Quark Configurations and Lie algebra Representations

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Introduction

Theories describing physical systems are always equipped with symmetries. Mathematically, a symmetry is introduced as a compact Lie group, say G, known as an "internal symmetry group" or "gauge group" for some *gauge theory*. Particles then live in representations of G on a finite-dimensional *Hilbert space* V. In other words, V can always be decomposed as a direct sum of irreducible representations (*irreps*) and particles are described as basis vectors of these irreps.

On the premise of recognizing the first generation quarks as the only fundamental particles and ignoring the existence of electrons and the electromagnetic interactions [which corresponds to the symmetry group $\mathrm{U}(1)$], we mathematically formulate the experimental existence of neutrons and pions, their interactions with pions and quark configurations the basic knowledge of representation theory; this gives us a taste of the striking simplicity of the representation theory involved in the mathematical formulation of the Standard Model without going into quantum field theory

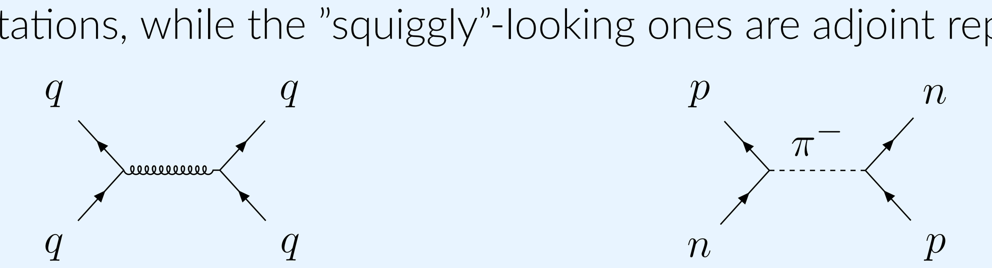
Starting with an investigation of the irreducible representations of $\mathfrak{sl}_2\mathbb{C}$ and $\mathfrak{sl}_3\mathbb{C}$, which are essentially the complexification of the Lie algebras of the isospin symmetry group $\mathrm{SU}(2)$ and the color symmetry group $\mathrm{SU}(3)$, we discuss the particle physics interpretation of these irreps and deduce from this the quark configurations of nucleons and pions.

Toolbox

- G be a matrix Lie group. A finite-dimensional **representation** of G is a Lie group homomorphism $\rho : G \to \mathrm{GL}(V)$ where V is a finite-dimensional complex vector space.
- A representation is **irreducible** if there is no proper subspace W of V such that W is invariant under ρ.
- **Weights add in tensor products** Let $V = \oplus V_\alpha$ and $W = \oplus W_\beta$. Then $V \otimes W = \oplus \left(V_\alpha \otimes W_\beta\right)$ and $V_\alpha \otimes W_\beta$ is a weight space for H with weights $\alpha + \beta$.
- **Roots** are weights in the special case of adjoint representations.

We also introduce some concepts from physics that come in handy.

- A **Hilbert Space** is essentially a vector space equipped with an inner product which defines a distance function for which it is a complete metric space.
- **Color charge** and **Isospin** are properties of quarks; they are **conserved** quantities under the corresponding symmetry groups. In the case of isospin, which is conserved under strong interactions, it is (for our limited purposes) the eigenvalue of H in our discussion of $\mathfrak{sl}_2\mathbb{C}$.
- In quantum mechanics, the state of any physical system is given by a unit vector in a complex Hilbert space that admits complex linear combinations. For general combinations of systems, we follow the heuristic relation that $\oplus =$ ***or*** and $\otimes =$ ***and***, e.g. the Hilbert space of one particle in two boxes and two particles in one box.
- **Feynman Diagrams.** Each line in the diagram is a representation. Straight lines stand for standard representations, while the "squiggly"-looking ones are adjoint representations.



The right diagram represents the interaction of two nucleons, living in the representation $\mathbb{C}^2$ of the gauge (symmetry) group $\mathrm{SU}(2)$, via the exchange of a pion, which lives in the complexified adjoint representation $\mathfrak{sl}_2\mathbb{C}$.

References

[1] John Baez and John Huerta. The algebra of grand unified theories. *Bulletin of the American Mathematical Society*, 47(3):483–552, 2010.

[2] William Fulton and Joe Harris. *Representation Theory*. Graduate Texts in Mathematics. Springer, 1st edition, 2004.

[3] Brian C. Hall. *Lie groups, lie algebras and representations*. Springer, 2nd edition, 2015.

Confinement: the Guiding Result

In quantum chromodynamics, *confinement* is the phenomenon that color-charged particles (quarks in our context) cannot be isolated. This amounts to the following decrees:

- The strong force does **not** appear in **macroscopic** observations;
- **Color-Neutral.** All observable particles must be invariant under the action of $\mathrm{SU}(3)$. In terms of color charges, observable states must be *white*.

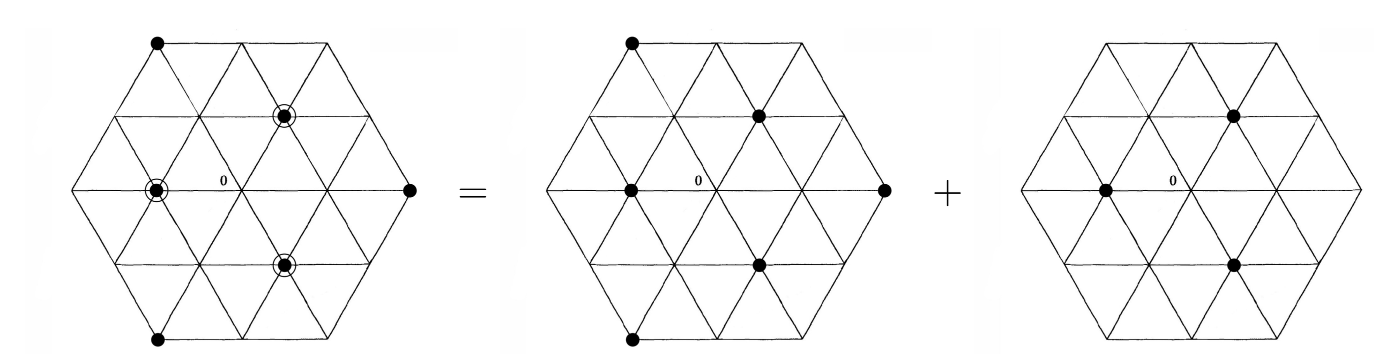
Pion-Nucleon Interactions and Group Actions

Experiments show that protons and neutrons are interchangeable via the interaction with pions. Treating them as two states of a *nucleon*, we can mix them as states in the Hilbert space $\mathbb{C}^2$, a two-dimensional irrep of $\mathrm{SU}(2)$. Mathematically, the standard basis vectors in $\mathbb{C}^2$ are interchangeable under the action of $\mathrm{SU}(2)$. Lie group representations induce Lie algebra representations; the action of $\mathrm{SU}(2)$ on itself gives the adjoint representation on $\mathfrak{su}(2)$. Thus pions need to span the 3-dim irrep $\mathrm{Sym}^2\,\mathbb{C}^2 \cong \mathfrak{sl}_2\mathbb{C} \cong \mathfrak{su}(2) \oplus \mathbb{C}$ of $\mathrm{SU}(2)$, which is unique as we have shown on the right.

The pions live in $\mathfrak{sl}_2\mathbb{C}$, *the complexification of the adjoint representation of* $\mathrm{SU}(2)$.

From the Quark Model to Hadrons

Quarks live in $\mathbb{C}^2 \otimes \mathbb{C}^3$, a representation of $\mathrm{SU}(2) \times \mathrm{SU}(3)$. This takes into account (first generation) *flavors* spanning $\mathbb{C}^2$ and *colors*, which is related to the strong interaction, spanning $\mathbb{C}^3$. The physical configurations of quarks are guided by confinement on the latter property. Since $\mathbb{C}^3$ transform nontrivially under $\mathrm{SU}(3)$, there exists no isolation of quarks. For $\mathbb{C}^3 \otimes \mathbb{C}^3 \cong \mathrm{Sym}^2\,\mathbb{C}^3 + \Lambda^2\mathbb{C}^3$, it contains no trivial representation and thus no unit vectors are fixed under $\mathrm{SU}(3)$:



Nucleons are made of three quarks. This is allowed as $\Lambda^3\mathbb{C}^3 \subset (\mathbb{C}^3)^{\otimes 3}$ is a trivial representation spanned by the single wedge product $r \wedge g \wedge b$, the color state of a nucleon.

- **qqq-states**. Isospin symmetry $\mathrm{SU}(2)$ as a subset serves a good approximate one for flavors. In addition to the conservation of isospin, the spin of quarks is also involved in the flavor states of nucleons. Up to normalization, they turn out to be

$$p = u \otimes u \otimes d + u \otimes d \otimes u + d \otimes u \otimes u \qquad n = u \otimes d \otimes d + d \otimes u \otimes d + d \otimes d \otimes u$$

- **q$\bar{\text{q}}$-states**. Pions are made from quark-antiquark pairs. Therefore, the flavor states of pions live in $\mathbb{C}^2 \otimes \mathbb{C}^{2*}$. Since pions live in $\mathfrak{sl}_2\mathbb{C}$, we can express pions π^0, π^+ and π^- as matrices H, X and Y in the natural basis of $\mathfrak{sl}_2\mathbb{C}$ respectively. Up to flavors, quarks live in $\mathbb{C}^2$ and antiquarks live in $\mathbb{C}^{2*}$. If we take up and down (anti)quarks as the standard basis vectors, the isospin symmetry of pions can be written as their linear combinations:

$$\pi^+ = u \otimes \bar{d} \qquad \pi^0 = u \otimes \bar{u} - d \otimes \bar{d} \qquad \pi^- = d \otimes \bar{u},$$

which indeed give the right I_3. These are the flavor states of pions.
The color states of pions live in $\mathbb{C}^3 \otimes \mathbb{C}^{3*}$, whose $\mathfrak{h}^*$−lattice is shown on the right. By confinement, pions need to live in a subspace on which $\mathrm{SU}(3)$ acts trivially. It turns out that $\mathbb{C}^3 \otimes \mathbb{C}^{3*}$ has a unique one-dimensional subspace spanned by

$$r \otimes \bar{r} + g \otimes \bar{g} + b \otimes \bar{b} \in \mathbb{C}^3 \otimes \mathbb{C}^{3*}$$

which is invariant under $\mathrm{SU}(3)$. This is then the color state of all pions.

Irreducible Representations of $\mathfrak{sl}_2\mathbb{C}$

Consider the natural basis for the Lie algebra $\mathfrak{sl}_2\mathbb{C}$:

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

with the Lie bracket relations

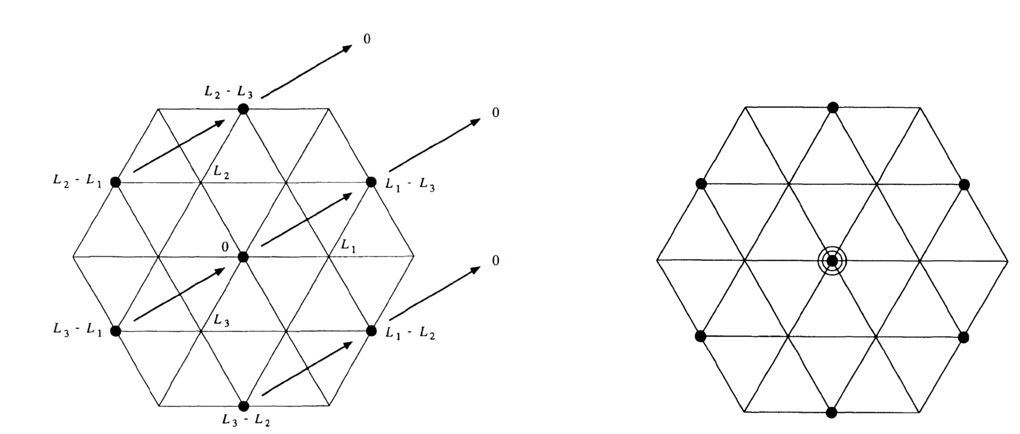
$$[H, X] = 2X \qquad [H, Y] = -2Y \qquad [X, Y] = H,$$

where ± 2 are known as the *roots* of X and Y. Let V be an irreducible finite-dimensional representation of $\mathfrak{sl}_2\mathbb{C}$. Then the action of H on V is diagonalizable and gives a decomposition $V = \oplus V_\lambda$, where the λ (i.e., *weights*) run over a collection of eigenvalues living in the complex numbers such that, for any $v \in V_\lambda$, we have $H \cdot v = \lambda \cdot v$: actions of H maps *weight spaces* V_λ into itself. For X and Y, their actions on V_λ carries it into another subspace $V_{\lambda'}$. Consider the action of X on $v \in V_\lambda$:

$$H \cdot (X \cdot v) = X \cdot (H \cdot v) + [H, X] \cdot v = (\lambda + 2) \cdot (X \cdot v).$$

Therefore, $X \cdot v$ is also an eigenvector for H with eigenvalue $(\lambda + 2)$. Similarly, we have $Y : V_\lambda \to V_{\lambda - 2}$. More generally, each λ must be congruent to every other one mod 2 by the irreducibility of V with a maximum ensured by the finiteness. Now that we cannot raise the weights any higher by applying X, we will apply Y to the vector $v \in V_n$ so that it forms a set $S = \{v, Y \cdot v, Y^2 \cdot v, \ldots, Y^m \cdot v\}$ which spans the representation V. One can also show via direct computation using Lie bracket relations that the λ are integers symmetric about 0. This leads to the important observation that the representation V is completely determined by the highest weight. By thinking of V as a space of homogeneous polynomials, we can inductively arrive at the following conclusion:

Theorem *The unique n-th dimensional irreducible representation of $\mathfrak{sl}_2\mathbb{C}$ is the n-th symmetric power of the standard representation $V \cong \mathbb{C}^2$.*



Irreducible Representations of $\mathfrak{sl}_3\mathbb{C}$

Proceeding by analogy to the study of $\mathfrak{sl}_2\mathbb{C}$, we note the difference that, instead of decomposing V into a direct sum of eigenspaces based on the action of a unique $H \in \mathfrak{sl}_2\mathbb{C}$, the role of H is taken by a two-dimensional subspace $\mathfrak{h} \subset \mathfrak{sl}_3\mathbb{C}$. Since commuting diagonalizable matrices are simultaneously diagonalizable, the finite dimensional representations of $\mathfrak{sl}_3\mathbb{C}$ also admits a decomposition $V = \oplus V_\alpha$, where $\alpha \in \mathfrak{h}^*$ is the linear functional sending H to its eigenvalue $\alpha(H)$.

The analogy for $\mathfrak{sl}_3\mathbb{C}$ of the other two matrices follows from the fact $X, Y \in \mathfrak{sl}_2\mathbb{C}$ are eigenvectors for the *adjoint action* of H on $\mathfrak{sl}_2\mathbb{C}$. With the decomposition $\mathfrak{sl}_3\mathbb{C} = \mathfrak{h} \oplus (\oplus \mathfrak{g}_\alpha)$, one can show in a similar manner that for any $X \in \mathfrak{g}_\alpha$ and $v \in V_\beta$, $X \cdot v$ is again an eigenvector for the action of $\mathfrak{h}$ with eigenvalue $\alpha + \beta$. Let L_i for $i = 1, 2, 3$ be three natural functionals on $\mathfrak{h}$ as in [2], we can explicitly write $\alpha \in \mathfrak{h}^*$ as six functionals $L_i - L_j$. The diagram for the adjoint representation is shown in the upper left, with the arrows indicating the action of the *root space* $\mathfrak{g}_{L_1 - L_3}$.

For any representation V of $\mathfrak{sl}_3\mathbb{C}$, the idea of extremal is admitted as the *highest weight vector*. By artificially defining the positive direction of the root space, we will arrive at the following theorem:

Theorem *For any pair of natural numbers a, b there exists a unique irreducible, finite-dimensional representation of $\mathfrak{sl}_3\mathbb{C}$ with highest weight $aL_1 - bL_3$.*