Schwarzschild Geometry and Black Holes

Daniel Halmrast

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1 Killing Fields, Symmetries, and Conserved Quantities

Suppose our spacetime has some symmetry to it. Ideally, we'd like to exploit this symmetry to simplify our analysis a bit. So, how are these symmetries expressed, and how do we exploit them?

1.1 Symmetries of Spacetime

Definition 1.1. *A* continuous symmetry *of a spacetime M is a one-parameter family of diffeomorphisms $\phi_t : M \to M$ such that $\phi_t^* g = g$ (i.e. ϕ_t is an isometry).*

Definition 1.2. *A* Killing field *on a spacetime M is a vector field X such that the flow of X is a continuous symmetry of M.*

Of course, every continuous symmetry gives rise to a Killing field. Furthermore, we have a way of determining whether or not a vector field is a Killing field.

Theorem 1.1. *A vector field X is a Killing field if and only if $L_X g = 0$.*

Proof. Suppose first that X is a Killing field. Let ϕ_t be its flow. Then,

$$L_X g = \lim_{t \to 0} \frac{\phi_{-t}^* g - g}{t}$$

and since X is a Killing field, $\phi_{-t}^* g = g$. Thus,

$$\begin{aligned} L_X g &= \lim_{t \to 0} \frac{\phi_{-t}^* g - g}{t} \\ &= \lim_{t \to 0} \frac{g - g}{t} = 0 \end{aligned}$$

Now suppose instead that $L_X g = 0$. Then, let's consider the map $t \mapsto \phi_t^* g$ and show it's constant. To do so, we'll calculate

$$\begin{aligned} \partial_t \phi_t^* g|_{t=t_0} &= \phi_{t_0} \partial_t^* \phi_t^* g|_{s=0} \\ &= \phi_{t_0}^* L_X g = 0 \end{aligned}$$

and so the map $t \mapsto \phi_t^* g$ is constant. Since $\phi_0^* g = g$, this implies that $\phi_t^* g = g$ for all t on which the flow is defined.

□

Another way of saying $L_X g = 0$ is that for all $Y, Z \in \mathfrak{X}(M)$,

$$g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = 0$$

This is known as Killing's equation, and can be proved quite easily.

Proof. We'll just follow some manipulations directly.

$$\begin{aligned} L_X(g(Y, Z)) &= X(g(Y, Z)) \\ &= \nabla_X(g(Y, Z)) \\ &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ &= g(\nabla_Y X + [X, Y], Z) + g(Y, \nabla_Z X + [X, Z]) \\ &= g(\nabla_Y X, Z) + g(Y, \nabla_Z X) + g([X, Y], Z) + g(Y, [X, Z]) \end{aligned}$$

Now, the left-hand side can be expanded. Namely, we know that L satisfies the Leibniz rule, so

$$\begin{aligned} L_X(g(Y, Z)) &= L_X g(Y, Z) + g(L_X Y, Z) + g(Y, L_X Z) \\ &= L_X g(Y, Z) + g([X, Y], Z) + g(Y, [X, Z]) \end{aligned}$$

Going back to our first calculation, we then have

$$\begin{aligned} L_X g(Y, Z) + g([X, Y], Z) + g(Y, [X, Z]) &= g(\nabla_Y X, Z) + g(Y, \nabla_Z X) + g([X, Y], Z) + g(Y, [X, Z]) \\ L_X g(Y, Z) &= g(\nabla_Y X, Z) + g(Y, \nabla_Z X) \end{aligned}$$

Thus, $L_X g = 0$ if and only if $g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0$. □

1.2 Conserved Quantities

Now, fix X a Killing field on a spacetime M. One would suspect that a Killing field would give rise to a conserved quantity of motion. As it turns out, this conserved quantity has a nice form.

Theorem 1.2. *For γ a geodesic in M, we have that $g(\gamma', X)$ is constant along γ.*

Proof. We can calculate directly

$$\begin{aligned}\nabla_{\gamma'} g(\gamma', X) &= g(\nabla_{\gamma'}\gamma', X) + g(\gamma', \nabla_{\gamma'} X)\\ &= g(0, X) + \frac{1}{2} L_X g = 0\end{aligned}$$

where the last equality comes from the geodesic equation on the first term, and the Killing equation on the second with $Y = Z = \gamma'$. □

As it turns out, these conserved quantities will be of much use to us throughout this analysis.

2 Schwarzschild Geometry

One of the simplest nontrivial setups we have is the static spacetime of a spherical body. We'll make three key assumptions.

First, we'll require our spacetime to be stationary. By stationary, we mean we want some sort of time-translation symmetry. In the language of Killing fields, we want a timelike Killing field X to exist in our spacetime. In the language of observer fields, this amounts to requiring that there exist an observer field U such that for some positive function $f > 0$, fU is a Killing field.

Next, we'll require our spacetime to be static. This amounts to requiring that there exists a hypersurface Σ orthogonal to our Killing field X. In the language of observer fields, this is saying that the observer field that generates X (defined above) has a restspace at each $p \in M$. Note that Frobenius theorem can guarantee such a restspace if the observer field (or the Killing field) is irrotational. That is, the subspace perpendicular to U is closed under the Lie bracket.

Finally, we'll require our spacetime to be spherically symmetric. This amounts to requiring that $SO(3)$ be a subgroup of the group of isometries of

M, and that the orbits of $SO(3)$ are S^2. From here, we could go two routes: First, we could require also that $SO(3)$ specifically acts on the restspace of U, or we could require that X is unique and derive that $SO(3)$ acts on the restspace of U.

I'll prove quickly why assuming the Killing field is unique gets our result.

Proof. Suppose X is unique. That is, for any timelike Killing field Y, $Y = X$. Now, since X is unique, it follows that for all $\phi \in SO(3)$, $\phi^*(X) = X$. So, denoting an orbit of $SO(3)$ as S^2_A, we see that $X|_{S^2_A}$ is invariant with respect to the action of $SO(3)$. However, the only vector field on S^2 that is invariant under the $SO(3)$ action is the zero field, so S^2_A is perpendicular to X, and is thus contained in our restspace. $\square$

Here we indexed the spheres by their surface area A, but we can define $r = \left(\frac{A}{4\pi}\right)^{\frac{1}{2}}$ so we can index them by their radius instead. We have to do this because these spheres don't have an intrinsic radius, and depending on their embedding it might be hard to find a good definition of radius...

Now, we're going to put some coordinates on our space. There's already a natural time coordinate given by the flow of X. That is, each point $p \in M$ around our restspace lies in some $\phi_t(\Sigma)$. So, we'll let our time coordinate of p be t. The orthogonality of X with Σ guarantees that the metric in these coordinates takes the form

$$ds^2 = -V^2(\vec{x})dt^2 + B_{\mu\nu}(\vec{x})dx^\mu dx^\nu$$

The fact that this has no time dependence comes from t being a Killing parameter, and the fact that there are no cross terms $dtdx^\mu$ comes from the fact that Σ is orthogonal to X.

Now, let's put some useful coordinates on Σ. We'll fix some S^2 and put spherical coordinates (θ, φ) on it. Then, we'll carry these coordinates to as much of Σ as we can via geodesics perpendicular to S^2.

Now, the metric on each S^2 is just

$$g|_{S^2_r} = r^2 d\Omega^2$$

and since the $SO(3)$ actions are isometries, in this coordinate system g should not depend on θ or φ. Thus,

$$g = -f(r)dt^2 + h(r)dr^2 + r^2 d\Omega^2$$

We can knock out the last bit of ambiguity by applying Einstein's equation, which says

$$G_{ab} = (R_{ab} - \frac{1}{2}Rg_{ab}) = 8\pi T_{ab} = 0$$

from which (taking the trace of each side and solving for R) we get

$$R_{ab} = 8\pi(T_{ab} - \frac{1}{2}g_{ab}T$$

The rest of the program is just solving this setup with the vacuum solution $R_{ab} = 0$, which will (with a bit of work) specify f and h up to some constant. If we assume that this metric agrees with the Newtonian gravity setup in the weak-field regime, then we have a unique f, h for a given "mass" at the center, and the metric takes the form

$$g = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2 d\Omega^2$$

Now the next task is to see how far we can extend these coordinates.

3 Coordinate singularities and Kruskal Coordinates

There are a couple striking features of this metric. First off, as $r \to \infty$, the metric reduces to the standard Minkowski metric. Second off, this metric has a couple of undefined points. Specifically, at $r = 2M$ and $r = 0$, our coordinate system breaks down.

Maybe these are just artifacts of our coordinate system! After all, at $\varphi = 0$ (i.e. the North pole) and $\varphi = \pi$ (the South pole) our coordinates break down, but we know that this is just because our charts don't cover all of S^2.

3.1 Eddington-Finkelstein Coordinates

So, let's do some more analysis. For now, we'll only study radial motion, so $\theta' = \varphi' = 0$ for our geodesics. First off, we note that for a null geodesic γ with tangent vector $\gamma'^a = U^a$,

$$U^a U_a = 0 = (1 - \frac{2M}{r})t'^2 + (1 - \frac{2M}{r})^{-1}r'^2$$

from which we get

$$\left(\frac{dt}{dr}\right)^2 = \left(1 - \frac{2M}{r}\right)^{-2}$$

or

$$\frac{dt}{dr} = \pm\left(1 - \frac{2M}{r}\right)^{-1}$$

So, our light cones will have slope $\frac{1}{1-\frac{2M}{r}} = \frac{r}{r-2M}$. This is a good characterization of the "singularity" at $r = 2M$, namely that the slope of the light cone in the $r - t$ plane approaches ∞ as r approaches $2M$. There's a lot of physics buried in here, which I won't have time to analyze, but this implies that anything falling into this region will not reach $r = 2M$ in finite time!

Let's try shifting coordinates a bit. If we solve for the equation of motion for the radial null geodesics, we get

$$t = \pm r^* + C$$

with the so-called "tortoise coordinate"

$$r^* = r + 2M \ln\left(\frac{r}{2M} - 1\right)$$

In these coordinates, our metric becomes

$$g = \left(1 - \frac{2M}{r}\right)(-dt^2 + dr^{*2}) + r^2 d\Omega^2$$

Which is certainly interesting, since now there's no singularity at $r = 2M$, but we have further problems. Most strikingly, $r = 2M$ is at $r^* = -\infty$.

Okay, let's keep pushing on. Define new coordinates

$$u = t - r^*$$
$$v = t - r^*$$

so that the null geodesics lie on constant u or v. In this coordinate system,

$$g = -\left(1 - \frac{2M}{r}\right) du dv + r^2 d\Omega^2$$

However, let's instead keep r as a radial coordinate, and let v take the place of t. With this replacement, the metric becomes

$$g = -\left(1 - \frac{2M}{r}\right) dv^2 + (dvdr + drdv) + r^2 d\Omega^2$$

and after all this, we've finally gotten rid of the coordinate singularity.

Let's examine the slopes of the null geodesics in this coordinate system. We have that

$$U^a U_a = 0 = -\left(1 - \frac{2M}{r}\right) v'^2 + 2v'r'$$

Or, dividing by r'^2,

$$\left(1 - \frac{2M}{r}\right)\left(\frac{v'}{r'}\right)^2 = 2\frac{v'}{r'}$$
$$\left(1 - \frac{2M}{r}\right)\left(\frac{dv}{dr}\right)^2 = 2\frac{dv}{dr}$$

So either $\frac{dv}{dr} = 0$ or $\frac{dv}{dr} = 2\left(1 - \frac{2M}{r}\right)^{-1}$. Plotting the light cones, we see how passing $2M$ makes all future-directed paths move closer to the singularity $r = 0$.

We could carry out the analysis for u instead of v, which would lead to another description of the space with past-directed paths moving towards the singularity… We won't cover this here.

3.2 Kruskal Coordinates and the Maximal Extension

Let's go back to our $u - v$ coordinate system. Recall that this coordinate system has the annoying feature that $r = 2M$ lies at $v = -\infty$ or $u = \infty$. So, let's try and define new coordinates (again) which will make $r = 2M$ an actual coordinatized region. To that extent, we define

$$\tilde{v} = \exp(\frac{v}{4M})$$
$$\tilde{u} = \exp(\frac{-u}{4M}$$

From which we get (trust me)

$$g = -\frac{16M^3}{r}\exp(\frac{-r}{2M})(d\tilde{v}d\tilde{u} + d\tilde{u}d\tilde{v}) + r^2 d\Omega^2$$

and finally, $r = 2M$ is no longer a problem. In fact, writing our new coordinates in terms of our old shows us that

$$\tilde{v} = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} \exp(\frac{r+t}{4M})$$
$$\tilde{u} = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} \exp(\frac{r-t}{2M})$$

from which it's clear that

- $\tilde{v}$ and $\tilde{u}$ are null (since null geodesics follow $r = \pm t$)
- $r = 2M$ is a very reasonable and well-defined spot on our coordinate chart.

To complete our analysis, let's shift back into timelike and spacelike coordinates

$$T = \frac{1}{2}(\tilde{v} + \tilde{u})$$
$$R = \frac{1}{2}(\tilde{v} - \tilde{u})$$

Which (for completeness' sake) are

$$T = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} \exp(\frac{r}{4M}) \sinh(\frac{t}{4M})$$
$$R = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} \exp(\frac{r}{4M}) \cosh(\frac{t}{4M})$$

with metric

$$g = \frac{16M^3}{r} \exp(\frac{-r}{2M})(-dT^2 + dR^2) + r^2 d\Omega^2$$

And we're done! These are the Kruskal coordinates for a spherically symmetric, static vacuum solution to Einstein's equation.

3.3 Properties of Kruskal Coordinates

The first important thing to notice is that the radial null geodesics are given by

$$T = \pm R + C$$

for some constant C. Furthermore, we can recover r from T and R via the relation

$$T^2 - R^2 = \left(1 - \frac{r}{2M}\right) \exp(\frac{r}{2M})$$

and so $r = 2M$ is at $T = \pm R$, and more generally surfaces of constant r are at $T^2 - R^2 = C$ for some constant C.

Surfaces of constant t are given by

$$\frac{T}{R} = \tanh(\frac{t}{4M})$$

We (somewhat boldly) claim these coordinates work for every value they can take without hitting $r = 0$, which amounts to R being unbounded, and T bounded by

$$T^2 < R^2 + 1$$

This claim is justified. We show that the regions I and II in the Kruskal plane (defined as $V > 0$, or $T > -R$) are isometrically diffeomorphic to the regions $r > 2M, 0 < r < 2M$ in the original coordinates. Of course, we really don't need to show anything, since our whole analysis has been using isometry transformations. What's interesting is that we have something like a double cover of our original spacetime, since (T, R) and $(-T, -R)$ correspond to the same (r, t) values.

Now this gives us a really interesting picture of what's going on. There's a lot more to say about the Kruskal diagram, but I won't detail it here.

4 Conformal Diagrams

I really liked how in the Kruskal picture the light cones were always at slope ± 1, no matter where you are on the diagram. However, I don't like how I can't see infinities on the diagram. To that end, we attempt to compactify our space (think Poincare disk model of H^2) while keeping our light cones at slope ± 1. The Schwarzschild case is a bit algebraically tough to handle, so we'll do the Minkowski case as an example, and skip straight to the result for Schwarzschild.

4.1 Conformal Diagrams for Minkowski Space

The basic idea is to find some null vectors, shove them into an arctangent, and see what happens.

So, lets take standard Minkowski space (with polar coordinates) which has metric

$$g = -dt^2 + dr^2 + r^2 d\Omega^2$$

Now, let's use

$$u = t - r$$

$$v = t + r$$

under which our metric becomes

$$g = \frac{-1}{2}(dudv + dvdu) + \frac{1}{4}(v - u)^2 d\Omega^2$$

Then, lets use the transformation

$$U = \arctan(u)$$
$$V = \arctan(v)$$

under which the metric becomes

$$g = \frac{1}{4\cos^2(U)\cos^2(V)}\left(-2(dUdV + dVdU) + \sin^2(V-U)d\Omega^2\right)$$

Finally, we transform back into spacelike/timelike coordinates with

$$T = V + U$$
$$R + V_U$$

to get

$$g = \omega^{-2}(T,R)(-dT^2 + dR^2 + \sin^2 R d\Omega^2)$$

with

$$\omega(T,R) = \cos(T) + \cos(R)$$

This is interesting! We now have g conformally related to $\tilde{g} = -dT^2 + dR^2 + \sin^2 R d\Omega^2$ the metric for $\mathbb{R} \times S^3$.

Now, U and V could only take on values in $(-\frac{\pi}{2}, \frac{\pi}{2})$ with $U \leq V$ (since $r \geq 0$), so our new coordinates have values

$$R \in [0, \pi)$$
$$|T| + R < \pi$$

and so we have a nice description of Minkowski space. Since we conformally transformed the metric, angles are preserved, and so in $\mathbb{R} \times S^3$, the light cones still have slope ± 1. Furthermore, we've compactified everything, since our coordinates have finite range. At this point it's helpful to draw diagrams, so I'll stop the text discussion here.

4.2 Conformal Diagrams for Schwarzschild Spacetime

I don't have a lot of textual stuff to say here... We'll use the coordinate transformation

$$\bar{u} = \arctan(\frac{\tilde{u}}{\sqrt{2M}})$$
$$\bar{v} = \arctan(\frac{\tilde{v}}{\sqrt{2M}})$$