{x-9}{x^2-7x-10} = \dfrac{x-9}{(x+5)(x-2)} = \dfrac{A}{x+5} + \dfrac{B}{x-2} = \boxed{\dfrac{2}{x+5} - \dfrac{1}{x-2}}$

By putting this over a common denominator we find

$$A(x-2)+B(x+5) = x-9$$

Rearranging (group in terms of x's) gives

$$x(A+B) - 2A+5B = 1x-9$$

$\Rightarrow$ $A+B=1 \Longrightarrow A=1-B$

$-2A+5B=-9 \Longrightarrow -2(1-B)+5B=-9 \Longrightarrow B=-1$ so $A=2$

2) <u>Repeated root</u>

ex. $\dfrac{2x+3}{(x+1)^2} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+1)^2} \longleftarrow$ (note the difference. if we proceeded as if the roots were distinct we would have $\dfrac{B}{x+1}$ not $\dfrac{B}{(x+1)^2}$)

Putting the fractions under a common den. gives

$$A(x+1)+B = 2x+3$$

or

$$x(A)+(A+B) = 2x+3$$

so

$$A=2$$
$$A+B=3 \Rightarrow B=1$$

and

$$\boxed{\frac{2x+3}{(x+1)^2} = \frac{2}{x+1} + \frac{1}{(x+1)^2}}$$

$$\int \frac{2x+3}{(x+1)^2}\,dx = 2\ln(x+1) + \underbrace{\int \frac{1}{(x+1)^2}\,dx}_{\int \frac{1}{u^2}du \;=\; -\frac{1}{u} \;=\; \frac{-1}{x+1}}$$

$u = x+1$
$du = dx$

$$= \boxed{2\ln(x+1) - \frac{1}{x+1}}$$

see pg 499 for a harder example

3) <u>Irreducible quadratic root</u>

ex. $\dfrac{2x^2-x+4}{x^3+4x} = \dfrac{2x^2-x+4}{x(x^2+4)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4} \longleftarrow$ make the numerator a linear function $Bx+C$ instead of a constant, B.

See pg 500 ex 5 for solution.

Summary.

Factor

$$\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$

Improper Integrals

Good things to know:

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \text{is convergent if } p>1, \text{ divergent if } p\le 1$$

$$\int_0^1 \frac{1}{x^p}\,dx \quad \text{is divergent if } p>1, \text{ convergent if } p\ge 1$$

ex. $\displaystyle\int_1^\infty \frac{1}{x^{.99}}\,dx = \lim_{t\to\infty}\int_1^t x^{-.99}\,dx$ but $\displaystyle\int_1^t x^{-.99}\,dx = \frac{1}{.001}x^{.001}\Big|_1^t = \frac{1}{.001} + \frac{t^{.001}}{.001}$

$$= \lim_{t\to\infty}\left(\frac{1}{.001} - \frac{t^{.001}}{.001}\right) \quad \text{but } t^{.001}\to\infty \text{ as } t\to\infty$$

$$= \infty$$

$\displaystyle\int_0^1 \frac{1}{x^{1.01}} = \lim_{t\to 0}\int_t^1 x^{-1.01}\,dx$ but $\displaystyle\int_t^1 x^{-1.01}\,dx = \frac{-1}{.01}x^{-.01}\Big|_t^1 = \frac{-1}{.01} + \frac{1}{.01\cdot t^{.01}}$

$$= \lim_{t\to 0}\left(\frac{-1}{.01} + \frac{1}{.01\cdot t^{.01}}\right) \quad \text{but } t^{.01}\to 0 \text{ as } t\to 0$$

$$= \infty$$

Comparison Test

if $f(x)\ge g(x)\ge 0$ for all $x\ge a$

$$\int_a^\infty f(x)\,dx \ge \int_a^\infty g(x)\,dx$$

ex. $\displaystyle\int_1^\infty \frac{\sin^2(x)}{x^2}\,dx$ $\quad \sin^2(x)\le 1$ for all x so

$$0\le \int_1^\infty \frac{\sin^2(x)}{x^2} \le \int_1^\infty \frac{1}{x^2}\,dx \quad \text{which is convergent since } p\ge 1$$

Arclength + Surface Area

$$\underline{\text{Arclength}} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

ex. Find length of curve $y=\ln(\sec(x))$ from $0 \le x \le \pi/4$

$$\frac{dy}{dx} = \frac{1}{\sec(x)} \cdot \sec(x)\tan(x) = \tan(x)$$

$$L = \int_0^{\pi/4} \sqrt{1+\tan^2(x)}\,dx = \int_0^{\pi/4} \sec(x)\,dx = \ln\left|\sec(x)+\tan(x)\right|\Big|_0^{\pi/4} = \ln\left(1+\frac{2}{\sqrt{2}}\right) - \underbrace{\ln(1)}_{0}$$

$$= \boxed{\ln\left(1+\frac{2}{\sqrt{2}}\right)}$$

$$\sec(\pi/4) = \frac{1}{\cos(\pi/4)} = \frac{2}{\sqrt{2}} \qquad \sec(0) = 1$$

$$\tan(\pi/4) = \frac{\sin(\pi/4)}{\cos(\pi/4)} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1 \qquad \tan(0) = 0$$

$$\underline{\text{Surface Area}} = \int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{(about x-axis)}$$

$$\int_a^b 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \qquad \text{(about y-axis)}$$

ex. Find the surface area of $y=\cos(2x)$ rotated about the x-axis $0 \le x \le \pi/6$

$$S = \int_0^{\pi/6} 2\pi\cos(2x)\sqrt{1+4\sin^2(2x)}\,dx \qquad \text{let } u = 2\sin(2x)$$

$$du = 4\cos(2x)\,dx$$

$$= \frac{\pi}{2}\int_0^1 \sqrt{1+u^2}\,du \qquad \text{when } x=0 \quad u = 2\sin(0) = 0$$

$$x = \pi/6 \quad u = 2\sin(\pi/6) = 1$$

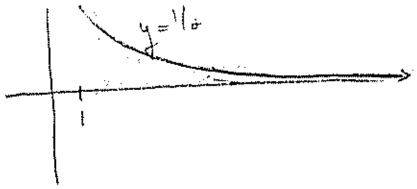
two ways to proceed, interpret $\int$ as area

of  $= \frac{\pi}{4}$ or let $u=\tan(\theta)$ and solve

$$= \boxed{\frac{\pi^2}{8}}$$

In conclusion let's try a WEIRD problem! pg 559 #25

Let $y = \frac{1}{x}$ $\quad 1 \le x \le \infty$



The volume of this curve when it is rotated around the x-axis is

$$V = \int_1^\infty \pi \left(\frac{1}{x}\right)^2 dx = \lim_{t\to\infty} \pi \int_1^t x^{-2} dx = \lim_{t\to\infty} \pi \left(\frac{-1}{x}\right)\Bigg|_1^t$$

↑
washers

$$= \lim_{t\to\infty} \frac{-\pi}{t} + \frac{\pi}{1}$$

$$= \pi \quad \text{since} \quad \frac{-\pi}{t} \longrightarrow 0$$

The surface area

$$S = \int_1^\infty 2\pi \left(\frac{1}{x}\right) \sqrt{1 + \ln^2(x)}\, dx$$

let $u = \ln(x)$

$du = \frac{1}{x} dx$

$$= 2\pi \int_0^\infty \sqrt{1+u^2}\, du$$

when $x=1$ $\quad u = \ln(1) = 0$

$x = \infty$ $\quad u = \infty$

$$\ge 2\pi \int_0^\infty \sqrt{u^2}\, du = 2\pi \int_0^\infty u\, du$$

which diverges!

This surface has finite volume and infinite surface area! Wow!
That means you can fill it with a finite amount of paint, but if you wanted to paint it, you'd never finish.