Functional Analysis

Manish Kumar, Daniel Halmrast

University of California, Santa Barbara

Introduction

The core idea in functional analysis is to treat functions as 'points' or 'elements' in some sort of abstract space, so that instead of working with individual functions we work with the structure of the space (as the tradition in classical analysis), we deal with functions as points in a space endowed with some kind of overall structure. This viewpoint, was an integral step in the process of transferring familiar concepts in finite-dimensional Euclidean space to (typically infinite-dimensional) 'function spaces'.

Basics

We will be concerned with complex Hilbert spaces. A Hilbert space (H) is defined as a vector space over $\mathbb{C}$, with an inner product such that it is complete with respect to the inner product. Specifically we will be looking at the $L^2(X,\mu)$, this is the Hilbert space of the square integrable functions, with associated measure μ. Note the norm in L^2 is given by:

$$||f||_{L^2} = \int |f|^2 d\mu$$

This is the familiar state space of a quantum particle, contained in a region X.
Let $T : X \longrightarrow Y$ be a linear map between normed linear spaces. T is called a bounded linear operator if $\exists C$ such that

$$||Tx||_Y \le C||x||_X$$

$\forall x \in X$. Now suppose $\mathscr{H}$ and $\mathscr{K}$ are hilbert spaces and $A : \mathscr{H} \longrightarrow \mathscr{K}$ a bounded linear operator. There is always a unique A^* such that:

$$\langle Ah, k\rangle_{\mathscr{K}} = \langle h, A^* k\rangle_{\mathscr{H}}$$

A^* is called the adjoint of A

Bounded Linear operators

In this section we will define spectrum on bounded linear operators. Let $T \in \mathscr{B}(X)$ ($T : X$ bounded), we say $\lambda \in \mathbb{C}$ is an eigenvalue, if for some x we have $Tx = \lambda x$.
Theorem: If $T \in \mathscr{B}(\mathscr{H})$ is self adjoint, then either $||T||$ or $-||T||$ is an eigenvalue. Note $||T|| = Sup(|\langle Tx, y\rangle| : ||x|| = ||y|| = 1)$. The eigenvalues are real and the eigenvectors for distinct eigenvalues are orthogonal.T is said compact if it is the limit of finite rank operators
Preliminary Spectral Theorem: Let T $\neq 0$ be a self-adjoint, compact operator in $\mathscr{B}(\mathscr{H})$, there exists a finite or countably infinite set of eigenvectors g_n, with corresponding eigenvalues λn such that

$$Tx = \Sigma \lambda_n \langle x, g_n\rangle g_n$$

When we move from compact operators to Banach algebras, the concept of eigenvalue gets generalized to the spectrum.

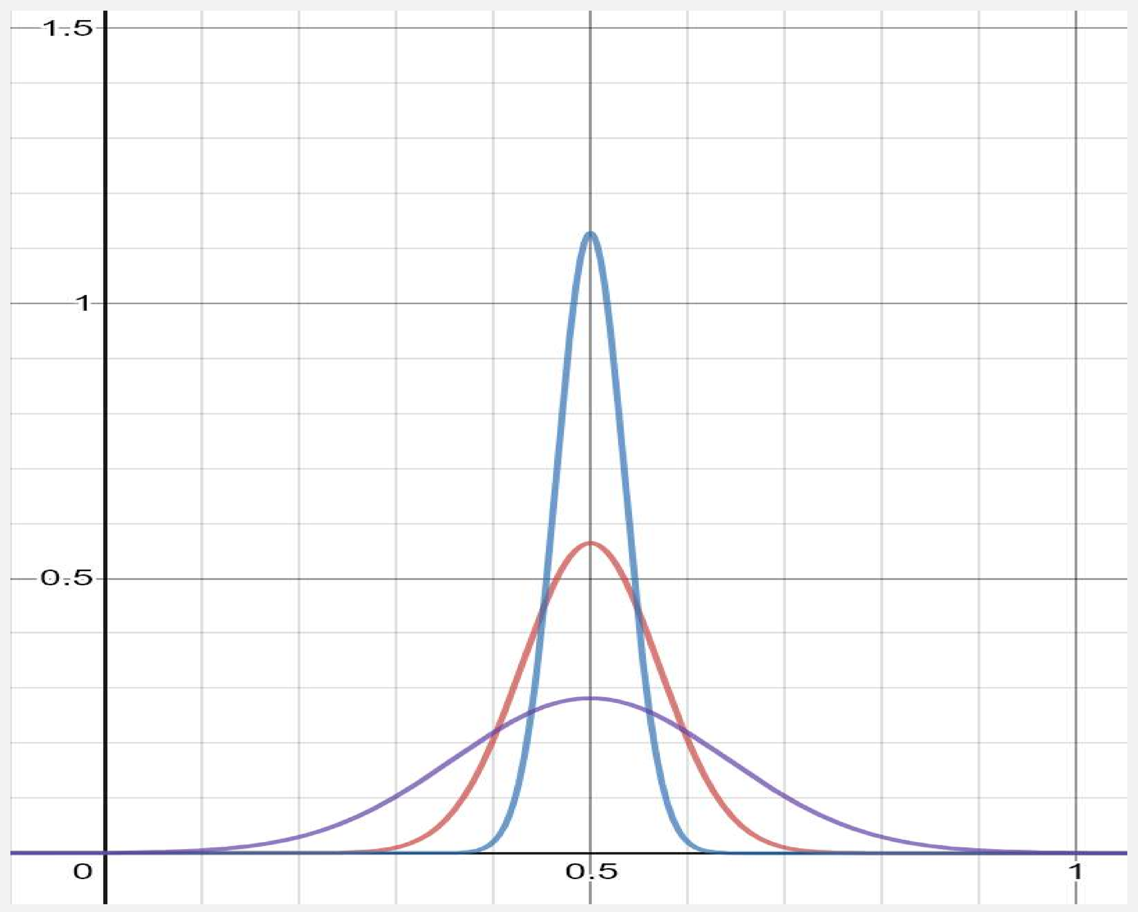

Figure: functions h_n approximating the eigenfunction for $\lambda = 0.5$

Spectrum

Spectrum of T: Let $T \in \mathscr{B}(X)$ be linear. The set of complex numbers λ such that $T - \lambda I$ is not invertible, is called the spectrum.
A bounded linear operator on a Hilbert space is invertible iff it is bounded below and has a dense range. So if either (or both) of those two conditions are not satisfied for $T - \lambda I$ then λ is in the spectrum. So we have:
Approximate point spectrum $\sigma_{ap}(x)$: $\{\lambda : T - \lambda I$ is not bounded below$\}$, this includes the eigenvalues.
Compression spectrum $\Gamma(x)$: $\{\lambda : T - I$ does not have a dense range$\}$
Spectral mapping theorem: Suppose $\mathscr{A}$ is a unital Banach algebra and $A \in \mathscr{A}$ and P is a polynomial. We have

$$\sigma(P(A)) = P(\sigma(A))$$

Our goal now is to generalize this to all continuous functions. For this we consider the set $\mathscr{M}_{\mathscr{A}}$ of *homomorphisms from $\mathscr{A}$ to $\mathbb{C}$ (in $\mathscr{B}(\mathscr{H})$ the * is just the adjoint).

Example:Position operator

Consider the operator M_x, multiplication by x. $M_x \in L^2[[0,1]]$. Clearly, this operator has no eigenvalues, but it has approximate point spectrum $\sigma_{ap} = [0,1]$. A key idea about $\lambda \in \sigma_{ap}(M_x)$ is that $(M_x - \lambda I)h_n \rightarrow 0$ for some sequence of normalized functions h_n. An example of such a sequence is :

$$h_n = \sqrt{\frac{n}{\pi}} e^{-n(x-\lambda)^2}$$

(If this did have a limit, it would be the delta function)

Gelfand Transform

If $\mathscr{A}$ is a commutative unital Banach algebra, then we can define a map:

$$\Gamma : \mathscr{A} \longrightarrow C(\mathscr{M}_{\mathscr{A}})$$

Theorem:If $\mathscr{A}$ is also a C^* algebra then Γ is a isometric-*isomorphism. Although we are limited by the commutativity requirement in the last theorem, we can generate a commutative C^* algebra, with $\{A, A^*, I\}$ for any $A \in \mathscr{A}$. The next result will be the central result of this poster, there is an isomorphism from $\mathscr{A}$ (which is a mysterious object) to $C(\sigma(A))$ (which are just continuous complex functions).
Theorem:Suppose $\mathscr{A}$ is some singly generated, commutative, unital C^* algebra with $\mathscr{A} = C^*(A)$ for some A which is necessarily normal. Then there is a unique * isomorphism between $\mathscr{A}$ and $C(\sigma(A))$, and maps A to the identity function on $\sigma(A)$.

Continuous spectral mapping

Theorem: Given a normal operator A as above, we have

$$\sigma(f(A)) = f(\sigma(A))$$

This theorem gives constraints on the spectra of operators based on their algebraic properties.

- $A^* = A$ iff $\sigma(A) \subset \mathbb{R}$
- $A^* = A^{-1}$ iff $\sigma(A) \subset \partial\mathbb{D}$
- $A^2 = A$ iff $\sigma(A) \subset \{0, 1\}$

References

[1] B. MacCluer.
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