# Midterm 1 Solutions

1. $X$ is NOT Hausdorff. Hausdorff condition fails on $x_1 = 0^+$ & $x_2 = 0^-$.

2. (a) We showed in HW that $A \subseteq X$ is open iff $\partial A \subseteq X - A$, and $A \subseteq X$ is closed iff $\partial A \subseteq A$.

So $A$ both open & closed iff

$$\partial A \subseteq X - A \quad \text{and} \quad \partial A \subseteq A.$$

This can happen if and only if $\partial A = \emptyset$. $\blacksquare$

(b) First, show $\bar{A} \subseteq A \cup \partial A$.

Let $x \in \bar{A}$.

We know $\bar{A} = A \cup A'$.

So $x \in \bar{A}$ implies that $x \in A$ or $x \in A'$

If $x \in A$, we are done.

Now suppose $x \in A' \setminus A$.

We know that $\partial A = \bar{A} \cap \overline{X-A}$.

$x \in X-A$ implies that $x \in \overline{X-A}$

Since $\overline{X-A} \supseteq X-A$.

So $x \in \bar{A} \cap \overline{X-A}$ i.e. $x \in \partial A$.

Now show $A \cup \partial A \subseteq \bar{A}$.

Assume $x \in A \cup \partial A$.

If $x \in A$, we are done since $\bar{A} \supseteq A$.

Suppose $x \notin A$ and $x \in \partial A$.

But $\partial A = \bar{A} \cap \overline{X-A}$, so $x \in \bar{A}$. $\boxed{3}$

3. Let $U \in \mathring{A}_X$, i.e. $U$ is a subset of $A$ which is open in $X$. Know that $U = U \cap Y$, so $U$ is also open in $Y$.

$\therefore$ $U \in \mathring{A}_Y$.

$\boxed{3}$

4. Recall that for $C \subseteq X$, a subset of $X$, $x \in \overline{C}$ if and only if every nbhd of $x$ intersects $C$. $\boxed{3}$

First show $\overline{A \times B} \subseteq \bar{A} \times \bar{B}$.

Let $(x,y) \in \overline{A \times B}$. wts: $(x,y) \in \bar{A} \times \bar{B}$

That is, wts $x \in \bar{A}$ & $y \in \bar{B}$.

Let $U$ be a nbhd of $x$ & $V$ be a nbhd of $y$. Then $U \times V$ is a nbhd of $(x,y)$. By hypothesis, $U \times V$ intersects $A \times B$. Say $(c,d) \in (U \times V) \cap (A \times B)$.

But $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$.

So $c \in U \cap A$ and $d \in V \cap B$.

Thus $x \in \bar{A} \times \bar{B}$.

Now show $\bar{A} \times \bar{B} \subseteq \overline{A \times B}$.

let $(x,y) \in \bar{A} \times \bar{B}$, ie. $x \in \bar{A}$ & $y \in \bar{B}$.

wts: $(x,y) \in \overline{A \times B}$.

let $W$ be a nbhd of $(x,y)$. By defn of product topology on $X \times Y$, there is a basis element $U \times V$ containing $(x,y)$ sitting inside of $W$, where $U$ is open in $X$ & $V$ is open in $Y$.

So $U$ is a nbhd of $x$ & $V$ is a nbhd of $y$. Since $x \in \bar{A}$, $y \in \bar{B}$, we know that $A \cap U$ contains some point $c$ & $B \cap V$ contains some point $d$.

So $(c,d) \in (A \cap U) \times (B \cap V) = (A \times B) \cap (U \times V)$.

Thus $(c,d) \in (A \times B) \cap W$, since $U \times V \subseteq W$.

$\blacksquare$

5. a. Let $U \subseteq Y$ be open.

Then $f^{-1}(U)$ is some subset of $X$.
Since $X$ has discrete topology,
$f^{-1}(U)$ must be open in $X$.
$\therefore$ $f$ is continuous.

b) Let $Y$ be the topological space
$(X, \mathcal{T}_d)$ where $\mathcal{T}_d$ is the discrete
topology on $X$.
Let $f: X \to Y$ be the identity map.

Let $U \subseteq X$ be some subset of $X$
which is not open in $X$. We know
there is such a thing since $X$ is
not discrete.

We know that $U \subseteq Y$ is open, since
$Y$ is discrete. But $f^{-1}(U) = U$ is
not open in $X$. Thus, $f$ is not continuous.

(c) Let $X$ be the topological space that is $(Y, \mathcal{T}_i)$ where $\mathcal{T}_i$ is the indiscrete topology.

Let $f: X \to Y$ be the identity map.

Since $Y$ is not indiscrete, we can find a set $U \subseteq Y$ which is open but $U \neq \phi$ and $U \neq Y$.

However, $f^{-1}(U) = U$ is <u>not</u> open in $X$ because the only sets open in $X$ are $\phi$ & $X$.

$\blacksquare$