Cyclic Quotient Singularities of 2-dimensional Affine Toric Varieties

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Some Definitions

- An ***affine algebraic variety*** is the common zero set of a set of complex polynomials on $\mathbb{C}^n$. We denote $V = \mathbf{V}(\{F_i\}_{i \in I}) \subseteq \mathbb{C}^n$.
- An ideal $I \subseteq A = \mathbb{C}[X_1, \ldots, X_n]$ gives an affine variety
$$\mathbf{V}(I) = \{p \in \mathbb{C}^n \mid f(p) = 0 \text{ for all } f \in I\},$$
and an affine variety $V \subseteq \mathbb{C}^n$ gives the ideal
$$\mathbf{I}(V) = \{f \in A \mid f(p) = 0 \text{ for all } p \in V\}.$$
The ***coordinate ring*** of a variety V is $\mathbb{C}[V] = A/\mathbf{I}(V)$. We write $V =$ spec$(\mathbb{C}[V])$ by identifying V with the set of maximal ideals of $\mathbb{C}[V]$.
- The closed sets of a variety V in the ***Zariski topology*** are the subvarieties of V, and the open sets are their complements.
- A ***torus*** is an affine variety isomorphic to $(\mathbb{C}^*)^n$. An ***affine toric variety*** is an affine variety containing a torus as a dense subset in the Zariski topology, such that the action of the torus on itself extends to an action of the torus on V.
- A ***character*** of a torus T is a group homomorphism $\chi : T \to \mathbb{C}^*$. A ***one-parameter subgroup*** of a torus is a group homomorphism $\lambda : \mathbb{C}^* \to T$.
- A ***lattice*** is a free abelian group of finite rank. A lattice of rank n is isomorphic to $\mathbb{Z}^n$. For example, a torus has lattices M and N, of characters and one-parameter subgroups, respectively.
- A ***semigroup*** is a set S with an associative binary operation and an identity element. An ***affine semigroup*** is a commutative finitely generated semigroup that can be embedded in a lattice.
- Given an affine semigroup, the ***semigroup algebra*** $\mathbb{C}[S]$ is the vector space over $\mathbb{C}$ with S as a basis and multiplication induced by the semigroup structure. We think of the embedding of S into the character lattice M of a torus T_N, so that $m \in M$ gives the character χ^m. Then $\mathbb{C}[S] = \{\sum_{m \in S} c_m \chi^m \mid c_m \in \mathbb{C} \text{ and } c_m = 0 \text{ for all but finitely many } m\}$, with multiplication induced by $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$. If $S = \mathbb{N}A$ for $A = \{m_1, \ldots, m_s\}$, then $\mathbb{C}[S] = \mathbb{C}[\chi^{m_1}, \ldots, \chi^{m_s}]$.
- A ***cone*** $\sigma \in N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ is a subset closed under positive scalar multiplication. A cone is called rational if it can be generated by elements of N, polyhedral if it is finitely generated, and strongly convex if it does not contain lines. The ***dual cone*** of σ is the cone $\sigma^\vee$ of linear functionals in M that are positive on σ.
- A ***face*** of the cone σ is defined $\tau = \sigma \cap u^\perp$ for some $u \in \sigma^\vee$.
- Each affine toric variety has associated to it a cone in its character lattice, and conversely any rational, strongly convex, polyhedral cone yields an affine toric variety. To obtain an affine toric variety from a cone, we consider a lattice N containing the cone, and its dual lattice $M =$ Hom$(N, \mathbb{Z})$. We take the intersection $S_\sigma = \sigma^\vee \cap M$ as the semigroup generating our semigroup algebra $A_\sigma = \mathbb{C}[S_\sigma]$. The affine toric variety is then given by $U_\sigma =$ spec(A_σ).

The first of these definitions can be found in [3], while the rest can be found in [1]. We will readily use the fact that the affine toric variety corresponding to a cone is singular if the cone is generated by part of a basis for the lattice N, whose proof can be found in [2].

References

[1] Hal Schenck David Cox John Little. *Toric Varieties.* Vol. 124. 2011.

[2] William Fulton. *Introduction to Toric Varieties.* Vol. 131. 1993.

[3] Karen E. Smith. *An Invitation to Algebraic Geometry.* Vol. 1. 2000.

An Example of a Singular Affine Toric Variety

Consider the lattice $N = \mathbb{Z}^2$ and the cone σ generated by $e_2, me_1 - e_2$, as well as its dual cone $\sigma^\vee$ in the dual lattice M.

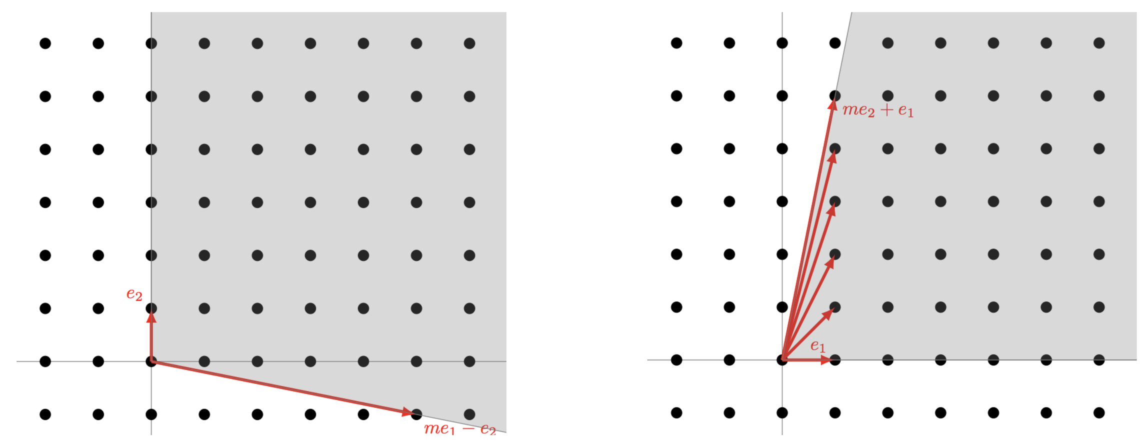

Fig. 1: σ and $\sigma^\vee$ with generators.

The dual cone $\sigma^\vee$ is generated by $\{e_1, e_1 + e_2, \ldots, e_1 + me_2\}$. The variety arising from σ must be singular, since $\sigma^\vee$ is not generated by part of a basis for N. The corresponding semigroup, $S_\sigma = \sigma^\vee \cap M$, gives the following semigroup algebra
$$\mathbb{C}[X, XY, \ldots, XY^m].$$
If we let $X = U^m$ and $Y = V/U$, we find that
$$A_\sigma = \mathbb{C}[S_\sigma] = \mathbb{C}[U^m, U^{m-1}V, \ldots, UV^{m-1}, V^m] \subset \mathbb{C}[U, V].$$
Then the corresponding affine toric variety $U_\sigma =$ spec(A_σ) is the cone over the rational normal curve of degree m. The natural inclusion $A_\sigma \hookrightarrow \mathbb{C}[U, V]$ corresponds to a mapping $\mathbb{C}^2 =$ spec$(\mathbb{C}[U, V]) \to$ spec$(A_\sigma) = U_\sigma$.

Examining the Singularity

Consider the group $G \cong \mathbb{Z}/m\mathbb{Z}$ of m-th roots of unity. G acts on $\mathbb{C}[U, V]$ by
$$(\zeta \cdot f)(U, V) = f(\zeta U, \zeta V).$$
The invariant elements under this action are exactly the monomials with degree divisible by m, and the field $\mathbb{C}$. It follows that the ring fixed by this action is $\mathbb{C}[U, V]^G = \mathbb{C}[U^m, U^{m-1}V, \ldots, UV^{m-1}, V^m] = A_\sigma$. Thus, the map $\mathbb{C}^2 \to U_\sigma$ mentioned in the last section is induced by the inclusion $\mathbb{C}[U, V]^G \hookrightarrow \mathbb{C}[U, V]$. Indeed, we have
$$\mathbb{C}^2 = \text{spec}(\mathbb{C}[U, V]) \to \text{spec}(\mathbb{C}[U, V]^G) = U_\sigma.$$
We've found an example of a singular toric surface whose singular point can be described as the quotient of $\mathbb{C}^2$ by the group $\mathbb{Z}/m\mathbb{Z}$ acting diagonally.

We can describe the singular point concretely as follows. Consider the embedding $U_\sigma \hookrightarrow \mathbb{A}^{m+1}$ induced by the quotient $q : \mathbb{C}[X_0, X_1, \ldots, X_m] \to \mathbb{C}[X_0, X_1, \ldots, X_m]/I(U_\sigma)$, By the ideal correspondence theorem, q sends maximal ideals to maximal ideals. We aim to verify that origin in U_σ, i.e., image of the maximal ideal $\langle X_0, X_1, \ldots, X_m \rangle / I(U_\sigma)$ in U_σ, is the singular point. The isomorphism of U_σ with spec$(\mathbb{C}[X_0, X_1, \ldots, X_m]/I(U_\sigma))$ is given by the following map of rings:
$$\begin{aligned} \mathbb{C}[X_0, X_1, \ldots, X_m] &\to \mathbb{C}[U, V]^G \\ X_i &\mapsto U^{m-i}V^i. \end{aligned}$$
Thus, the maximal ideal $\mathfrak{m}_{U_\sigma} = \langle U^m, U^{m-1}V, \ldots, V^m \rangle$ is the point corresponding to the origin in U_σ. We then expect the Zariski cotangent space $\mathfrak{m}_{U_\sigma}/\mathfrak{m}^2_{U_\sigma}$ to have dimension 2. However since $\mathfrak{m}_{U_\sigma}$ is the ideal of polynomials in U, V for which the degree of each term is divisible by m, $\mathfrak{m}^2_{U_\sigma}$ is the ideal of polynomials for which the degree of each term is divisible by $2m$. It follows that $\mathfrak{m}^2_{U_\sigma}/\mathfrak{m}^2_{U_\sigma}$ is the vector space of polynomials in U, V for which the degree of each of its terms is *exactly* m, which has dimension $m + 1$. Since the cotangent space is not 2-dimensional, we can confirm that $\mathfrak{m}_{U(\sigma)}$ is the singular point.

Resolving the Singularity

We attempt to resolve the singularity by finding a fan containing σ, whose corresponding toric variety is smooth. We can insert an edge along the e_1 axis to obtain a fan Σ composed of two cones σ_0, σ_1.

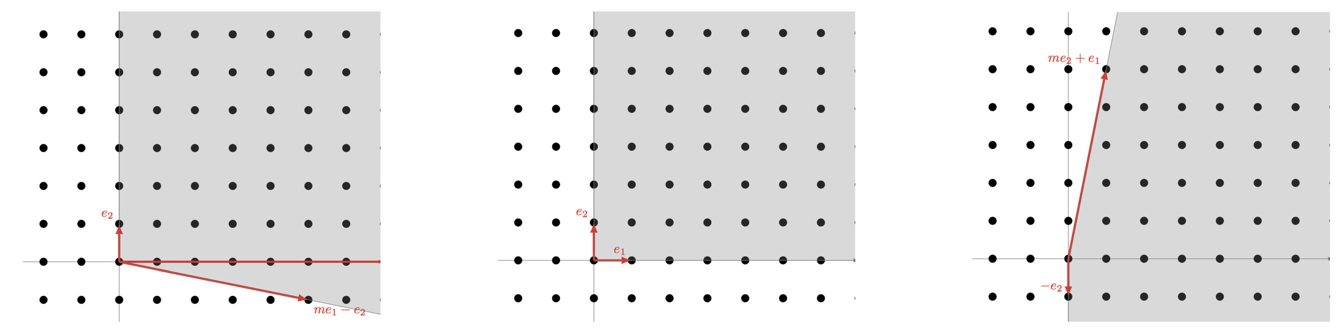

Fig. 2: The fan Σ, and the dual cones $\sigma_0^\vee, \sigma_1^\vee$

Since each of σ_0 and σ_1 are each generated by part of a basis for N, each patch is nonsingular. We see that σ_0 is self-dual, and $\sigma_0^\vee$ is generated by e_1^*, e_2^*. It's corresponding semigroup algebra is then $\mathbb{C}[X, Y]$. $\sigma_1^\vee$ is generated by $-e_2^*, e_1^* + me_2^*$, giving the corresponding semigroup algebra $\mathbb{C}[Y^{-1}, XY^m]$. We find that spec(A_{σ_0}) = spec$(A_{\sigma_1}) = \mathbb{C}^2$. The maps $U_{\sigma_i} \to U_\sigma$ can be described as mappings of $\mathbb{C}^2$ into U_σ. Let $\varphi : N \to N$ be the identity map, defining $\varphi_* : X(\Sigma) \to X(\Sigma)$, where $X(\Sigma)$ denotes the toric variety corresponding to Σ. φ_* is proper, since φ preserves the support of Σ. [2] To see that φ_* is an isomorphism away from the singular point, we show that it preserves the torus in $X(\Sigma)$. To see this, we observe that the diagram and its algebraic analogue,

$$\begin{array}{ccc} U_{\sigma_0} & \longrightarrow & U_\sigma \\ \uparrow & & \uparrow \\ T & = & T, \end{array} \quad \Longleftrightarrow \quad \begin{array}{ccc} \mathbb{C}[X, XY, \ldots, XY^m] & \longrightarrow & \mathbb{C}[X, Y] \\ \downarrow & & \downarrow \\ \mathbb{C}[X, Y, X^{-1}, Y^{-1}] & = & \mathbb{C}[X, Y, X^{-1}, Y^{-1}], \end{array}$$

commute since every arrow in the algebraic diagram denotes a trivial inclusion. Now let τ be a proper face of σ_0. τ generates an affine subvariety $U_\tau \hookrightarrow U_{\sigma_0}$. Define $U_\sigma(\tau)$ as the affine toric variety defined by restricting $\sigma^\vee$ to $\tau^\perp$. $\tau^\perp$ is the Y-axis, so we set the semigroup elements with positive X-coordinate to zero giving $U_{\sigma_0}(\tau) =$ spec$(\mathbb{C}[Y]) = \mathbb{C}$. Since $\mathfrak{m}$ contains X as a generator, it contains the maximal ideal $\langle X, XY, \ldots, XY^m \rangle$. Since $\mathfrak{m}$ is maximal, we must have $\varphi_*(\mathfrak{m}) = \langle X, XY, \ldots, XY^m \rangle$. Therefore, any point in $U_{\sigma_0}(\tau)$ maps to the singular point in U_σ. Having calculated $U_{\sigma_0}(\tau)$, a similar construction shows that $U_{\sigma_1}(\tau) =$ spec$(\mathbb{C}[Y^{-1}]) = \mathbb{C}$. Gluing these together gives a copy of the projective line $\mathbb{CP}^1$. Thus, inserting the edge τ has the effect of resolving the singularity at the origin, at the cost of introducing a copy of $\mathbb{CP}^1$ in place of the origin.

Generalities

We now consider a more general case. Let $\sigma \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ be a strongly convex rational polyhedral cone. Any minimal generator along an edge of σ is part of a basis for N, so we can take $(0, 1)$ and $(m, -k)$ as generators, where m is a positive integer and m, k are coprime. It can be shown similarly that the singular point of the corresponding toric variety can be described as a quotient of $\mathbb{C}^2$ by the group $G = \mathbb{Z}/m\mathbb{Z}$, acting on $\mathbb{C}[U, V]$ by $(\zeta \cdot f)(U, V) = f(\zeta U, \zeta^k V)$. We can also show that for the particular case of $\zeta^k = \overline{\zeta} = \zeta^{m-1}$, resolving the singularity of the corresponding toric variety yields m copies of the projective line, each with self intersection -2. Each of these copies intersects the next exactly once. This gives a chain of spheres, represented by the following graph for which edges represent a copy of $\mathbb{CP}^1$ and each vertex represents an intersection.

$\mathrm{A_n}$ 

This is referred to as the Type A_m Rational Double Point singularity.