Algebraic Geometry and Moduli Spaces

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Algebraic Varieties

An **affine algebraic variety** is the common zero set of a collection $\{F_i\}_{i\in I}$ of complex polynomials. In particular, the zero sets of homogeneous polynomials can be viewed as a **projective variety** in a quotient of $\mathbb{C}^{n+1}$ known as the projective space $\mathbb{P}^n$. These varieties form **Zariski topology**, where the open sets are the complement of the varieties. These varieties are completely determined by their **coordinate rings**, defined as $\mathbb{C}[V] = \mathbb{C}[x_1, ..., x_n]/\mathbb{I}(V)$, and converesy every reduced, finite type $\mathbb{C}$-algebra gives an affine/projective variety. The passage from a $\mathbb{C}$-algebra to its variety is denoted by $Spec$, which consists of all the prime ideals of the algebra.

Veronese Map

One useful relationship between projective spaces is the followling: All homogeneous degree d polynomial in the polynomial ring $\mathbb{C}[x_0, ..., x_n]$ form a finite dimensional $\mathbb{C}-$vector space with the basis consisting of $\binom{d+n}{d}$ monomials: $x_0^{d_0}...x_n^{d_n}$ with $\sum d_i = d$. This motivates the **Veronese embedding** of the projective space $\mathbb{P}^n$ into $\mathbb{P}^m(m = \binom{d+n}{d} - 1)$, which is the morphism:

$$[x_0 : ... : x_n] \xrightarrow{\nu_d} [x_0^d : x_0^{d-1}x_1 : ... : x_n^d]$$

Five Points Determine a Conic

A conic in projective space $\mathbb{P}^2$ is the zero set of the polynomial:

$$F(x, y, z) = ax^2 + by^2 + cz^2 + dxy + exz + fyz$$

where the coefficients are not all 0. Hence each line through $\mathbb{C}^6$, denoted by $[a : b : c : d : e : f]$ uniquely determines a conic. Therefore we can identify sets of conics in $\mathbb{P}^2$ with points in $\mathbb{P}^5$, and we say that $\mathbb{P}^5$ parameterizes conics in $\mathbb{P}^2$. This is an example of a solution to a **moduli problem**, which I will talk about later.
Now consider a fixed point $[x_0 : y_0 : z_0]$ in $\mathbb{P}^2$, $F(x_0, y_0, z_0) = 0$ now defines a linear equation satisfied by a, b, c, d, e, f. Hence each point in $\mathbb{P}^2$ defines a hyperplane in $\mathbb{P}^5$ through F! Therefore five points(we require there can be no more than three collinear points) $p_1, p_2, p_3, p_4, p_5 \in \mathbb{P}^2$ determines five hyperplanes $H_1, ..., H_5 \subset \mathbb{P}^5$. The intersection of five linearly independent hyperplanes is nothing but a point in $\mathbb{P}^5$, since intersecting once reduce the dimension by one. So there is exactly one conic passing through five fixed point.

References

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Hilbert Polynomial

A **graded ring** is a ring that decomposes into direct sum of its subrings. The **Hilbert function** is defined on a graded ring $R = R_0 \oplus R_1 \oplus R_2 ... \oplus R_m$ by:

$$m \longrightarrow \dim R_m$$

Let $V \subset \mathbb{P}^n$ be a projective variety, whose coordinate ring is clearly a graded ring. In this way we can define the Hilbert function of a projective variety.
For large m Hilbert function agrees with a polynomial, called the **Hilbert polynomial**:

$$P(m) = e_0 m^d + ... + e_{d-1}m + e_d$$

with degree $d = \dim V$ and $e_0 = \frac{degV}{d!}$. $degV$ is the **degree** of V, which is defined to be the largest possible number of intersections between V and a codimension $\dim V$ linear subvariety of $\mathbb{P}^n$.

The Hilbert Scheme

Fixing an arbitrary polynomial P, the set of all subvarieties with P as its Hilbert polynomial naturally forms a variety, or more precicely, a **scheme**(a generalization of a variety) in its own right. We call this the **Hilbert scheme**. To contruct the Hilbert scheme, note that any projective variety $V \in \mathbb{P}^n$ is uniquely defined by a homogeneous radical ideal $I = \mathbb{I}(V) \subset \mathbb{C}[x_0, ..., x_n]$. Grothendieck showed that for any P, there exists a positive integer r(depending on P) such that for all ideals I defining a variety with Hilbert polynomial P, I is the radical of the subideal generated by its elements of degree r. Hence, to every Hilbert Polynomial P, one can associate a vector subspace $I_r \subset S_r$, where S_r is the vector space of all homogeneous polynomials of degree r. One can compute the dimension of the vector subspace I_r by:

$$d_r = \dim I_r = \dim S_r - \dim S_r/I_r = \binom{r+n}{r} - P(r)$$

In this way, a Hilbert polynomial, together with r, uniquely specifies a **Grassmannian** $G(\binom{r+n}{r}, d_r)$, which consists of all the d_r-dimensional vector subspaces of a $\binom{r+n}{r}$-dimensional vector space S_r. And a variety uniquely determines a single point in the Grassmannian. Therefore, the Hilbert scheme is a very good way to classify and parameterize subvarieties(or more generally, subschemes) of projective space.

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Category, Natural Transformation and the Yoneda Lemma

If F, G are functors between categories $\mathcal{A}, \mathcal{B}$, then a **natural transformation** $\eta : F \implies G$ is a set of morphisms that satisfies:

- The natural transformation must associate a morphism $\eta_A : F(A) \longrightarrow G(A)$ to every object $A \in \mathcal{A}$. This morphism is called a **component** of A.
- For every morphism $f : A_1 \longrightarrow A_2$, we have:

  $$\eta_{A_2} \circ F(f) = G(f) \circ \eta_{A_1}$$

  In other words, the following diagram must commute:

$$\begin{array}{ccc} F(A_1) & \xrightarrow{\eta_{A_1}} & G(A_1) \\ \downarrow F(f) & & \downarrow G(f) \\ F(A_2) & \xrightarrow{\eta_{A_1}} & G(A_2) \end{array}$$

In category theory, one of the most important results regarding natural transformation is called the **Yoneda lemma**. Given a fixed category $\mathcal{A}$, each object $X \in \mathcal{A}$ naturally gives a functor h_X defined by:

$$h_X = Hom(-, X)$$

Hence for any objects $Y \in \mathcal{A}$, $h_X(Y) = Hom(Y, X)$, which is the set of all morphisms from Y to X. The Yoneda lemma states that the set of natural transformation between h_X and h_Y is isomorphic to the set of morphisms from Y to X. In other words:

$$Hom(h_x, h_y) \cong Hom(Y, X)$$

The Yoneda lemma allows us to completely determine any object by looking at the morphisms that maps into it. This is very powerful in the context of moduli problem, where the structure of the moduli space is not obvious.

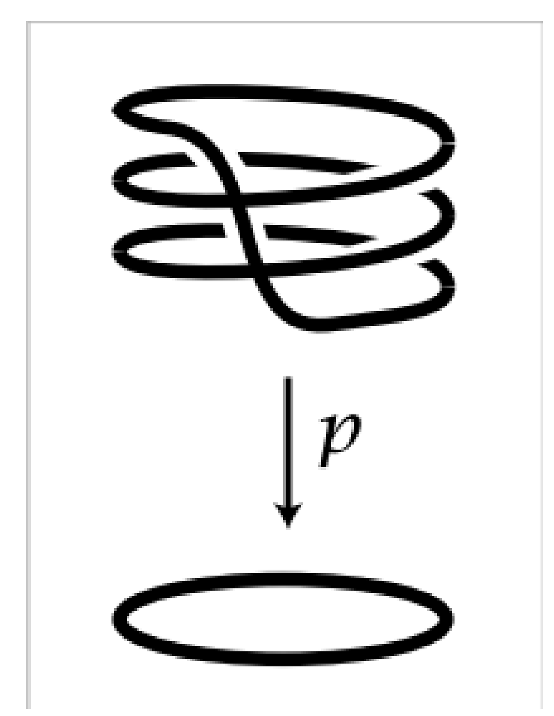
Hilbert functor and Moduli Space

In a more categorical term, the Hilbert scheme is a representation of a functor that sends topological spaces to sets. **Hilbert Functor** It can be defined as:

$$Hilb_X^d : Top \longrightarrow Sets$$

$$Hilb_X^d(Y) = \left\{ Z \subset X \times Y : \begin{array}{c} Z \xrightarrow{\pi_Y} Y \text{is finite and} \\ \text{locally free of rank } d \end{array} \right\}$$

In particular, π_y is analogous to a finite and locally free covering map:



To give a more concrete example, let X be a topological space. Consider the Hilbert functor $Hilb_X^1$. It sends any topological space Y to the set where the elements are topological subspaces $Z \subset X \times Y$ such that the projection from the Z to Y is a homeomorphism. Interestingly enough, $Hilb_X^1$ is in fact naturally isomorphic to the functor h_X and the components of the natural transformation map the set Z to the function $\pi_X \circ \pi_Y^{-1}$. If you let X be $\mathbb{R}$, then $Hilb_X^1$ simply sends any function $g : \mathbb{R} \longrightarrow \mathbb{R}$ to its graph.
Another example concerns the previously mentioned process of five points determining a conic.The Hilbert functor corresponding to a Hilbert polynomial $P(m)$ and subvarieties of $\mathbb{P}^n$ is denoted by: $Hilb_{\mathbb{P}^n}^{P(m)}$. Conics in $\mathbb{P}^2$ has Hilbert polynomial $2m + 1$ and degree 2. The veronese map of degree d associates the set of all degree d hypersurfaces in $\mathbb{P}^n$ with the set of all linear hyperplanes in $\mathbb{P}^M(M = \binom{d+n}{d} - 1)$, which is isomorphic to $\mathbb{P}^M$. Hence we have:

$$Hilb_{\mathbb{P}^2}^{2m+1} \cong h_{\mathbb{P}^5}$$

We say $\mathbb{P}^5$ **represents** the Hilbert functor $Hilb_{\mathbb{P}^n}^{2m+1}$.
These are the simple examples of **moduli spaces**, whose points represent algebraic subvarieties, or more generally subschemes, up to isomorphisms. In the language of moduli spaces, one can parameterize different classes of interesting geometric objects. More often than not, the moduli spaces themselves can have interesting structures beyond merely being a set of points representing classes of objects. And the Yoneda lemma is presicely the tool to study abstract objects like moduli spaces: one can probe the structure of moduli spaces by looking at how other topological spaces map into them. For instance, the map from $\mathbb{A}^1$ into any moduli spaces can give us information about their path connectedness. Hence, solving moduli problems not only helps one classify interesting objects, but give insight into how these classes relate to each other. This makes the study of moduli spaces a very active area in mathematics and physics.